

6. LR(k) Parsing

LR(k) parsing:

The most general deterministic parsing method in which the input string is parsed

- (1) in a single Left-to-right scan,
- (2) producing a Right parse, and
- (3) using lookahead of length k.

Generalization of

- (1) nondeterministic shift-reduce parser
- (2) the simple precedence parser

stack symbols:

grammar symbols are divided up into one or more “context dependent” symbols

Two stack strings $\gamma_1 X$ and $\gamma_2 X$ are equivalent, if exactly same set of parsing actions are valid in the context of $\gamma_1 X$ and $\gamma_2 X$.

Replacing X by equivalent class $[\gamma X]$
refinement of stack symbol

$$X \in V \quad [\gamma X] \in 2^{V^*}$$

6.1 Viable Prefixes

G_{ab} :

$$S \rightarrow aA \mid bB$$

$$A \rightarrow c \mid dAd$$

$$B \rightarrow c \mid dBd$$

$$L(G_{ab}) = \{a, b\}\{d^n cd^n \mid n \geq 0\}.$$

$\$ \alpha c \mid y \$$, where $\$ \alpha:1 \in \{a, b, d\}$ and $1:y\$ \in \{\$, d\}$
 reduce-reduce conflict for $A \rightarrow c$ and $B \rightarrow c$.
 $(1:\alpha = a) \quad (1:\alpha = b)$

Extending lookahead and lookback into length k .

$$\alpha c \mid x \rightarrow \alpha A \mid x, \quad \beta c \mid y \rightarrow \beta B \mid y$$

$$\alpha, \beta \in V^*:k, \quad x, y \in k:\Sigma^* \$.$$

but

$$ad^k c \mid d^k \rightarrow ad^k A \mid d^k, \quad bd^k c \mid d^k \rightarrow bd^k B \mid d^k$$

reduce-reduce conflict for any k !

A string γ is a **viable stack string** of pda M , if

$$\$ \gamma_s \mid w \$ \Rightarrow^* \$ \gamma \mid y \$ \Rightarrow^* \$ \gamma_f \mid \$ \text{ in } M.$$

stack string in some accepting computation M .

Not arbitrary string is a **viable** stack string.

$$2^{V^*} \text{ vs. } 2^{VS} \text{ where } VS \subseteq V^*.$$

Viable stack strings of G_{ab} :

$$\begin{aligned} & \{\varepsilon\} \cup \{ad^n \mid n \geq 0\} \cup \{ad^n c \mid n \geq 0\} \\ & \cup \{ad^n A \mid n \geq 0\} \cup \{ad^n Ad \mid n \geq 1\} \\ & \cup \{bd^n \mid n \geq 0\} \cup \{bd^n c \mid n \geq 0\} \\ & \cup \{bd^n B \mid n \geq 0\} \cup \{bd^n Bd \mid n \geq 1\} \\ & \cup \{S\} \end{aligned}$$

*Not every **action** is valid, for viable stack string*
 $ad^n c \mid \Rightarrow_{\text{valid}} ad^n A \mid$, $bd^n c \mid \Rightarrow_{\text{valid}} bd^n B \mid$; *but*
 $ad^n c \mid \not\Rightarrow_{\text{valid}} ad^n B \mid$, $bd^n c \mid \not\Rightarrow_{\text{valid}} bd^n A \mid$.

*An action r is **valid** for viable stack string γ of M if*
 $\$ \gamma \mid y \$ \Rightarrow^r \$ \gamma' \mid y' \$ \Rightarrow^* \$ \gamma_f \mid \$$ *in M*

*The set of viable stack strings are **infinite**. But we can divide the set of viable stack strings in to a **finite** number of **equivalent classes**.*

*Two viable stack string belongs to the **same equivalent class** if they have same set of **valid actions**.*

Since for any $G = (N, \Sigma, P, S)$ in shift-reduce parser
number of distinct actions = $|\Sigma| + |P| \leq |G|$
number of equivalent classes $\leq 2^{|G|}$.
 \therefore number of equivalent classes is finite.

<i>equivalent classes:</i>	<i>valid actions:</i>
$\{\epsilon\}$	<i>shift a, shift b</i>
$\{ad^n \mid n \geq 0\} \cup \{bd^n \mid n \geq 0\}$	<i>shift c, shift d</i>
$\{ad^n c \mid n \geq 0\}$	<i>reduce by $A \rightarrow c$</i>
$\{bd^n c \mid n \geq 0\}$	<i>reduce by $B \rightarrow c$</i>
$\{aA\}$	<i>reduce by $S \rightarrow aA$</i>
$\{bB\}$	<i>reduce by $S \rightarrow bB$</i>
$\{ad^n A \mid n \geq 1\} \cup \{bd^n B \mid n \geq 1\}$	<i>shift d</i>
$\{ad^n Ad \mid n \geq 1\}$	<i>reduce by $A \rightarrow dAd$</i>
$\{bd^n Bd \mid n \geq 1\}$	<i>reduce by $B \rightarrow dBd$</i>
$\{S\}$	—

stack symbols: equivalent classes (grammar symbol)

$X \Rightarrow [\delta X]: \delta X: \text{viable stack string}$

$[\delta X]: \text{equivalent class of } \delta X$

shift a

$[\delta] \mid a \rightarrow [\delta][\delta a] \mid$

reduce by $A \rightarrow X_1 \dots X_n$

$[\delta][\delta X_1] \dots [\delta X_1 \dots X_n] \mid \rightarrow [\delta][\delta A] \mid$

$\gamma_s = [\epsilon] \text{ and } \gamma_f = \{[\epsilon][S]\}$

$\therefore [\epsilon] \mid yz \Rightarrow^* [\epsilon][Y_1] \dots [Y_1 \dots Y_k] \mid z \Rightarrow^* [\epsilon][S] \mid.$

$Y_1 \dots Y_i$ are viable stack string for $0 \leq \forall i \leq k.$

Regular expression for valid viable stack strings

$\varepsilon, ad^* \mid bd^*, ad^*c, aA, ad^+A \mid bd^+B, ad^+Ad, bd^*c, bB, bd^+Bd, S$

For regular expression E , we define

$$[E] \equiv \cup_{w \in L(E)} [w].$$

$\therefore L(E) \subseteq [E]$, in fact usually $L(E) = [E]$.

equivalent classes:

valid actions:

