7. Construction and Implementation of LR(1) Parsers

Implementation methods of LR(1) parsers:
table-driven,
set of program statements of rules

7.1 Construction of SLR(1) Parsers

To construct SLR(1) parser:
construct canonical LR(0) collection and deterministic (canonical) LR(0) machine,
simply add the lookahead symbols to the rules of the LR(0) parser.

In deterministic LR(0) machine for G;

$Valid_0(\varepsilon) = {}_1^{*}\{[S \to \cdot\omega] \mid S \to \omega \in P\}$,

$Goto(Valid_0(\gamma), X) = Valid_0(\gamma X)$.

Let $G = (N, \Sigma, P, S)$. $M_0 = (I_0 \cup \{q_s\}, V, q_s, I_f, \delta_0)$ *is the* ***nondeterministic LR(0) machine*** *for G where*

I_0: *set of 0-items,* V: *input alphabet,*

$q_s \notin I_0$: *initial state,* $I_f = I_0$: *set of final states,*

$\delta_0: (I_0 \cup \{q_s\}) \times (V \cup \{\varepsilon\}) \to 2^{I_0}$ *of the form;*

$q_s\varepsilon \to [S \to \cdot\omega]$,

$[A \to \alpha\cdot X\beta]X \to [A \to \alpha X\cdot\beta]$, $X \in V$, *and*

$[A \to \alpha\cdot B\beta]\varepsilon \to [B \to \cdot\omega]$

where $\beta \Rightarrow^* w, \exists w \in \Sigma^*$.

Nondeterministic LR(0) machine for G is computed in time $O(|G|^2)$ and of size $O(|G|^2)$.

Lemma 7.1 *The set of viable prefixes of G is the language accepted by the nondeterministic LR(0) machine M_0 for G, and for any viable prefix γ,*

$$Valid_0(\gamma) = \{q \mid \exists q \in I_0 : q \in \delta_0'(q_s, \gamma)\}$$

where $\delta_0'(q, \alpha X) = \{p \mid \exists \alpha \in V^*, X \in V, r \in \delta_0'(q, \alpha) : p \in \delta_0(r, X)\}$.

Proof. *By construction,*

$$Valid_0(\varepsilon) = {}_1{}^*\{[S \to \cdot\omega] \mid S \to \omega \in P\}$$

$$= \{q \mid \exists q \in I_0 : q \in \delta_0'(q_s, \varepsilon)\}.$$

Assume as i.h. for γ of length n, $n \geq 0$,

$I \neq \emptyset$.э. $I = \{q \mid \exists q \in I_0 : q \in \delta_0'(q_s, \gamma)\}$

iff γ *is viable prefix and* $I = Valid_0(\gamma)$.

$I' \neq \emptyset$.э. $I' = \{q \mid \exists q \in I_0 : q \in \delta_0'(q_s, \gamma X)\}$

iff γ *is viable prefix and* $I' = Goto(Valid_0(\gamma), X) \neq \emptyset$.

Since $Goto(Valid_0(\gamma), X) = Valid_0(\gamma X)$, *and*

$Valid_0(\gamma X) \neq \emptyset$ *iff* γX *is viable prefix,*

$I' \neq \emptyset$ *iff* γX *is viable prefix and* $I' = Valid_0(\gamma X)$. *Q.E.D.*

Nondeterministic LR(0) machine has $O(|G|)$ states.

Theorem 7.2 *Deterministic LR(0) machine for any grammar G can be computed in time* $O(2^{|G|+2\log|G|})$.

Theorem 7.3 *The LR(0) parser of any grammar G can be constructed in time* $O(2^{|G|+2\log|G|})$.

Actions of an SLR(1) parser:

(sa) $[\delta] \mid a \rightarrow [\delta][\delta a] \mid$

(ra) $[\delta][\delta X_1]...[\delta X_1...X_m] \mid a \rightarrow [\delta][\delta A] \mid a$

where $[A \rightarrow X_1...X_m\cdot] \in Valid_0(\delta X_1...X_m)$ *and*

$a \in Follow'_1(A)$.

Theorem 7.4 *The SLR(1) parser of any grammar G can be constructed in time* $O(2^{|G|+2\log|G|+\log|T|})$.

A grammar G is not SLR(1) iff for some state q

(1) $\exists [A_1 \rightarrow \omega_1\cdot], [A_2 \rightarrow \omega_2\cdot] \in q$ *where*

$Follow'_1(A_1) \cap Follow'_1(A_2) \neq \emptyset$, *or*

(2) $\exists [A_1 \rightarrow \alpha\cdot a\beta], [A_2 \rightarrow \omega\cdot] \in q$ *where*

$a \in Follow'_1(A_2)$.

SLR(1) test can be performed in time

$O(|G|^2 \cdot 2^{|G|+\log|G|}) = O(2^{|G|+3\log|G|})$.

7.2 Construction of Canonical LR(1) Parsers

$[A \rightarrow \alpha{\cdot}\beta, \{a_1,..., a_n\}]$ *represents;*

set of 1-items $\{[A \rightarrow \alpha{\cdot}\beta, a_1],..., [A \rightarrow \alpha{\cdot}\beta, a_n]\}$.

Let $G = (N, \Sigma, P, S)$. $M_1 = (I_1 \cup \{q_s\}, V, q_s, I_f, \delta_1)$ *is the* ***nondeterministic LR(1) machine*** *for G where*

I_1: *set of 1-items,* V: *input alphabet,*

$q_s \notin I_1$: *initial state,* $I_f = I_1$: *set of final states,*

$\delta_1: (I_1 \cup \{q_s\}) \times (V \cup \{\varepsilon\}) \rightarrow 2^{I_1}$ *of the form;*

$q_s\varepsilon \rightarrow [S \rightarrow {\cdot}\omega, \varepsilon]$,

$[A \rightarrow \alpha{\cdot}X\beta, y]X \rightarrow [A \rightarrow \alpha X{\cdot}\beta, y], X \in V$,

$[A \rightarrow \alpha{\cdot}B\beta, y]\varepsilon \rightarrow [B \rightarrow {\cdot}\omega, z]$

where $\beta \Rightarrow^* w, \exists w \in \Sigma^*$, *and* $z \in First_1(\beta y)$.