$[\varepsilon]$

shift a , shift b

$[S]$

—

$[ad^* \mid bd^*]$

shift c , shift d

$[ad^*c]$

reduce by $A \rightarrow c$

$[bd^*c]$

reduce by $B \rightarrow c$

$[aA]$

reduce by $S \rightarrow aA$

$[bB]$

reduce by $S \rightarrow bB$

$[ad^+A \mid bd^+B]$

shift d

$[ad^+Ad]$

reduce by $A \rightarrow dAd$

$[bd^+Bd]$

reduce by $B \rightarrow dBd$

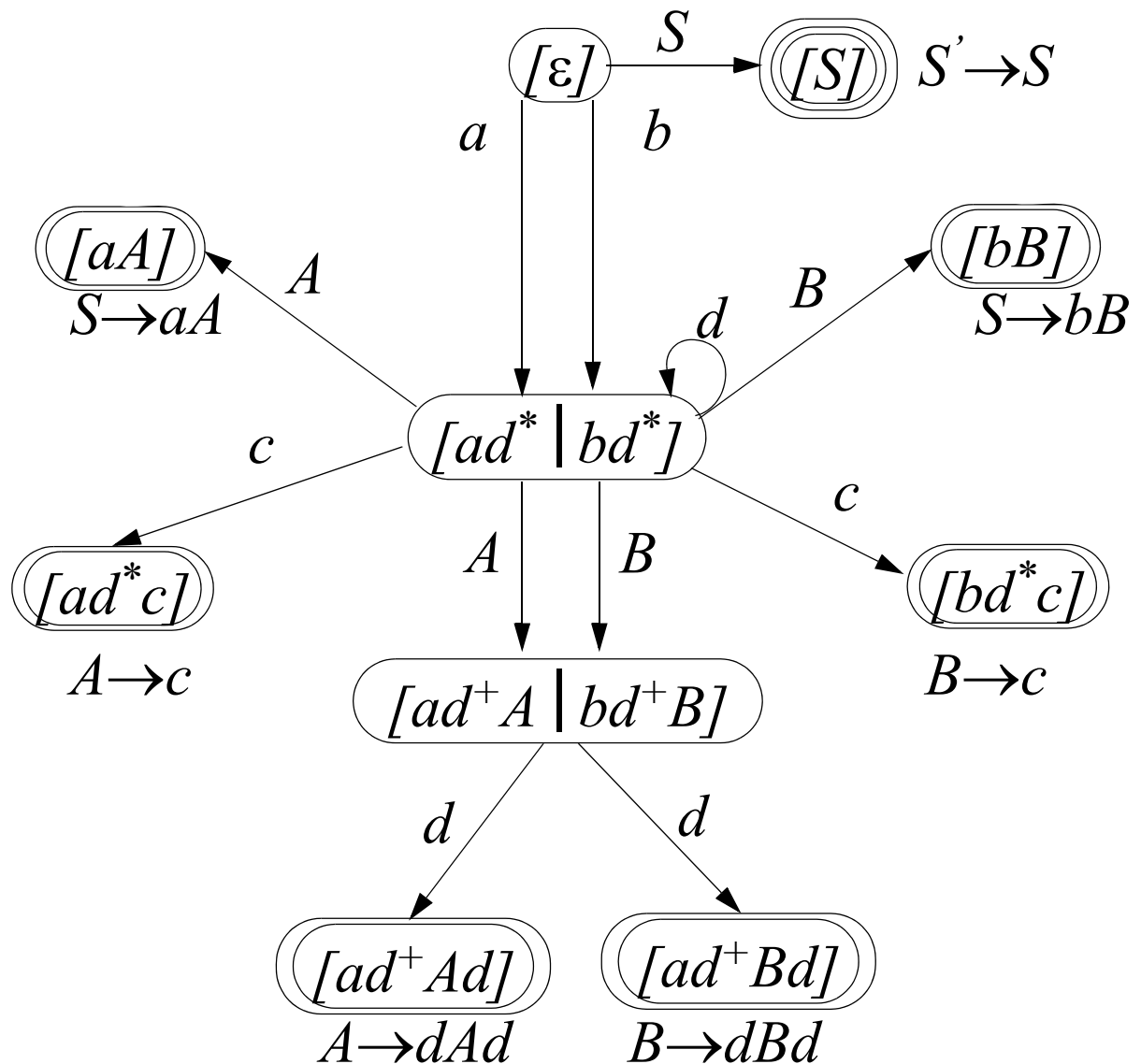
Regular expressions \Rightarrow finite automata

regular expression over $N \cup \Sigma$.

\therefore finite automaton with input alphabet $N \cup \Sigma$.

Characteristic finite state machine

Deterministic parsing of context-free languages?



No “reduce-reduce conflicts” by $A \rightarrow c$ and $B \rightarrow c$.

$$[ad^* | bd^*][ad^*c] \mid \rightarrow [ad^* | bd^*][aA] \mid$$

(reduce by $A \rightarrow c$),

$$[ad^* | bd^*][bd^*c] \mid \rightarrow [ad^* | bd^*][bB]$$

(reduce by $B \rightarrow c$).

note that $[aA] \neq [ad^+A | bd^+B] \neq [bB]$.

But “shift-shift conflict”

$$[ad^*|bd^*] \mid c \rightarrow [ad^*|bd^*][ad^*c] \mid \quad (\text{shift } c),$$

$$[ad^*|bd^*] \mid c \rightarrow [ad^*|bd^*][bd^*c] \mid \quad (\text{shift } c),$$

and

$$[ad^+A|bd^+B] \mid d \rightarrow [ad^+A|bd^+B][ad^+Ad] \mid,$$

$$[ad^+A|bd^+B] \mid d \rightarrow [ad^+A|bd^+B][bd^+Bd] \mid.$$

Consider ad^n , ad^nA , and ad^nB for $n \geq 0$.

$ad^n \in [ad^*|bd^*]$. But

$$ad^nA \in [aA] \text{ and } [ad^+A|bd^+B].$$

$$bd^nA \in [aB] \text{ and } [ad^+A|bd^+B].$$

$[ad^*|bd^*]$ is split into $[a]$, $[ad^+]$, $[b]$, and $[bd^+]$

Since $[ad^*c] \neq [bd^*c]$, $[ad^*] \neq [bd^*]$.

Since $[aA] \neq [ad^+A]$, $[a] \neq [ad^+]$.

Since $[bA] \neq [bd^+A]$, $[b] \neq [bd^+]$.

$[ad^+A|bd^+B]$ is split into $[ad^+A]$ and $[bd^+B]$

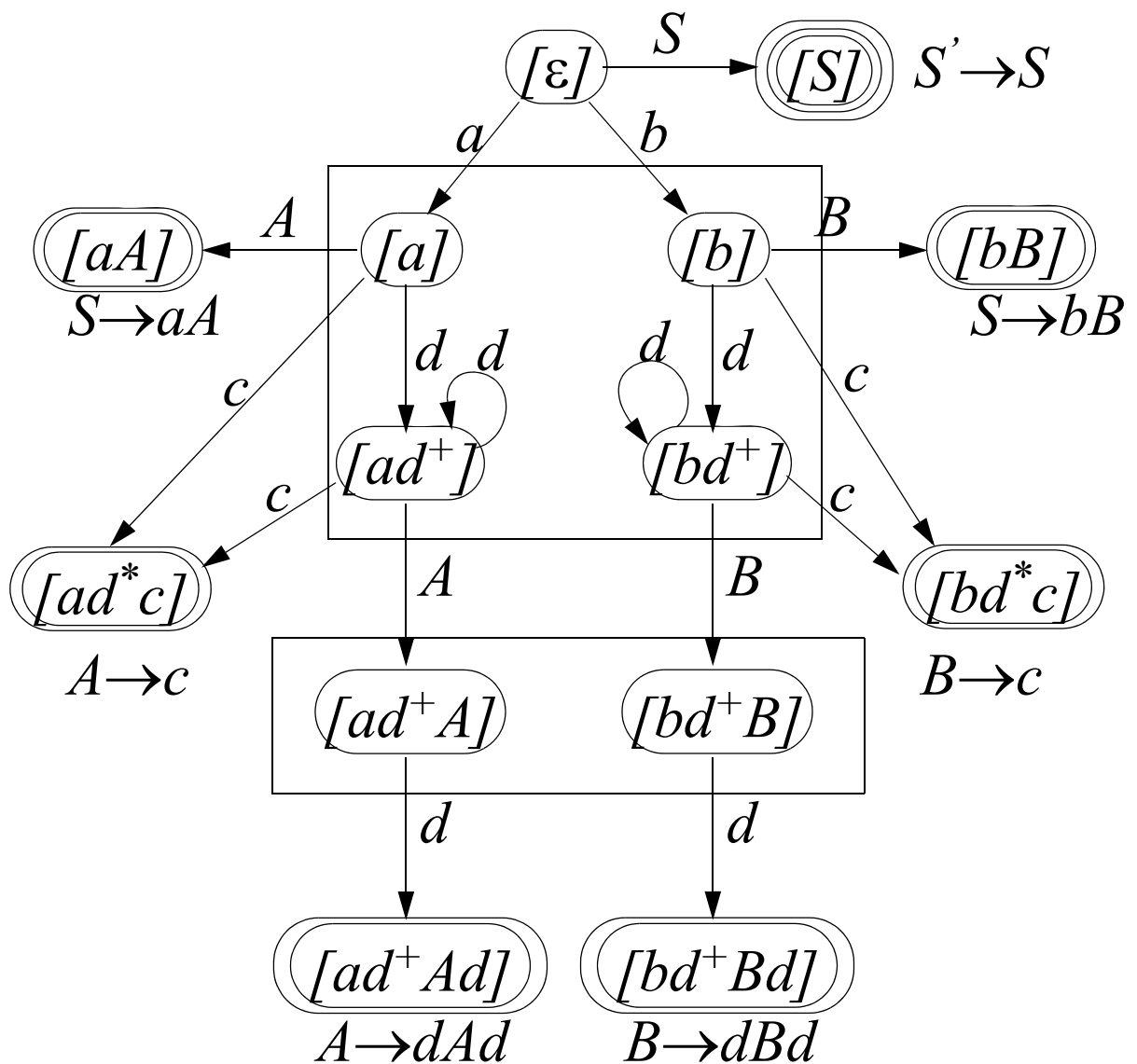
Since $[ad^+Ad] \neq [bd^+Bd]$, $[ad^+A] \neq [bd^+B]$.

$[ad^* \mid bd^*]$

$[a], [b], [ad^+], [bd^+]$

$[ad^+A \mid bd^+B]$

$[ad^+A], [bd^+B]$

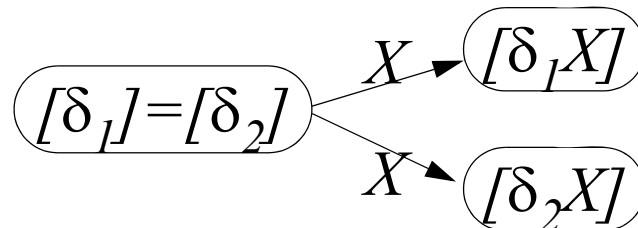


Right invariant(unique outgoing symbol)

If two stack string δ_1 and δ_2 are equivalent, they remain equivalent when they are lengthened.

If $[\delta_1] = [\delta_2]$, $[\delta_1 X] = [\delta_2 X]$.

Otherwise “shift-shift” conflict.

**Unique entry symbol**

Two equivalent stack string should end with same symbols. If $[\gamma_1] = [\gamma_2]$, $\gamma_1:1 = \gamma_2:1$.

Otherwise, reduce action is not uniquely defined.

Consider $[\delta][\delta X_1] \dots [\delta X_1 \dots X_n] \mid \rightarrow [\delta A] \mid$.

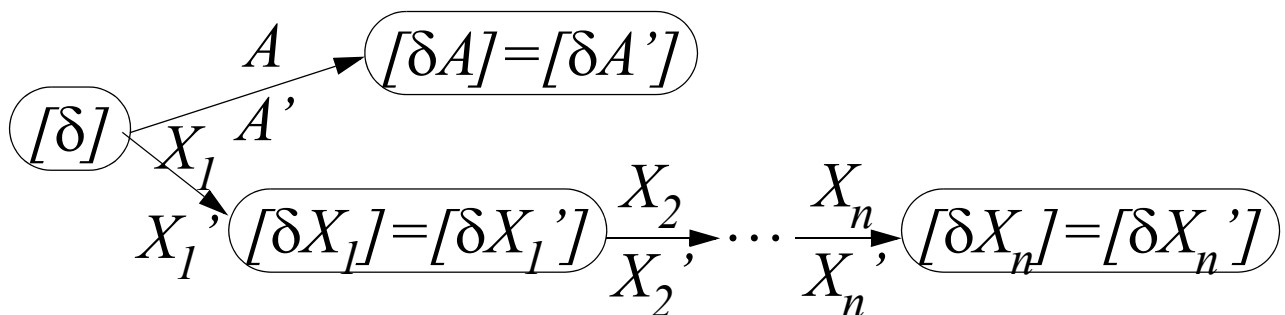
the rule $A \rightarrow X_1 \dots X_n$ is uniquely defined, if

$[\delta][\delta X_1'] \dots [\delta X_1' \dots X_n'] \mid \rightarrow [\delta A'] \mid$,

$\exists A' \rightarrow X_1' \dots X_n' \in P$

$\exists. [\delta A] = [\delta A'], [\delta X_1] = [\delta X_1'], \dots,$

$[\delta X_1 \dots X_n] = [\delta X_1' \dots X_n']$.



Let $G = (N, \Sigma, P, S)$ be a grammar. String $\gamma \in V^*$ is a **viable prefix** of G , if

$$S \Rightarrow_{rm}^* \delta A y \Rightarrow_{rm} \delta \alpha \beta y (= \gamma \beta y)$$

where $\delta \in V^*$, $y \in \Sigma^*$, and $A \rightarrow \alpha \beta \in P$.

γ is a **complete viable prefix**, if $\beta = \varepsilon$.

Fact 6.1 Any viable prefix is a **prefix** of some complete viable prefix.

Lemma 6.4 Any **prefix** of a viable prefix is a **viable prefix**.

Proof $S \Rightarrow_{rm}^n \delta A y \Rightarrow_{rm} \delta \alpha \beta y = \underline{\gamma}_1 \underline{\gamma}_2 \beta y$

i) δ is a prefix of γ_1 .

$\gamma_1 = \delta \alpha'$ where $\alpha = \alpha' \gamma_2$. $\therefore \gamma_1$ is a viable prefix.

ii) γ_1 is a prefix of δ . ($\delta \neq \varepsilon$)

$\delta A = \gamma_1 \eta$. $n > 0$, γ_1 is a viable prefix. **(L6.2)**

Lemma 6.2 Let $G = (N, \Sigma, P, S)$ be a grammar, $\pi \in P^+$, $\gamma, \eta, \delta \in V^*$, $A \in N$, and $y \in \Sigma^*$.

If $S \Rightarrow_{rm}^\pi \gamma \eta y = \delta A y$ in G , and $\pi \neq \varepsilon$. Then

$$S \Rightarrow_{rm}^\pi \delta' A' y'$$

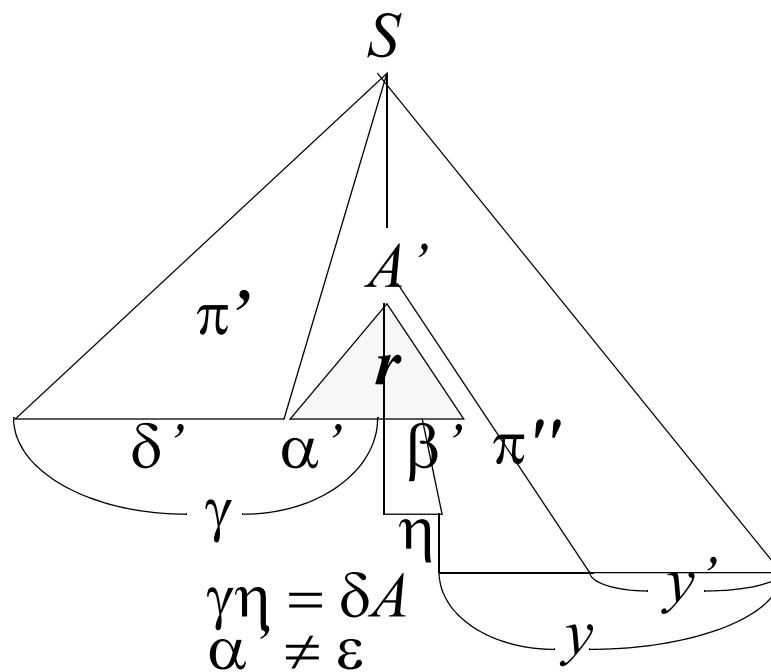
$$\Rightarrow_{rm}^r \underline{\delta'} \underline{\alpha'} \underline{\beta'} y' = \underline{\gamma} \beta' y'$$

$$\Rightarrow_{rm}^{\pi''} \gamma \eta y = \delta A y,$$

$\pi' r \pi'' = \pi$, and $\alpha' \neq \varepsilon$ ($\alpha':1 = \gamma:1$).

If γ is a **prefix** of some nontrivially derived right sentential form (not extending over the last nonterminal), the derivation contains a segment rule (r) that proves γ to be a **viable prefix**, even so that the right-hand side of the rule r cuts γ properly.

Any prefix of nontrivially derived right sentential form (not extending over the last nonterminal) is a **viable prefix**.



Proof induction on the length of π .

i) $|\pi| = 1$. $\pi = S \rightarrow \gamma\eta y = A' \rightarrow \alpha' \beta' y$.

$$\delta' = y' = \varepsilon, (\gamma = \alpha', \eta y = \beta')$$

ii) $|\pi| > 1$. Assume that IH holds for π_1 where $\pi = \pi_1 r_1$.

$$S \Rightarrow_{rm}^{\pi_1} \gamma_1 \eta_1 y_1 = \delta_1 A_1 y_1$$

$$\Rightarrow_{rm}^{r_1} \delta_1 \omega y_1 = \gamma \eta y = \delta A y, \pi = r \pi_1.$$

$$\text{Then } S \Rightarrow_{rm}^{\pi_1} \delta_1' A_1' y_1'$$

$$\Rightarrow_{rm}^{r_1} \underline{\delta_1'} \underline{\alpha_1'} \underline{\beta_1'} y_1' = \gamma_1 \beta_1' y_1'$$

$$\Rightarrow_{rm}^{\pi_1} \gamma_1 \eta_1 y_1 = \delta_1 A_1 y_1,$$

$$\pi_1' r_1 \pi_1'' = \pi_1, \text{ and } \delta_1' \alpha_1' = \gamma_1.$$

Note that $\gamma = \delta_1 \alpha''$ where $\alpha'' \neq \varepsilon$ or $\gamma \alpha = \delta_1$

a) $\gamma = \delta_1 \alpha''$, it is trivial, since

$$\delta' = \delta_1, y' = y_1, \pi' = \pi_1, \pi'' = \varepsilon, r = r_1.$$

b) $\gamma \alpha = \delta_1$

$$S \Rightarrow_{rm}^{\pi_1} \gamma \eta_1 y_1 = \delta_1 A_1 y_1.$$

Lemma 6.3

$S \Rightarrow_{rm}^+ \delta Ay$. Then δ is a *viable prefix*.