Nondeterministic LR(1) machine for G is computed in time $O(|T|^2 \cdot |G|^2)$ *and of size* $O(|T|^2 \cdot |G|^2)$.

Lemma 7.5 *The set of viable prefixes of G is the language accepted by the nondeterministic LR(1) machine* M_1 *for G, and for any viable prefix* γ,

$$Valid_1(\gamma) = \{q \mid \exists q \in I_1 : q \in \delta_1'(q_s, \gamma)\}$$

where $\delta_1'(q, \alpha X) = \{p \mid \exists \alpha \in V^*, X \in V, r \in \delta_1'(q, \alpha):$
$p \in \delta_1(r, X)\}$.

Theorem 7.6 *The deterministic LR(1) machine for any grammar G can be computed in time*

$$O(2^{|G|^2+4\log|G|}).$$

Theorem 7.7 *The canonical LR(1) parser of any grammar G can be constructed in time*

$$O(2^{|G|^2+4\log|G|}).$$

A grammar G is not SLR(1) iff for some state q

(1) $\exists [A_1 \rightarrow \omega_1\cdot, a], [A_2 \rightarrow \omega_2\cdot, a] \in q$, or

(2) $\exists [A_1 \rightarrow \alpha\cdot a\beta, b], [A_2 \rightarrow \omega\cdot, a] \in q$ where

$a \in \Sigma\cup\{\$\}$.

Theorem 7.8 *A grammar G with terminal alphabet* Σ *can be tested for the LR(1) property in deterministic time* $O(2^{|G|^2+4\log|G|})$.

7.3 Construction of LALR(1) Parsers

To construct LALR(1) parser:

construct LR(0) collection (I_0),

add lookahead symbols in appropriate 0-items.

Let $G = (N, \Sigma, P, S)$ be a reduced grammar and $G' = (N \cup \{S'\}, \Sigma \cup \{\$\}, P \cup \{S' \rightarrow \$S\$\}, S')$ its \$-augmented grammar.

For state q in the LR(0) collection for G' and production rule $A \rightarrow \omega$ of G:

$$LALR(q, A \rightarrow \omega) = \{a \in \Sigma \cup \{\$\} \mid \exists \gamma: q = Valid_0(\gamma), [A \rightarrow \omega\cdot, a] \in Valid_1(\gamma)\}$$

where $LALR(q, A \rightarrow \omega)$ is ***LALR(1) lookahead set for the reduce action by rule $A \rightarrow \omega$ at state q****.*

LALR(1) lookahead sets can be determined directly from the transitions of the deterministic LR(0) machine.

Let Q be the set of states of the deterministic LR(0) machine for the \$-augmented grammar G' for grammar $G = (N, \Sigma, P, S)$.

$(q, A) \in Q \times N$ *where* $\exists p \in I_0, A \in N: q \xrightarrow{A} p$,

$(q, A \rightarrow \omega) \in Q \times P$ *where* $[A \rightarrow \omega\cdot] \in q$.

(q, A) ***goes-to*** $Goto(q, A)$, if $^\exists p \in I_0, A \in N$: $q \xrightarrow{A} p$;

q ***has-transition-on*** X, if $^\exists p \in I_0, X \in V$: $q \xrightarrow{X} p$;

q ***has-null-transition*** (q, A), if $^\exists p \in I_0, A \in N$: $q \xrightarrow{A} p \wedge A \Rightarrow^+ \varepsilon$;

$(Goto(q, \alpha), A)$ ***includes*** (q, B), if $^\exists p \in I_0, A, B \in N$: $q \xrightarrow{B} p \wedge B \rightarrow \alpha A \beta \in P, \beta \Rightarrow^* \varepsilon$;

$(Goto(q, \omega), A \rightarrow \omega)$ ***lookback*** (q, A), if $^\exists p \in I_0, A \in N$: $q \xrightarrow{A} p$;

Lemma 7.9 *Let q be a state in Q that has a transition on a nonterminal B and let A be a nonterminal, and* α *a string in* V^* *such that* $B \Rightarrow^n_{rm} \alpha A$. *Then*

$Goto(q, \alpha A) \neq \emptyset$ and $(Goto(q, \alpha), A)$ ***includes**** (q, B).

Proof. *By induction on n*

1) base: clear

2) $^\exists m < n, \delta' \in V^*, A' \rightarrow \alpha' A \beta'$:

$B \Rightarrow^m_{rm} \delta' A' \Rightarrow_{rm} \delta' \alpha' A \beta' = \alpha A \beta', \beta' \Rightarrow^* \varepsilon$.

As i.h. $Goto(q, \delta' A') \neq \emptyset$ and

$(Goto(q, \delta'), A')$ ***includes**** (q, B).

Then $Goto(q, \delta' \alpha' A) \neq \emptyset$ and

$(Goto(q, \delta' \alpha'), A)$ ***includes*** $(Goto(q, \delta'), A')$.

Q.E.D.

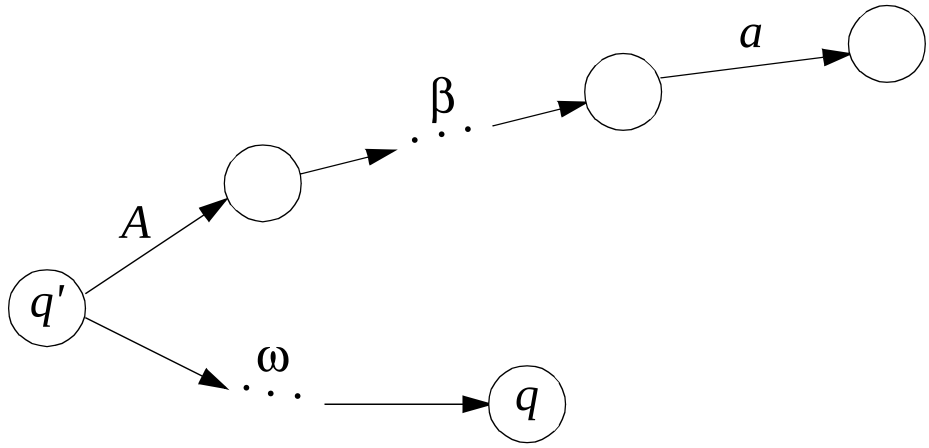
Lemma 7.10 *If for $n \geq 0$, (q, A)* ***includes^{n}*** *(q', B) then for some α in V^{*}, $q = Goto(q', \alpha)$ and $B \Rightarrow^{*}_{rm} \alpha A$.*

directly-reads = goes-to has-transition-on terminal,
reads = goes-to has-null-transition.