Proof. $\eta = \varepsilon$.

Lemma 6.5 Let $A \rightarrow \alpha\beta \in P$. Then if γA is a viable prefix of G , then so is $\gamma\alpha$.

Lemma 6.6 If

$\$ \mid w\$ \Rightarrow^\pi \$\gamma\eta \mid y\$$. Then

$\$ \mid w\$ \Rightarrow^{\pi'} \$\gamma \mid z\$ \Rightarrow^{\pi''} \$\gamma\eta \mid y\$$, and $\pi = \pi' \pi''$.

Proof induction on $|\pi|$.

i) $\pi = \varepsilon$, $\gamma = \eta = \varepsilon$.

ii) $\pi \neq \varepsilon$ and $\eta \neq \varepsilon$, $\pi = \pi_1 r_1$.

(1) $\$ \mid w\$ \Rightarrow^{\pi_1} \$\psi \mid ay\$ \Rightarrow^{r_1} \$\psi a \mid y\$$, or

(2) $\$ \mid w\$ \Rightarrow^{\pi_1} \$\delta\omega \mid y\$ \Rightarrow^{r_1} \$\delta A \mid y\$$.

γ is a prefix of ψ in (1), and a prefix of δ in (2).

Theorem 6.7

Let $G = (N, \Sigma, P, S)$, M be a shift-reduce parser of G .

Any viable stack string of M is

either S or *viable prefix* of G .

Conversely, any viable prefix of G is

a *viable stack string* of M ,

provided that G is reduced.

Proof from lemma 5.17, 5.19.

(shift-reduce parser = right parser)

Given a grammar $G = (N, \Sigma, P, S)$,

Let $G_{VP} = (N_{VP}, \Sigma_{VP}, P_{VP}, [S])$ where

$$N_{VP} = \{[A] \mid A \in N\},$$

$$\Sigma_{VP} = N \cup \Sigma, \text{ and}$$

$$P_{VP} = \{[A] \rightarrow \alpha \mid A \rightarrow \alpha\beta \in P\} \cup \\ \{[A] \rightarrow \alpha[B] \mid A \rightarrow \alpha B\beta \in P, B \in N\}.$$

Example)

$(G_{ab})_{VP}$:

$$[S] \rightarrow \varepsilon \mid a \mid aA \mid b \mid bB \mid a[A] \mid b[B]$$

$$[A] \rightarrow \varepsilon \mid c \mid d \mid dA \mid dAd \mid d[A]$$

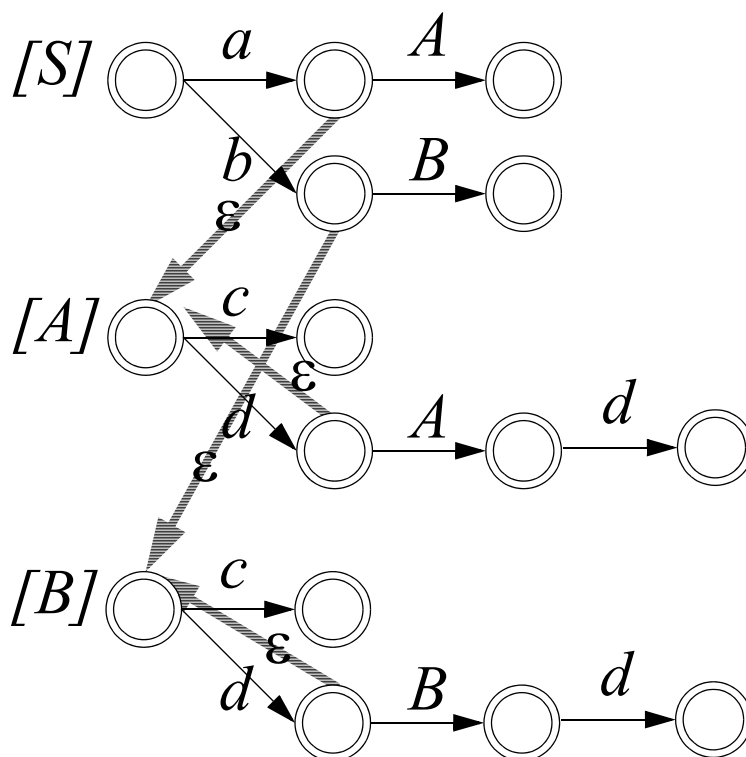
$$[B] \rightarrow \varepsilon \mid c \mid d \mid dB \mid dBd \mid d[B]$$

G_{ab} :

$$S \rightarrow aA \mid bB$$

$$A \rightarrow c \mid dAd$$

$$B \rightarrow c \mid dBd$$



Rule automaton

Lemma 6.8 Let $S \Rightarrow_{rm}^n \delta Ay$ in G . Then

$$[S] \Rightarrow^* \delta[A] \text{ in } G_{VP}.$$

Proof

i) $n=0$: it is clear ($A=S$, $\delta=y=\varepsilon$).

ii) $0 \leq \forall m < n$:

$$S \Rightarrow_{rm}^m \delta' A' y' \Rightarrow_{rm} \delta' \alpha A \beta y' = \delta A \beta y' \text{ in } G. \text{ (L6.2)}$$

$[A'] \rightarrow \alpha[A] \in P_{VP}$, since $A' \rightarrow \alpha A \beta \in P$

$$[S] \Rightarrow^* \delta'[A'] \Rightarrow \delta' \alpha[A] = \delta[A] \text{ in } G_{VP}.$$

Lemma 6.9 Let $[S] \Rightarrow^n \delta[A]$ in G_{VP} . Then

$$S \Rightarrow^* \delta Ay \text{ in } G.$$

Proof

i) $n=0$: $\delta=\varepsilon$, $A=S$, $y=\varepsilon$.

ii) $0 \leq \forall m < n$:

$$S \Rightarrow^m \delta'[A'] \Rightarrow \delta' \alpha[A] = \delta[A] \text{ in } G_{VP},$$

$A' \rightarrow \alpha A \beta \in P$, $\beta \Rightarrow^* x \in \Sigma^*$, since $[A'] \rightarrow \alpha[A] \in P_{VP}$.

$$S \Rightarrow^* \delta' A' y' \Rightarrow \delta' \alpha A \beta \Rightarrow^* \delta' \alpha A x y' = \delta Ay \text{ in } G.$$

Theorem 6.10 *The grammar G_{VP} generates the set of viable prefixes of G . And G_{VP} is right linear.*

Proof.

If $S \Rightarrow_{rm}^* \delta Ay \Rightarrow_{rm} \delta \alpha \beta y (= \gamma \beta y)$ in G ($A \rightarrow \alpha \beta \in P$),
 $[S] \Rightarrow^* \delta[A] \Rightarrow \delta \alpha (= \gamma)$.

If $[S] \Rightarrow^* \delta[A] \Rightarrow \delta \alpha \in V^*$ and $A \rightarrow \alpha \beta \in P$, then
 $S \Rightarrow_{rm}^* \delta Ay \Rightarrow_{rm} \delta \alpha \beta y$.

Theorem 6.11 *For any grammar $G = (N, \Sigma, P, S)$, the set of all viable prefixes is a **regular expression** over V .*

*viable prefixes = valid stack strings
 = regular expression*

G_{VP} is a regular grammar generating the set of viable prefixes of G .

C_0 in G is the **dfa** for G_{VP} .

6.2 Valid LR(k) Items

Let $A \rightarrow \alpha\beta \in P$. Then $[A \rightarrow \alpha\bullet\beta, y]$ is a **k-item**, if

$A \rightarrow \alpha\bullet\beta$ is a position of G and $y \in \Sigma^k$.

0-item $[A \rightarrow \alpha\bullet\beta, \varepsilon] \equiv [A \rightarrow \alpha\bullet\beta]$

$A \rightarrow \alpha\bullet\beta$ is **core** of the item,
 y is the **lookahead** of the item.

A k-item $[A \rightarrow \alpha\bullet\beta, y]$ is **LR(k)-valid** (or **valid**) for string $\gamma(=\delta\alpha) \in V^*$ if

$S \Rightarrow_{rm}^* \delta Az \Rightarrow_{rm} \delta\alpha\beta z (= \gamma\beta z)$ and $y = k:z\k .

Let R_k denotes the set of whole valid LR(k) items.

Fact 6.12 If $[A \rightarrow \alpha\bullet\beta, y]$ is a LR(k) valid item for string $\gamma(=\delta\alpha)$, then γ is a **viable prefix** and

$y \in \text{Follow}_k(\delta\alpha\beta) = \text{Follow}_k(\delta A) \subseteq \text{Follow}_k(A)$.

Conversely, if a string γ is a viable prefix, then some item is LR(k)-valid for γ .

Define $Valid_{LR(k)}^G: V^* \rightarrow 2^{R_k}$.

Let $\gamma \in V^*$. Then

$$Valid_k(\gamma)_{LR(k)}^G =$$

$$\{[A \rightarrow \alpha.\beta, x] \mid S \xRightarrow{rm^*} \delta Az \xRightarrow{rm} \delta \alpha \beta z = \gamma \beta z, x = k:z\$^k\}$$

Valid LR(k) items for the **viable prefix** γ

$$Valid_{LR(k)}^G \equiv Valid_{LR(k)} \equiv Valid_k \equiv Valid$$

$$Valid_k: V^* \rightarrow 2^{R_k}.$$

Define $\rho_{LR(k)} \subseteq V^* \times V^*$

γ_1 is **LR(k)-equivalent** to γ_2 ,

$$\text{written } \gamma_1 \rho_{LR(k)} \gamma_2 \text{ (or } \gamma_1 \rho_k \gamma_2),$$

$$\text{if } Valid_k(\gamma_1) = Valid_k(\gamma_2).$$

The relation ρ_k is called the **LR(k)-equivalence** for G .

ρ_k is an **equivalent** relation.

$[\gamma]_{\rho_k}$ denotes an **equivalent class** of γ under ρ_k

$$[\gamma]_{\rho_k} = \{\delta \mid \gamma \rho_k \delta\}$$

$$[\gamma]_{\rho_k} \equiv [\gamma]_k \equiv [\gamma].$$

We denote $[\gamma]_{\rho_k}$ by $[\gamma]_k$ (or even $[\gamma]$).

We extend the domain of $Valid_k$ from V^* to 2^{V^*} :

$$Valid_k(L) = \{I \in R_k \mid I \in Valid_k(\alpha), \alpha \in L \subseteq V^*\}$$

$$Valid_k([\gamma]_k) = \{I \mid I \in Valid_k(\delta), \delta \in [\gamma]_k\}$$

Since $Valid_k(\gamma_1) = Valid_k(\gamma_2)$, if $\gamma_1, \gamma_2 \in [\gamma]_k$ or $\gamma_1 \rho_k \gamma_2$

We may write $Valid_k(\gamma)$ to denote $Valid_k([\gamma]_k)$.

$$\begin{aligned} Valid_k(\gamma) &= Valid_k([\gamma]_k) \\ &= \{I \in R_k \mid I \in Valid_k(\delta), \delta \in [\gamma]_k\} \end{aligned}$$

$[\gamma]_k$: denotes an **equivalent class** of

γ (viable prefixes) under ρ_k .

may be **infinite** ($[\gamma]_k \subseteq V^*$)

$Valid_k(\gamma)$: denotes a set of

$[A \rightarrow \alpha \cdot \beta, x]$ (LR(k) items) under ρ_k .

always be **finite** ($Valid_k(\gamma) \subseteq R_k$)

$[\gamma_1]_k = [\gamma_2]_k$ iff $Valid_k(\gamma_1) = Valid_k(\gamma_2)$,

bijection correspondence between

$[\gamma]_k$ and $Valid_k(\gamma)$.

We may write $\langle \gamma \rangle_k$ instead of $Valid_k(\gamma)$

Is it possible that $\gamma = \delta$ implies $[\gamma, x]_k = [\delta, y]_k$?

$$\text{Valid}_k(\varepsilon) = \{[S \rightarrow .aA, \$^k], [S \rightarrow .bB, \$^k]\}$$

$$\text{Valid}_k(a) = \{[S \rightarrow a.A, \$^k], [A \rightarrow .c, \$^k], [A \rightarrow .dAd, \$^k]\}$$

$$\text{Valid}_k(aA) = \{[S \rightarrow aA., \$^k]\}$$

$$\text{Valid}_k(ad^{n+1}) = \{[A \rightarrow d.Ad, k:d^n \$^k], [A \rightarrow .c, k:d^{n+1} \$^k], \\ [A \rightarrow .dAd, k:d^{n+1} \$^k]\}$$

$$\text{Valid}_k(ad^n c) = \{[A \rightarrow c., k:d^n \$^k]\}$$

$$\text{Valid}_k(ad^{n+1} A) = \{[A \rightarrow dA.d, k:d^n \$^k]\}$$

$$\text{Valid}_k(ad^{n+1} Ad) = \{[A \rightarrow dAd., k:d^n \$^k]\}$$

...

$$1 + 2 \cdot (2 + 4(k+1)) = 8k + 13 \text{ (LR(k) states)}$$

$$[ad^+]_0 = [ad]_k \cup \dots \cup [ad^k]_k \cup [ad^{k+1}d^*]_k$$

$$[ad^+]_0 = [add^*]_0$$

$$= [ad]_1 \cup [add^*]_1$$

$$= [ad]_2 \cup [add]_2 \cup [adddd^*]_2$$

$$= [ad]_3 \cup [add]_3 \cup [adddd]_3 \cup [addddd^*]_3$$

$$[ad^*c]_0 = [ac]_k \cup \dots \cup [ad^{k-1}c]_k \cup [ad^kcd^*]_k$$

$$[ad^+A]_0 = [adA]_k \cup \dots \cup [ad^kA]_k \cup [ad^{k+1}d^*A]_k$$

$$[ad^+Ad]_0 = [adAd]_k \cup \dots \cup [ad^kd]_k \cup [ad^{k+1}d^*d]_k$$

...

$[\gamma, \varepsilon]_{\rho_0}$ denotes the an equivalent class of γ ,
under ρ_0 .

$[\gamma]_{\rho_0} \equiv [\gamma]_0 \equiv [\gamma]$ denotes an equivalent class of
valid prefixes under LR(0) equivalence.

$Valid_k(\langle \gamma, x \rangle)$ denotes an equivalent class of
 $[A \rightarrow \alpha.\beta, x]$ under ρ_k .

$Valid_k(\gamma)$ can denotes an equivalent class of
valid LR(k)-items under LR(k) equivalence.

$\rho_k: V^* \times V^*, \iota_k: I \times I.$ equivalent relation

$[\gamma]_{\rho_k}: 2^{V^*}, [I]_{\iota_k}: 2^I.$ equivalent class

$Valid_k: V^* \rightarrow 2^I, \text{ or } 2^{V^*} \rightarrow 2^I.$

$Valid_k^{-1}: I \rightarrow 2^{V^*}, \text{ or } 2^I \rightarrow 2^{V^*}.$

$Valid_k(\{[\gamma]_{\rho_k}\}) = Valid_k^{-1}(\{[I]_{\iota_k}\})$

iff $I \in Valid_k(\gamma)$ and/or $\gamma \in Valid_k^{-1}(I)$

Theorem 6.13 *The LR(k)-equivalence ρ_k for G is an equivalence relation on V^* , ρ_k is a finite index, and the index of ρ_k is at most $2^{|G| \cdot (|\Sigma| + 1)^k}$.*

One of the equivalent class under ρ_k is

$\{\gamma \mid \gamma \text{ is not a viable prefix of } G\}$.

Proof.

As $[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}$ iff $\langle \gamma_1 \rangle_{\rho_k} = \langle \gamma_2 \rangle_{\rho_k}$

bijjective correspondence: $[\gamma]_{\rho_k}$ and $\langle \gamma \rangle_{\rho_k}$.

\therefore index of ρ_k = number of distinct sets $\langle \gamma \rangle_{\rho_k}$

At most distinct $|G|$ item cores in G and

$$|\Sigma|^k + |\Sigma|^{k-1} + \dots + |\Sigma|^1 + 1 \leq (|\Sigma| + 1)^k.$$

A string γ is a viable prefix iff $[\gamma]_{\rho_k} \neq \emptyset$

\therefore set of non viable prefixes forms a single equivalent class under ρ_k .

Lemma 6.14 *Let $k \leq l$. Then*

$$\langle \gamma \rangle_k = \{[A \rightarrow \alpha.\beta, k:y] \mid [A \rightarrow \alpha.\beta, y] \in \langle \gamma \rangle_l\}.$$

Lemma 6.15 *Let $k \leq l$. Then LR(l)-equivalence is a refinement of LR(k)-equivalence. More specifically*

$$[\gamma]_k = \cup [\delta]_l.$$

$[\gamma]_k$ are **bijective** correspondence with $\langle \gamma \rangle_k$.