Fact 7.11 *For $n > 0$, (q, A)* ***reads^{n}*** *(q', B) iff B is nullable and there is a nullable string δ of length (n-1) such that $q' = Goto(q, A\delta)$ and q' has a transition on B.*

has-LALR-lookahead = lookback includes* reads* directly-reads.

$(q, A \rightarrow \omega)$ ***has-LALR-lookahead*** *a iff*
$a \in LALR(q, A \rightarrow \omega)$. In other words,
$LALR(q, A \rightarrow \omega)$ = ***has-LALR-lookahead*** *$(q, A \rightarrow \omega)$.*

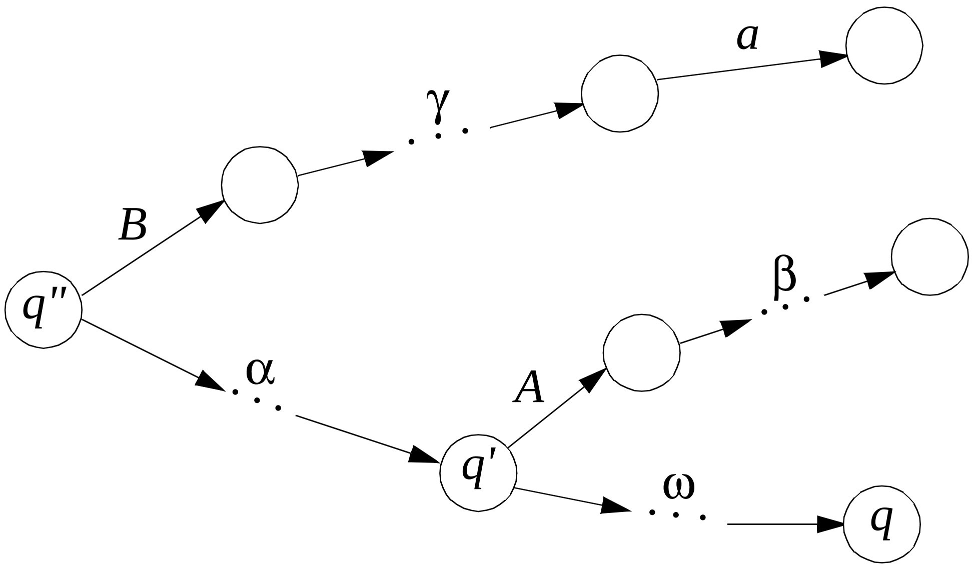


$A \to \omega \in P$, $\beta \Rightarrow^*_{rm} \varepsilon$,

$\exists\gamma, \omega \in V^*$: $Valid_0(\gamma\omega) = q$, $[A \to \omega\cdot, a] \in Valid_1(\gamma\omega)$,

$a \in LALR(q, A\to\omega)$,

$(q, A\to\omega)$ **lookback** (q', A) **reads*** **directly-reads** a.



$A \to \omega \in P$, $B \to \alpha A\beta \in P$, $\beta \Rightarrow^*_{rm} \varepsilon$, $\gamma \Rightarrow^*_{rm} \varepsilon$,

$a \in LALR(q, A\to\omega)$,

$(q, A\to\omega)$ **lookback** (q', A) **includes** (q'', B) **reads*** **directly-reads** a.

Lemma 7.12 *If* $\gamma\beta$ *is a viable prefix of a reduced grammar and* $\beta \Rightarrow_{rm}^{n} az$*, where* $n \geq 0$*, a is a terminal and z is a terminal string, then there is a viable prefix* $\gamma\delta a$*, where* δ *is nullable.*

Proof*. By induction on n.*

1) base: $\beta = \delta a\psi$*, where* $\delta \Rightarrow_{rm}^{*} \varepsilon$.

2) $\beta = \delta' B\psi$*, where* $\delta' \Rightarrow_{rm}^{*} \varepsilon$*,* $B \Rightarrow \beta' \Rightarrow_{rm}^{n-1} az'$.

Since $\gamma\delta' B$ *is viable,* $\gamma\delta'\beta'$ *is also viable.*

As $\beta' \Rightarrow_{rm}^{n-1} az'$*, by applying i.h. to viable prefix* $\gamma'\beta'$*, where* $\gamma' = \gamma\delta'$*. Q.E.D.*

Lemma 7.13 *In a reduced grammar, (q, A)* ***reads**** ***directly-reads*** *a iff q contains an item* $[B \rightarrow \alpha \cdot A\beta]$ *with* $a \in First_1(\beta)$.

Proof. *Let* γ *be a viable prefix .э.* $q = Valid_0(\gamma)$.

$\Rightarrow$: *Assume (q, A)* ***reads**** ***directly-reads*** *a,*

then $\exists \delta \in V^*: \delta \Rightarrow^*_{rm} \varepsilon$, $Goto(q, A\delta a) \neq \emptyset$.

$\gamma A\delta a$ is viable prefix of G' and

$S' \Rightarrow^*_{rm} \gamma' By \Rightarrow_{rm} \gamma'\alpha'\beta' y = \gamma A\delta a\beta' y$, $B \rightarrow \alpha'\beta'$.