$\langle \gamma \rangle_k$: finite representation of the class $[\gamma]_k$

collection of all sets $[\gamma]_k$

finite representation of the entire LR(k) equivalence

canonical collection of set of LR(k)-valid items for G
canonical LR(k) collection for G : C_k

canonical LR(k) machine M

(or **deterministic LR(k) machine**)

$$M = (C_k, V, \{[\gamma]_k \cdot X \rightarrow [\gamma \cdot X]_k\}, [\varepsilon]_k, \emptyset)$$

ε -free, normal-form, completely specified, and deterministic fa

(1) **right-invariance** of the LR(k)-equivalence.

Since dfa, if $[\gamma_1]_k = [\gamma_2]_k$, $[\gamma_1 \cdot X]_k = [\gamma_2 \cdot X]_k$.

(2) $[\gamma]_k$ has a **unique entry symbol**.

Since $[\gamma]_k \cdot X \rightarrow [\gamma \cdot X]_k \in P$,

if $[\gamma_1]_k = [\gamma_2]_k$, $\gamma_1 \cdot 1 = \gamma_2 \cdot 1$

$$[A \rightarrow \alpha \cdot B \beta, y] \partial_{LR(k)} [B \rightarrow \cdot \omega, z], z \in First_k(\beta y)$$

$$\partial_{LR(k)} \equiv \partial_k \equiv \partial.$$

I_2 is an **immediate LR(k)-descendant** of I_1 , if $I_1 \partial I_2$.

I_2 is an **LR(k)-descendant** of I_1 , if $I_1 \partial^* I_2$.

I_1 is an **(immediate) LR(k)-ancestor** of I_2 ,

if I_2 is an **(immediate) LR(k)-descendant** of I_1 .

$[B \rightarrow \cdot \omega, z]$ is **immediate LR(k)-descendant** of
 $[A \rightarrow \alpha \cdot B \beta, y]$, if $z \in First_k(\beta y)$

$$\langle \gamma \rangle_k^n = \{[A \rightarrow \alpha \cdot \beta, y] |$$

$$S \Rightarrow_{rm}^n \delta A z \Rightarrow_{rm} \delta \alpha \beta z (= \gamma \beta z), y = k:z\}$$

Fact 6.16 $\langle \gamma \rangle_k = \bigcup_{n=0}^{\infty} \langle \gamma \rangle_k^n = \langle \gamma \rangle_k^*$

Lemma 6.17 If

$[A \rightarrow \alpha \cdot B \beta, y] \in \langle \gamma \rangle_k^n$ and $\beta \Rightarrow^m v \in \Sigma^*$. Then

$$[B \rightarrow \cdot \omega, k:vy] \in \langle \gamma \rangle_k^{n+m+1}(\gamma).$$

Lemma 6.18 $\langle \gamma \rangle_k$ is closed under ∂_k , i.e.,

$$\partial_k^*(\langle \gamma \rangle_k) = \langle \gamma \rangle_k.$$

Lemma 6.19 *If*

$[B \rightarrow \cdot \omega, z] \in \langle \gamma \rangle_k^n$ *and* $n > 0$. *Then*

$$[A \rightarrow \alpha \cdot B \beta, y] \in \langle \gamma \rangle_k^m, \beta \Rightarrow^{n-m-1} \nu, k: \nu y = z.$$

Fact 6.20 $\langle \gamma \rangle_k^0 = \{[S \rightarrow \gamma \cdot \omega, \varepsilon] \mid S \rightarrow \gamma \omega \in P\}$

$[A \rightarrow \alpha \cdot \beta]$ *is* **LR-essential** *(or* **essential***), if* $\alpha \neq \varepsilon$ **inessential**, *otherwise.*

$$Ess_{LR}(q) = \{I \in q \mid I \text{ is LR-essential}\}.$$

Lemma 6.21 *Let* $I \in \langle \gamma \rangle_k^n$, $k, n \geq 0$.

(1) $n = 0$, $\gamma = \varepsilon$, $I = [S \rightarrow \cdot \omega, \varepsilon]$.

(2) $\gamma \neq \varepsilon$ *and* I **is essential**.

(3) $n > 0$, I **is inessential**

and $\exists J, J \partial_k I, J \in \langle \gamma \rangle_k^m, m < n$.

Lemma 6.22

$$\langle \varepsilon \rangle_k^n \subseteq \partial_k^* (\{[S \rightarrow \cdot \omega, \varepsilon] \mid S \rightarrow \omega \in P\})$$

$$\langle \gamma \rangle_k^n \subseteq \partial_k^* (Ess(\langle \gamma \rangle_k)), \text{ if } \gamma \neq \varepsilon.$$

Lemma 6.23 (F.6.16, L6.18, and L6.22)

$$\langle \varepsilon \rangle_k = \partial_k^* (\{[S \rightarrow \cdot \omega, \varepsilon] \mid S \rightarrow \omega \in P\})$$

$$\langle \gamma \rangle_k = \partial_k^* (Ess(\langle \gamma \rangle_k)), \text{ if } \gamma \neq \varepsilon.$$

χ_k^X : relation on set of LR(k) items.

$$[A \rightarrow \alpha \cdot X \beta, y] \chi_k^X [A \rightarrow \alpha X \cdot \beta, y],$$

pass-X, or χ^X for short

$$\begin{aligned} \text{Basis}_{LR}(q, X) &= \{[A \rightarrow \alpha X \cdot \beta, y] \mid [A \rightarrow \alpha \cdot X \beta, y] \in q\} \\ &\equiv \chi_k^X(q). \end{aligned}$$

δ_k^X : relation on set of LR(k) items.

$$\begin{aligned} \text{Goto}_{LR}(q, X) &= \partial_k^*(\text{Basis}_{LR}(q, X)) = \partial_k^*(\chi_k^X(q)) \\ &= \chi_k^X \cdot \partial_k^* \equiv \delta_k^X(q). \end{aligned}$$

X-successor, δ_k^X for short

Fact 6.24

If $[A \rightarrow \alpha \cdot \omega \beta, y] \in \langle \gamma \rangle_k^n$, $[A \rightarrow \alpha \omega \cdot \beta, y] \in \langle \gamma \omega \rangle_k^n$.

If $[A \rightarrow \alpha \omega \cdot \beta, y] \in \langle \gamma \rangle_k^n$, $[A \rightarrow \alpha \cdot \omega \beta, y] \in \langle \delta \rangle_k^n$, $\gamma = \delta \omega$.

Lemma 6.25 $\text{Ess}(\langle \gamma X \rangle_k) = \text{Basis}(\langle \gamma \rangle_k, X)$

$$\text{Ess}(\langle \gamma X \rangle_k) = \chi_k^X(\langle \gamma \rangle_k).$$

Lemma 6.26 $\langle \gamma X \rangle_k = \text{Goto}(\langle \gamma \rangle_k, X)$

$$\langle \gamma X \rangle_k = \partial_k^*(\chi_k^X(\langle \gamma \rangle_k)) = \delta_k^X(\langle \gamma \rangle_k).$$

$$\delta_k^\varepsilon(q) = \partial_k^*(q)$$

$$\delta_k^{\gamma \cdot X}(q) = \partial_k^*(\chi_k^X(\delta_k^\gamma(q))), \quad \gamma \neq \varepsilon.$$

$$\begin{aligned} \therefore \delta_k^\gamma(q_s) &= \delta_k^{X_1}(\delta_k^{X_2}(\dots(\delta_k^{X_n}(\delta_k^\varepsilon(q_s)))\dots)) \\ &= \partial_k^*(\chi_k^{X_1}(\partial_k^*(\chi_k^{X_2}(\dots(\partial_k^*(\chi_k^{X_n}(\partial_k^*(q_s)))\dots))))). \end{aligned}$$

$$[\varepsilon]_k = \delta_k^\varepsilon(q_s) = \partial_k^*([S' \rightarrow .S, \$^k])$$

$$[X]_k = \partial_k^*(\chi_k^X([\varepsilon]_k))$$

$$[\gamma X]_k = \partial_k^*(\chi_k^X([\gamma]_k))$$

Algorithm Compute $M = (C_k, V, P, q_s, \emptyset)$

$$q_s := \partial_k^*([S' \rightarrow .S, \$^k]);$$

$$C_k := \{q_s\};$$

$$P := \emptyset;$$

repeat

for $q \in C_k$ **and** $X \in V$ **do**

$$p := \partial_k^*(\chi_k^X(q));$$

$$C_k := C_k \cup \{p\};$$

$$P := P \cup \{q \cdot X \rightarrow p\}$$

od

until nothing is added to C_k .

Lemma 6.27 Let $M = (Q_M, V, P_M, q_s, F)$ be a canonical LR(k) machine for $G = (V, \Sigma, P, S)$. Then

(a) M is deterministic.

(b) $q \in Q_M$, $\text{Goto}(p, X) = q$, unique X .

(c) $q_s \mid \gamma \Rightarrow^* \Phi q \mid$, iff $q = \langle \gamma \rangle_k$

(d) If $F = \{\langle \gamma \rangle_k\}$ for some γ , $L_M = [\gamma]_k$

(e) If $F = \{\langle \gamma \rangle_k \mid \langle \gamma \rangle_k \neq \emptyset\}$,

$L_M = \text{Set of viable prefixes of } X.$

(f) If $F = \{\langle \gamma \rangle_k\}$ for all γ , $L_M = V^*$.

Proof

Assume $\langle \gamma_1 \rangle_k \cdot X \rightarrow \langle \gamma_1 X \rangle_k$ and

$\langle \gamma_2 \rangle_k \cdot X \rightarrow \langle \gamma_2 X \rangle_k$

where $\langle \gamma_1 \rangle_k = \langle \gamma_2 \rangle_k$

Then $\langle \gamma_1 X \rangle_k = \text{Goto}(\langle \gamma_1 \rangle_k, X)$

$= \text{Goto}(\langle \gamma_1 \rangle_k, X) = \langle \gamma_2 X \rangle_k$

$\therefore \langle \gamma_1 X \rangle_k = \langle \gamma_2 X \rangle_k \therefore M$ is deterministic. (a)

Assume $\langle \gamma_1 \rangle_k \cdot X_1 \rightarrow \langle \gamma \rangle_k$ and $\langle \gamma_2 \rangle_k \cdot X_1 \rightarrow \langle \gamma \rangle_k$. Then

$\langle \gamma \rangle_k = \langle \gamma_1 X_1 \rangle_k = \partial_k^*(\text{Basis}(\langle \gamma_1 \rangle_k, X_1)),$

$= \langle \gamma_2 X_2 \rangle_k = \partial_k^*(\text{Basis}(\langle \gamma_2 \rangle_k, X_2)).$

$\therefore \text{Basis}(\langle \gamma_1 \rangle_k, X_1) = \text{Basis}(\langle \gamma_2 \rangle_k, X_2) \neq \emptyset.$

$\therefore X_1 = X_2.$ (b)

$[\varepsilon]_k \cdot \gamma_1 \gamma_2 \Rightarrow^* [\gamma_1]_k \gamma_2$ in M .

Since M is deterministic, $[\gamma]_k$ is the only state. (c)

Theorem 6.28

(a) The LR(k) equivalence of G is the equivalence induced by the canonical LR(k) machine of G .

$$[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k},$$

iff $q_s \mid \gamma_1 \Rightarrow^* q_s \dots q \mid$ and $q_s \mid \gamma_2 \Rightarrow^* q_s \dots q \mid$.

(b) The LR(k) equivalence of G is **right invariance**.

$$\text{If } [\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}, [\gamma_1 \cdot X]_{\rho_k} = [\gamma_2 \cdot X]_{\rho_k}.$$

(c) The LR(k) equivalence of G is **ends with same symbols**.

$$\text{If } [\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}, \gamma_1 \cdot 1 = \gamma_2 \cdot 1.$$

$$[\gamma]_k = \delta_M^\gamma(q_s) = \delta_M^\gamma([\varepsilon]_k) \stackrel{?}{=} (\partial_k^* \chi_k^\gamma)^* (\partial_k^*([\varepsilon]_k)).$$

$$[\varepsilon]_k = \partial_k^*([S' \rightarrow .S, \$^k]).$$

$$[\gamma \cdot X]_k = \partial_k^*(\chi_k^X([\gamma]_k)).$$

$$\equiv \delta_k^X([\gamma]_k).$$

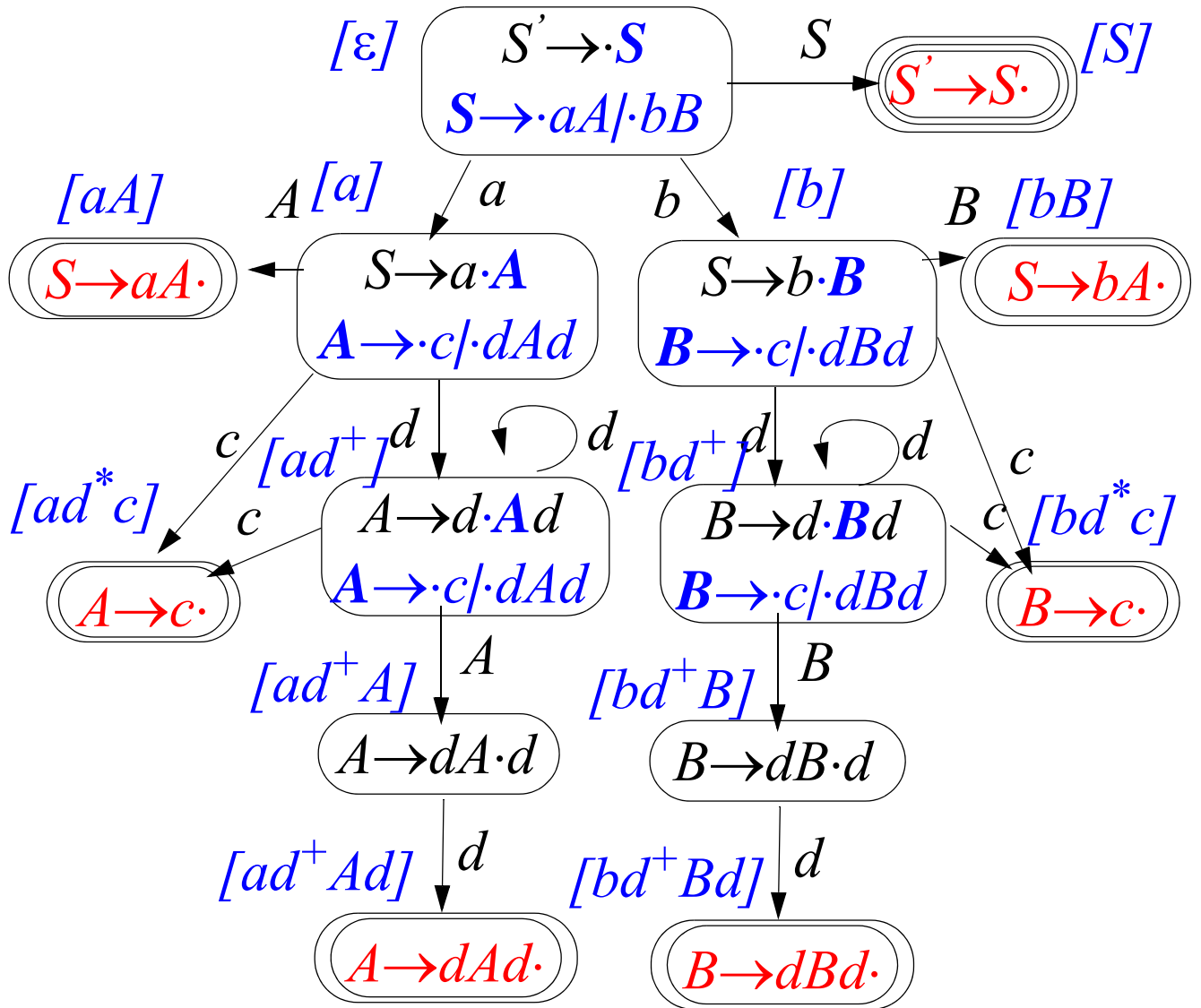
$$\therefore [\gamma]_k = \delta_k^\gamma([\varepsilon]_k).$$

$$\delta_k^\varepsilon(q) = \{q\}.$$

$$\delta_k^{\gamma \cdot X}(q) = \partial_k^*(\chi_k^X(\delta_k^\gamma(q))).$$

$$[\varepsilon]_k = \partial_k^*([S' \rightarrow .S, \$^k]).$$

$$[\gamma \cdot X]_k = \delta_k^X(\delta_k^\gamma(q)).$$



$\$[\epsilon] \mid addcdd\$$

LR(0) Parser

$\Rightarrow^a \$[\epsilon][a] \mid ddcdd\$$

$\Rightarrow^d \$[\epsilon][a][ad] \mid dcdd\$$

$\Rightarrow^d \$[\epsilon][a][ad][add] \mid cdd\$$

$\Rightarrow^c \$[\epsilon][a][ad][add][addc] \mid dd\$$

$\Rightarrow^{A \rightarrow c} \$[\epsilon][a][ad][add][addA] \mid dd\$$

$\Rightarrow^d \$[\epsilon][a][ad][add][addA][addAd] \mid d\$$

$\Rightarrow^{A \rightarrow dAd} \$[\epsilon][a][ad][addA] \mid d\$$

$\Rightarrow^d \$[\epsilon][a][ad][addA][addAd] \mid \$$

$\Rightarrow^{A \rightarrow dAd} \$[\epsilon][a][aA] \mid \$$

$\Rightarrow^{S \rightarrow aA} \$[\epsilon][S] \mid \$$ accept "addcdd"!