(1) In case $\alpha' = \alpha A\delta a$, where α is suffix of γ:

$S' \Rightarrow^*_{rm} \gamma' By \Rightarrow_{rm} \gamma'\alpha A\beta y = \gamma A\beta y$, where $\beta = \delta a\beta'$.

$\therefore [B \rightarrow \alpha \cdot A\beta] \in Valid_0(\gamma)$ and $a \in First_1(\beta)$.

(2) In case $\gamma' = \gamma A\eta$ and $S' \Rightarrow^*_{rm} \gamma' By = \gamma A\eta By$, where $a \in First_1(\eta\beta)$. Then

$S' \Rightarrow^*_{rm} \delta' B' y' \Rightarrow_{rm} \delta'\alpha'' A\beta'' y' = \gamma A\beta'' y'$,

$\beta'' y' \Rightarrow^* \eta By$, $B' \rightarrow \alpha'' A\beta''$.

$\therefore [B' \rightarrow \alpha'' \cdot A\beta''] \in Valid_0(\gamma)$,

$\beta'' \Rightarrow^* \eta Bz$, where z is prefix of y. $a \in First_1(\beta'')$.

$\Leftarrow$: *Assume* $[B \rightarrow \alpha \cdot A\beta] \in q$ *and* $a \in First_1(\beta)$.

$\gamma A\beta$ is a viable prefix,

$\gamma A\delta a$ is also viable prefix, where $\delta \Rightarrow^* \varepsilon$.

$Goto(q, A\delta a) \neq \emptyset$.

$\therefore$ *(q, A)* ***reads**** ***directly-reads*** *a.*

Lemma 7.14 *Let $G = (N, \Sigma, P, S)$ be a grammar. Further let A be a nonterminal, X and Y in V, γ and ψ strings in V^*, y a string in Σ^*, and π a rule string in P^* such that*

$$A \underset{rm}{\overset{\pi}{\Rightarrow}} \gamma X \psi Y y \text{ and } \psi \Rightarrow^* \varepsilon.$$

Then there are symbols X' and Y' in V, a rule $r' = B \to \alpha X' \psi' Y' \beta$ in P, and strings γ', α', β' in V^* and y' in Σ^* such that

$$A \underset{rm}{\overset{\pi'}{\Rightarrow}} \gamma' B y' \underset{rm}{\overset{r'}{\Rightarrow}} \gamma' \alpha X' \psi' Y' \beta y', \gamma' \alpha \alpha' = \gamma,$$

$$X' \Rightarrow^*_{rm} \alpha' X, \psi' \Rightarrow^*_{rm} \varepsilon, \text{ and } Y' \Rightarrow^*_{lm} Y \beta',$$

where $\pi' r'$ is a prefix of π. In other words, in the rightmost derivation of $\gamma X \psi Y y$ from A there is a step showing that the symbols X and Y "originate" from a pair of adjoining symbols in the right-hand side of the same rule.

Lemma 7.15 *In a reduced grammar, terminal a belongs to LALR(q, $A \rightarrow \omega$) iff there is a rule $C \rightarrow \alpha B\beta$ and state q' such that*

(q, $A \rightarrow \omega$) **lookback includes*** *(q', B),*
$[C \rightarrow \alpha \cdot B\beta] \in q'$, and $a \in First_1(\beta)$.

Proof.

$\Rightarrow$: *Assume $a \in LALR(q, A \rightarrow \omega)$, then $^\exists\gamma$ of G', y:*

$S' \Rightarrow^*_{rm} \gamma A a y \Rightarrow_{rm} \gamma\omega a y$, *where* $Valid_0(\gamma\omega) = q$.

$S' \Rightarrow^*_{rm} \gamma' C y' \Rightarrow_{rm} \gamma'\alpha B\beta y'$, $\gamma'\alpha\alpha' = \gamma$,

$B \Rightarrow^*_{rm} \alpha' A$, $\beta \Rightarrow^* a\beta'$,

$[C \rightarrow \alpha \cdot B\beta] \in Valid_0(\gamma'\alpha) = q'$, $a \in First_1(\beta)$,

$\gamma'\alpha B$ is a viable prefix.

$\therefore$ *$Goto(q', \alpha' A) \neq \emptyset$, ($Goto(q', \alpha')$, A)* **includes*** *(q', B)*

$Goto(q', \alpha') = Valid_0(\gamma'\alpha\alpha') = Valid_0(\gamma)$.

γA is a viable prefix, thus

($Valid_0(\gamma, \omega)$, $A \rightarrow \omega$) **lookback** *($Valid_0(\gamma)$, A),*

$\therefore$ *(q, $A \rightarrow \omega$)* **lookback includes*** *(q', B).*

$\Leftarrow$: *Assume* $(q, A\rightarrow\omega)$ ***lookback includes***** (q', B),
$[C \rightarrow \alpha\cdot B\beta] \in q'$, $a \in First_1(\beta)$.

$S' \Rightarrow^*_{rm} \delta C y \Rightarrow_{rm} \delta\alpha B\beta y$, $Valid_0(\delta\alpha) = q'$.

Since $a \in First_1(\beta)$, $\beta \Rightarrow^*_{rm} az$.

$\therefore S' \Rightarrow^*_{rm} \delta\alpha B\beta y \Rightarrow^*_{rm} \delta\alpha Bazy$.

$((q, A\rightarrow\omega)$ ***lookback*** (q_1, A) ***includes***** (q', B),
where $Goto(q_1, \omega) = q$.

$\therefore\ ^{\exists}\alpha', q_1 = Goto(q', \alpha'), B \Rightarrow^*_{rm} \alpha' A$:

$S' \Rightarrow^*_{rm} \delta\alpha Bazy \Rightarrow^*_{rm} \delta\alpha\alpha' Aazy \Rightarrow_{rm} \delta\alpha\alpha'\omega azy$,
where $Valid_0(\delta\alpha\alpha'\omega)=Goto(q', \alpha'\omega)=Goto(q_1, \omega)=q$.

$\therefore a \in LALR(q, A\rightarrow\omega)$.

Theorem 7.16 *Let G be a reduced grammar and G' its \$-augmented grammar. Terminal a of G' is in the LALR(1) lookahead set for the reduce action by rule* $A \rightarrow \omega$ *of G at state q in the deterministic LR(0) machine for G' iff*

$$(q, A\rightarrow\omega) \textbf{ has-LALR-lookahead } a.$$

Theorem 7.17 *Let DM be the deterministic LR(0) machine for the \$-augmented grammar G' for a reduced grammar G. The collection of all LALR(1) lookahead sets* $LALR(q, A\rightarrow\omega)$*, where q is a state of DM and* $A \rightarrow \omega$ *is a rule of G, can be computed in time* $O(t\cdot|G|\cdot|Q|)$*, where Q is the set of states of DM and t is the time taken by one set operation (assignment or union) on subsets of* Σ*.*

Fact 7.18 *Reduce actions of an LALR(1) parser:*

$$[\delta]_0[\delta X_1]_0...[\delta X_1...X_m]_0 \mid a \rightarrow [\delta]_0[\delta A]_0 \mid a,$$

where $A \rightarrow X_1...X_m \in P$*,* δA *is a viable prefix of G', and* $a \in LALR(Valid_0(\delta X_1...X_m), A \rightarrow X_1...X_m)$*.*

Theorem 7.19 *The LALR(1) parser of any grammar G can be constructed in time*

$$O(2^{|G|+2log|G|+log|T|}).$$

LALR(1) test can be performed in time

$$O(2^{|G|+2log|G|+log|T|}).$$

7.5 Optimization of LR(1) Parsers

Inessential Error Entries

$^{\exists}q$, a: *Action[q,a] = "error",*

(q, a) is an ***essential error entry****, if*

$^{\exists}w, y \in \Sigma^*$, $\phi \in Q^*$: $\$[\$] \mid w\$ \underset{M}{\Rightarrow}^* \$\phi q \mid y\$$, $1{:}y\$ = a$.

Otherwise (q, a) is ***inessential****.*

Fact 7.24 *For state q = [γb], where γb is a viable prefix ending with a terminal, all error entries (q,a),* $a \in \Sigma \cup \{\$\}$*, are essential.*

Fact 7.25 *Let q = [δA] for some viable prefix ending with a nonterminal in* $\Sigma \cup \{\$\}$ *such that (q,a) is an error entry. The error entry (q, a) is essential iff*

$$\$\phi' q' \mid ay \underset{M}{\Rightarrow}^* \$\phi q \mid ay,$$

where q' = [γb] for some terminal b in $\Sigma \cup \{\$\}$*, and the configuration* $\$\phi' q' \mid ay$ *is accesible from some initial configuration.*

(q, A → α·Bβ), (q, ·B), (q, B·)

(Goto[q,α], B·) ***symbol-in*** *(q, A → αB·β)*
(q, A → α·Bβ) ***points*** *(Goto[q,α], ·B)*
(q, ·B) ***expands*** *(q, B → ·ω)*
(q, A → αX·β) ***entered-by*** *X, where X ∈ V*

(q, B → ω·) ***on-a-reduce-to*** *(q, B·) whenever Action[Goto[q, ω], a] = "reduce by B → ω"*

(q, A → α·Bβ) ***directly-on-a-passes-null*** *(q, A → αB·β) whenever Action[Goto[q, α], a] = "reduce by B → ε"*

(q, A → α·β) ***error-entry-on-a*** *(Goto[q, α], a) whenever Action[Goto[q, α], a] = "error"*

directly-descends = points expands

may-on-a-access = (on-a-reduces-to symbol-in ∪ directly-descends* ·directly-on-a-passes-null)*

may-imply-a-essential = terminal entered-by^{-1} ·may-on-a-access error-entry-on-a.

Lemma 7.28 *Let* $n \geq 0$ *and*

$$\$[X_1][X_1X_2] \ldots [X_1 \ldots X_m] \mid a \overset{n}{\underset{M}{\Rightarrow}} \$[Y_1][Y_1Y_2] \ldots [Y_1 \ldots Y_p] \mid a .$$

Then for all $[B \to Y_{j+1} \ldots Y_p \cdot \beta] \in Valid(Y_1 \ldots Y_p), j < p$, *there is an item* $[A \to X_{i+1} \ldots X_m \cdot \alpha] \in Valid(X_1 \ldots X_m)$ *such that*

$([X_1 \ldots X_i], A \to X_{i+1} \ldots X_m \cdot \alpha)$ ***may-on-a-access*** $([Y_1 \ldots Y_j], B \to Y_{j+1} \ldots Y_p \cdot \beta)$.

Proof. *By induction on n.*

1) base: trivial

2) Induction:

i) case 1.

$$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \overset{n-1}{\underset{M}{\Rightarrow}} \$[Y_1] \ldots [Y_1 \ldots Y_{p-1}] \mid a$$
$$\Rightarrow \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a$$

i-1) hypothesis:

$\forall [C \to Y_{k+1} \ldots Y_{p-1} \cdot \gamma] \in Valid(Y_1 \ldots Y_{p-1}), k<(p-1)$

$\exists [A \to X_{i+1} \ldots X_m \cdot \alpha] \in Valid(X_1 \ldots X_m)$,

$([X_1 \ldots X_i], A \to X_{i+1} \ldots X_m \cdot \alpha)$ ***may-on-a-access***

$([Y_1 \ldots Y_k], C \to Y_{k+1} \ldots Y_{p-1} \cdot \gamma)$,

i-2) step:

$Action[[Y_1 \ldots Y_{p-1}],a]=Y_p \to \cdot$, $[Y_p \to \cdot] \in Valid(Y_1 \ldots Y_{p-1})$

$([Y_1 \ldots Y_k], C \to Y_{k+1} \ldots Y_{p-1} \cdot \gamma)$

directly-descends* directly-on-a-passes-null

$([Y_1 \ldots Y_j]\ ,\ B \rightarrow Y_{j+1} \ldots Y_p \cdot \beta)$

ii) case 2.

$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \overset{n}{\underset{M}{\Rightarrow}} \$[Y_1] \ldots [Y_1 \ldots Y_{p-1} Z_1 \ldots Z_l] \mid a$

$\Rightarrow \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a$

ii-1) hypothesis:

$\forall [\beta], B \rightarrow \gamma \cdot \delta$ *such that* $\beta\gamma = Y_1 \ldots Y_{p-1} Z_1 \ldots Z_l$, $|\gamma| \geq 1$, *and* $[B \rightarrow \gamma \cdot \delta] \in Valid(\beta\gamma)$,

$\exists\ [A \rightarrow X_{i+1} \ldots X_m \cdot \alpha] \in Valid(X_1 \ldots X_m)$,

$([X_1 \ldots X_i], A \rightarrow X_{i+1} \ldots X_m \cdot \alpha)$ ***may-on-a-access***

$([\beta], B \rightarrow \gamma \cdot \delta)$

ii-2) step:

$Action[[Y_1 \ldots Y_{p-1} Z_1 \ldots Z_l], a] = Y_p \rightarrow Z_1 \ldots Z_l$

$[Y_p \rightarrow Z_1 \ldots Z_l \cdot] \in Valid(Y_1 \ldots Y_{p-1} Z_1 \ldots Z_l)$,

$([X_1 \ldots X_i]\ , A \rightarrow X_{i+1} \ldots X_m \cdot \alpha)$ ***may-on-a-access***

$([Y_1 \ldots Y_{p-1}]\ ,\ Y_p \rightarrow Z_1 \ldots Z_l \cdot)$

$([Y_1 \ldots Y_{p-1}], Y_p \rightarrow Z_1 \ldots Z_l \cdot)$ ***on-a-reduce-to symbol-in***

$([Y_1 \ldots Y_j]\ ,\ B \rightarrow Y_{j+1} \ldots Y_p \cdot \beta)$

Lemma 7.29 *If an error (q,a) is essential, then*
b ***may-imply-a-essential*** *(q,a)*
for some $b \in \Sigma \cup \{\$\}$.

Lemma 7.30 *The set of essential error entries is included in the set* ***may-imply-essential*** $(\Sigma \cup \{\$\})$.

$(q, B\cdot)$ ***left-corner-in*** $(q, A \rightarrow B\cdot\beta)$
$(q, A \rightarrow \alpha\cdot B\beta)$ ***on-*** *a* ***-passes-null*** $(q, A \rightarrow \alpha B\cdot\beta)$
if $(q, A \rightarrow \alpha\cdot B\beta)$ ***may-on-*** *a* ***-access*** $(q, A \rightarrow \alpha B\cdot\beta)$

on- *a* ***-access = (on-*** *a* ***-reduces-to symbol-in*** $\cup$ ***on-*** *a* ***-passes-null)***$^*\cdot$ ***(on-*** *a* ***-reduces-to left-corner-in*** $\cup$ ***directly-descends*** $\cup$ ***on-*** *a* ***-passes-null)****

imply-a-essential = terminal entered-by$^{-1}\cdot$ ***on-*** *a* ***-access error-entry-on-*** *a*

Lemma 7.34 *Let* $n \geq 0$ *and*
$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \underset{M}{\Rightarrow}^n \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a.$
Then for all $[B \rightarrow Y_{j+1} \ldots Y_p\cdot\beta] \in Valid(Y_1 \ldots Y_p)$, $j<p$ *there is an item* $[A \rightarrow X_{i+1} \ldots X_m\cdot\alpha] \in Valid(X_1 \ldots X_m)$ *such that*

$([X_1 \ldots X_i], A \rightarrow X_{i+1} \ldots X_m\cdot\alpha)$ ***on-*** *a* ***-access***
$([Y_1 \ldots Y_j], B \rightarrow Y_{j+1} \ldots Y_p\cdot\beta)$.

Lemma 7.35 *Let* $Y_1 \ldots Y_p$, $p>0$, *be a viable prefix of G' and* $[B \rightarrow Y_{j+1} \ldots Y_p\cdot\beta]$, $j<p$, *be an item in Valid*$(Y_1 \ldots Y_p)$. *If*
$([Y_1 \ldots Y_j], B \rightarrow Y_{j+1} \ldots Y_p\cdot Z\beta)$ ***on-*** *a* ***-passes-null***
$([Y_1 \ldots Y_j], B \rightarrow Y_{j+1} \ldots Y_p Z\cdot\beta)$, *then*

$$\$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a \underset{M}{\Rightarrow}^* \$[Y_1] \ldots [Y_1 \ldots Y_p][Y_1 \ldots Y_p Z] \mid a.$$

Lemma 7.36 *Let $n \geq 0$, and*
*$(q, A \rightarrow \alpha\cdot\beta)$ (**on**-a-**reduces-to symbol-in** $\cup$ **on**-a-**passes-null**)^n $(q', B \rightarrow \gamma\cdot\delta)$*
where $\alpha \neq \varepsilon$ and $\gamma \neq \varepsilon$. Then for any viable prefix $Y_1 \ldots Y_p$, $p \geq 1$, of G' and $j<p$ such that

$[Y_1 \ldots Y_j] = q'$ and $Y_{j+1} \ldots Y_p = \gamma$,

there is a viable prefix $X_1 \ldots X_m$, $m \geq 1$, and $i<m$ such that

$X_1 \ldots X_i = q$, $X_{i+1} \ldots X_m = \alpha$ and

$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \overset{}{\underset{M}{\Rightarrow}} \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a$.*

Proof. *By induction on n.*
1) base: trivial ($n = 0$)
2) hypothesis:
i) case 1.

*$(q, A \rightarrow \alpha\cdot\beta)$ (**on**-a-**reduces-to symbol-in** $\cup$ **on**-a-**passes-null**)$^{n-1}$ $(q, C \rightarrow \omega\cdot)$ **on**-a-**reduces-to symbol-in** $(q', B \rightarrow \gamma\cdot\delta)$*

$\exists$viable prefix $X_1 \ldots X_m$, $m \geq 1$, of G' and $i<m$ such that

$X_1 \ldots X_i = q$, $X_{i+1} \ldots X_m = \alpha$ and

$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \overset{}{\underset{M}{\Rightarrow}} \$[Z_1] \ldots [Z_1 \ldots Z_l] \mid a$*

for a viable prefix $Z_1 \ldots Z_l$ and $k<l$ such that

$[Z_1 \ldots Z_k] = q''$, $Z_{k+1} \ldots Z_l = \omega$.

i-1) step:

By the def. of ***on-a-reduces-to symbol-in***

for any viable prefix $Y_1 \ldots Y_p$, $p \geq 1$, *of G' and* $j<p \ni$

$[Y_1 \ldots Y_p] = q'$ *and* $Y_{j+1} \ldots Y_p = \gamma$

$\exists$ *a viable prefix* $Z_1 \ldots Z_l$ *and* $k<l$ *such that*

$[Z_1 \ldots Z_k] = q''$, $Z_{k+1} \ldots Z_l = \omega$ *and*

$\$[Z_1] \ldots [Z_1 \ldots Z_m] \mid a \overset{*}{\underset{M}{\Rightarrow}} \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a.$

$\therefore$ $X_1 \ldots X_i = q$, $X_{i+1} \ldots X_m = \alpha$ *and*

$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \overset{*}{\underset{M}{\Rightarrow}} \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a.$

ii) case 2.

$(q, A \rightarrow \alpha \cdot \beta)$ (***on-a-reduces-to symbol-in*** $\cup$ ***on-a-passes-null***)$^{n-1}$$(q', B \rightarrow \gamma' \cdot Y\delta)$ ***on-a-passes-null*** $(q', B \rightarrow \gamma' Y \cdot \delta) = (q', B \rightarrow \gamma \cdot \delta)$

By hypothesis and ***Lemma 7.35***.

Lemma 7.37 *Let n Š 0, and*
$(q, A \rightarrow \alpha\cdot\beta)$ (***on-a-reduces-to left-corner-in*** $\cup$
directly-descends $\cup$ ***on-a-passes-null***$)^n$ $(q',B \rightarrow \gamma\cdot\delta)$
Then there is a vaible prefix $Y_1 \ldots Y_p$, $p \geq 1$, *of G' and* $j \leq p, k \leq j$ *such that*

$[Y_1 \ldots Y_j] = q'$, $Y_{j+1} \ldots Y_p = \gamma$, $[Y_1 \ldots Y_k] = q$ *and*

$\$[Y_1] \ldots [Y_1 \ldots Y_k\alpha] \mid a \underset{M}{\Rightarrow}^* \$[Y_1 \ldots Y_p] \mid a$.

Proof. *By induction on n.*

1) base: $(n = 0)$ $q = q'$, $A \rightarrow \alpha\cdot\beta = B \rightarrow \gamma\cdot\delta$

2) induction:

i) case 1: $(j+1) = p$ and

$(q, A \rightarrow \alpha\cdot\beta)$ (***on-a-reduces-to left-corner-in*** $\cup$
directly-descends $\cup$ ***on-a-passes-null***$)^{n-1}$
$(q', C \rightarrow \omega\cdot)$ ***on-a-reduces-to left-cornet-in***
$(q', B \rightarrow C\cdot\delta) = (q', B \rightarrow \gamma\cdot\delta)$

i-1) hypothesis: $\exists$ viable prefix $Y_1 \ldots Y_r$ and $j \leq r$, $k \leq j$ э

$[Y_1 \ldots Y_j] = q'$, $Y_{j+1} \ldots Y_r = \gamma$, $[Y_1 \ldots Y_k] = q$ *and*

$\$[Y_1] \ldots [Y_1 \ldots Y_k\alpha] \mid a \underset{M}{\Rightarrow}^* \$[Y_1 \ldots Y_r] \mid a$.

$\therefore$ $\$[Y_1] \ldots [Y_1 \ldots Y_r] \mid a$

$= [Y_1] \ldots [Y_1 \ldots Y_k] \ldots [Y_1 \ldots Y_j] \ldots [Y_1 \ldots Y_j\omega] \mid a$

$\underset{M}{\Rightarrow}$ $[Y_1] \ldots [Y_1 \ldots Y_k] \ldots [Y_1 \ldots Y_j] \ldots [Y_1 \ldots Y_j\omega] \mid a$

ii) case 2: $\gamma = \varepsilon$ *and*

$(q, A \rightarrow \alpha\cdot\beta)$ (**on-***a***-reduces-to left-cornet-in** $\cup$ **directly-descends** $\cup$ **on-***a***-passes-null**)$^{n-1}$

$(q'', C \rightarrow \eta\cdot\psi)$ ***directly-descends***

$(Goto[q'',\eta], B \rightarrow \cdot\delta) = (q', B \rightarrow \gamma\cdot\delta)$

where $[C \rightarrow \eta\cdot\psi] \in Goto[q'',\eta]$

ii-1) hypothesis: $^{\exists}$*viable prefix* $Y_1 ... Y_p$ *and* $l \le p$, $k \le l \ni$

$[Y_1 ... Y_l] = q''$, $Y_{l+1} ... Y_p = \eta$, $[Y_1 ... Y_k] = q$ *and*

$\$[Y_1] ... [Y_1 ... Y_k\alpha] \mid a \underset{M}{\Rightarrow}^{*} \$[Y_1 ... Y_p] \mid a$.

$\therefore$ *when* $j = p$, *since*

$q' = Goto[q'',\eta] = [Y_1 ... Y_p]$ *and* $\gamma = \varepsilon$

iii) case3: Exercise

$(q, A \rightarrow \alpha\cdot\beta)$ (**on-***a***-reduces-to left-cornet-in** $\cup$ **directly-descends** $\cup$ **on-***a***-passes-null**)$^{n-1}$

on-*a***-passes-null** $(q', B \rightarrow \gamma\cdot\delta)$

Lemma 7.38 *Let q and q' be state of M and $[A \rightarrow \alpha \cdot \beta]$ and $[B \rightarrow \gamma \cdot \delta]$ items such that*

$(q, A \rightarrow \alpha \cdot \beta)$ ***on-a-passes-null*** $(q', B \rightarrow \gamma \cdot \delta)$.

Then there are viable prefixes $X_1 \ldots X_m$, mŠ1, $Y_1 \ldots Y_p$, $p \geq 1$, of G' and $i \leq m$ and $j \leq p$ such that

$[X_1 \ldots X_i] = q, X_{i+1} \ldots X_m = \alpha$

$[Y_1 \ldots Y_j] = q', Y_{j+1} \ldots Y_p = \gamma$ *and*

$\$[X_1] \ldots [X_1 \ldots X_m] \mid a \underset{M}{\Rightarrow}^{*} \$[Y_1] \ldots [Y_1 \ldots Y_p] \mid a.$

Proof: *By* ***Lemma 7.36*** *and* ***7.37****.*

Lemma 7.39 *An error entry (q,a) is* ***essential*** *iff b* ***implies-****a****-essential*** *(q,a) for some $b \in \Sigma \cup \{\$\}$.*
Proof: *By* ***Lemma 7.34, 7.38*** *and* ***Fact 7.25****.*

implies-essential = ***implies-****a_1****-essential*** $\cup \ldots \cup$ ***implies-****a_n****-essential****.*

where $\{a_1, \ldots, a_n\} = \Sigma \cup \{\$\}$

Lemma 7.40 *The set of essential error entries is obtained as the set* ***implies-essential*** $(\Sigma \cup \{\$\})$.
Proof: *By* ***Lemma 7.39****.*

Reducing the number of state in an LR(1) parser

*For a deterministic LR(0)-based LR(1) parser M of G, two state q_1 and q_2 are **compatible**, if*

(1) $Action[q_1, a] = Action[q_2, a]$, or either (q_1, a) or (q_2, a) is an inessential error entry, for all $a \in \Sigma \cup \{\$\}$

(2) $Goto[q_1, A] = Goto[q_2, A]$, or either (q_1, A) or (q_2, A) is an error entry, for all $A \in N$

Let [G'] be the set of states of M

*$\rho = \{Q_1, ..., Q_m\}$ is a **compatible partition** of [G'] if Q_i in ρ contains only pairwise compatible states*

Theorem 7.43 *Let $G = (N, \Sigma, P, S)$ be an LALR(1) grammar and M its LALR(1) parser. Let ρ be a compatible partition of the set of state of M, and let Action' and Goto' be tables defined by:*

$\forall Q \in r, a \in S \cup \{\$\}$:

$Action'[Q,a]$ = error, if $\forall (q,a), q \in Q$, are error

= $Action[q,a]$, where $q \in Q$ and (q,a) is not error entry, otherwise;

$\forall X \in V, Goto'[Q,X] = \bigcup_{q \in Q} Goto[q,X]$

Then Action' and Goto' form a parsing table that represents a right parser of G which behave in the same way as M.

Eliminating reduction by unit rules

Let G be an LALR(1) grammar and M be its parser.
Let $A \rightarrow B$ be a unit rule of G.
q_1= Goto[q, A] and q_2= Goto[q, B] are (A,B)-compatible, if
(1) for all $a \in \Sigma \cup \{\$\}$, Action[$q_1$, a] = Action[$q_2$, a] or one of the three is true

(a) Action[q_2, a] = reduce by $A \rightarrow B$

(b) (q_2, a) is an inessential error entry, or

(c) (q_1, a) is an inessential error entry

(2) for all $C \in N$, one of the following statement is true

(d) Goto[q_1, C] = Goto[q_2, C],

(e) Goto[q_1, C] = ø, or

(f) Goto[q_2, C] = ø.

If q_1, q_2 are (A,B)-compatible,

(1) replace Goto[q, B] by q_1.

$\forall a \in \Sigma \cup \{\$\}$, whenever Action[$q_1$, a] = "error" and Action[q_2, a] $\neq$ "reduce by $A \rightarrow B$",

(2) replace Action[q_1, a] by Action[q_2, a].

Theorem 7.45 *Resulting parser behaves exactly in the same way as the original parser, except that it possibley bypass some reductions by unit rules.*

Let G be an LALR(1) grammar and M be its parser.
Let $A_1 \rightarrow A_2, A_2 \rightarrow A_3, \ldots A_{p-1} \rightarrow A_p$ *be a sequence of unit rules of G.*
$q_1 = Goto[q, A_1]$, ... , $q_p = Goto[q, A_p]$
Assume q_1 *no reduction by a unit rule*

Lemma 7.46 *Let* $A_1, A_2, \ldots, A_p$ *and* $q_1, \ldots, q_p$ *as above. For all i, i=1, ... , p-1, and for all* $a \in \Sigma \cup \{\$\}$*, if* (q_i) *is not an error entry, then* $Action[q_{i+1}, a]$ = *"reduce by* $A_i \rightarrow A_{i+1}$*"*

Lemma 7.47 *Let* $A_1, A_2, \ldots, A_p$ *and* $q_1, \ldots, q_p$ *as above. For any two distinct state* q_i *and* q_j *and for any* $X \in V$, $Goto[q_i, X] = ø$ *or* $Goto[q_j, X] = ø$.

Each new state q_i', $2 \le i \le p$ is defined by extending Action and Goto table as follows.

(1) for all $a \in \Sigma \cup \{\$\}$,
$Action[q_i', a]$ =

1) $Action[q_1, a]$ if $Action[q_1, a]$ is not error
2) $Action[q_j, a]$ where $1 < j < i$, if
    $Action[q_j, a]$ not in {"error",
        "reduced by $A_{j-1} \rightarrow A_j$"}
3) $Action[q_1, a]$ otherwise

(2) for all $X \in N$,
$Goto[q_i', X]$ =

$Goto[q_j, X]$, where $1 \le j < i$,
    if $Goto[q_j, X]$ is not ø
$Goto[q_i, X]$, otherwise