$\$ \mid addcdd\$$ Right Parser

$\Rightarrow^a \$a \mid ddcdd\$$

$\Rightarrow^d \$ad \mid dcdd\$$

$\Rightarrow^d \$add \mid cdd\$$

$\Rightarrow^c \$addc \mid dd\$$

$\Rightarrow^{A \rightarrow c} \$addA \mid dd\$$

$\Rightarrow^d \$addAd \mid d\$$

$\Rightarrow^{A \rightarrow dAd} \$adA \mid d\$$

$\Rightarrow^d \$adAd \mid \$$

$\Rightarrow^{A \rightarrow dAd} \$aA \mid \$$

$\Rightarrow^{S \rightarrow aA} \$S \mid \$$

6.3 Canonical LR(k) Parser

Let $G = (N, \Sigma, P, S)$. The **canonical LR(k) parser** for G is a pushdown transducer $M = ([G]_k, \Sigma, \Gamma, P, \tau, [\varepsilon]_k, \{[\varepsilon]_k[S]_k\}, \$, |)$ where

$$[G]_k = \{[\delta]_k \mid \delta \in V^*\}$$

$$\Gamma = \{[\delta]_k[\delta X_1]_k \dots [\delta X_1 \dots X_n]_k \mid y \rightarrow [\delta]_k[\delta A]_k \mid y \\ \mid [A \rightarrow X_1 \dots X_n \bullet, y] \in \langle \delta X_1 \dots X_n \rangle_k\} \quad (ra)$$

$$\cup \{[\delta]_k \mid ay \rightarrow [\delta]_k[\delta a]_k \mid y \\ \mid a \in \Sigma, [A \rightarrow \alpha \bullet a \beta, z] \in \langle \delta \rangle_k \\ y \in First_{\max\{k-1, 0\}}(\beta z)\} \quad (sa)$$

$$\tau([\delta]_k[\delta X_1]_k \dots [\delta X_1 \dots X_n]_k \mid y \rightarrow [\delta]_k[\delta A]_k \mid y) \\ = A \rightarrow X_1 \dots X_n, \\ \tau([\delta]_k \mid ay \rightarrow [\delta]_k[\delta a]_k \mid y) = \varepsilon.$$

$$[B \rightarrow \alpha \bullet A \beta, x] \in \langle \delta \rangle_k \quad [B \rightarrow \alpha A \bullet \beta, x] \in \langle \delta A \rangle_k$$

$$[A \rightarrow \bullet X_1 \dots X_n, y] \in \langle \delta \rangle_k$$

$$[A \rightarrow X_1 \bullet X_2 \dots X_n, y] \in \langle \delta X_1 \rangle_k$$

...

$$[A \rightarrow X_1 \dots X_n \bullet, y] \in \langle \delta X_1 \dots X_n \rangle_k$$

Canonical LR(k) Parser right parser

LR(k) parser \Leftrightarrow right parser

T6.34 (**T5.21**: s/r par., **T5.65**: simple-prec. par)

LR(k) parser \Rightarrow right parser

L6.29, L6.30 (**L5.17, 5.18** and **L5.60, L5.61**)

homomorphism h :

action in LR(k) parser

\rightarrow action in shift-reduce parser

$$h([\delta]_k[\delta X_1]_k \dots [\delta X_1 \dots X_n]_k \mid y \rightarrow [\delta]_k[\delta A]_k \mid y)$$

$$= X_1 \dots X_n \mid \rightarrow A \mid,$$

$$h([\delta]_k \mid a \rightarrow [\delta]_k[\delta a]_k \mid) = \mid a \rightarrow a \mid.$$

Furthermore, h :

configuration in LR(k) parser

\rightarrow configuration in shift-reduce parser

$$h([\varepsilon]_k[X_1]_k \dots [X_1 \dots X_n]_k \mid w\$^k) = \$X_1 \dots X_n \mid w\$^k.$$

$$\therefore [\varepsilon]_k[X_1]_k \dots [X_1 \dots X_n]_k \mid x\k$

$$\Rightarrow^\theta [\varepsilon]_k[Y_1]_k \dots [Y_1 \dots Y_m]_k \mid y\$^k \text{ implies}$$

$$\$X_1 \dots X_n \mid x\$^k \Rightarrow^{h(\theta)} \$Y_1 \dots Y_m \mid y\$^k.$$

LR(k) parser \Leftarrow right parser

L6.31, L6.32, L6.33

(**L5.19, 5.20** and **L5.63, L5.64**)

Lemma 6.29 Let M be a LR(k) parser for G . If

$$[\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid xy \k$

$$\Rightarrow^\theta [\varepsilon]_k [X_1]_k \dots [X_1 \dots X_m]_k \mid y \$, \theta \in \Gamma^* \text{ in } M,$$

then

$$X_1 \dots X_m \Rightarrow_{rm}^{\tau(\theta)^R} Y_1 \dots Y_n x \text{ in } G,$$

$$\text{and } |\theta| = |\tau(\theta)| + |x|.$$

Proof Induction on the length of action string θ .

i) $\theta = \varepsilon$. $x = \varepsilon$, $Y_1 \dots Y_n = X_1 \dots X_m$, and $\tau(\varepsilon) = \varepsilon$.

ii) $\theta = r\theta'$.

ii.1) r is reduce by $A \rightarrow Y_p \dots Y_n$, $1 \leq p \leq n$.

$$[\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid xy \k$

$$\Rightarrow^r [\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_{p-1}]_k [Y_1 \dots Y_{p-1} A]_k \mid xy \k$

$$\Rightarrow^{\theta'} [\varepsilon]_k [X_1]_k \dots [X_1 \dots X_m]_k \mid y \$^k.$$

$$X_1 \dots X_m \Rightarrow_{rm}^{\tau(\theta')^R} Y_1 \dots Y_{p-1} A x \Rightarrow_{rm}^{A \rightarrow \omega} Y_1 \dots Y_n x,$$

and

$$|\theta'| = |\tau(\theta')| + |x|.$$

$$\therefore X_1 \dots X_m \Rightarrow_{rm}^{(\tau(\theta') \cdot A \rightarrow \omega)^R} Y_1 \dots Y_n x, \text{ and}$$

$$|\theta| = 1 + |\theta'| = 1 + |\tau(\theta')| + |x| = |\tau(\theta)| + |x|.$$

ii.2) $r = [\delta]_k \mid a \rightarrow [\delta]_k [\delta a]_k \mid \in \Gamma$,

$$[\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid xy \$$$

$$= [\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid ax' y \$$$

$$\Rightarrow^r [\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k [Y_1 \dots Y_n a]_k \mid x' y \$$$

$$\Rightarrow^{\theta'} [\varepsilon]_k [X_1]_k \dots [X_1 \dots X_m]_k \mid y \$$$

and $|\theta'| = |\tau(\theta')| + |x'|$.

$$X_1 \dots X_m ax'y \Rightarrow_{rm}^{\tau(\theta')R} Y_1 \dots Y_n ax'y$$

$$\therefore \delta \Rightarrow_{rm}^{\tau(\theta') \cdot \varepsilon} \gamma x \text{ in } G, \text{ and}$$

$$|\theta| = 1 + |\theta'| = |\tau(\theta')| + 1 + |x'| = |\tau(\theta)| + |x|$$

Lemma 6.30 *Let M be a canonical LR(k) parser for G . Then*

- (1) $L(M) \subseteq L(G)$,
- (2) $\forall \theta$: actions in M , $\tau(\theta)$ is a **right parse** of w ,
- (3) $Time_G(w) \leq Time_M(w) - |w|$.

Lemma 6.31 Let M be a LR(k) parser.

If $[A \rightarrow \alpha \cdot \beta, z] \in \langle \gamma a_1 \dots a_n \rangle_k$ and $k:y\$^k \in \text{First}_k(\beta z)$,

then $[\varepsilon]_k \dots [\gamma]_k \mid a_1 \dots a_n y \k

$\Rightarrow^\theta [\varepsilon]_k \dots [\gamma]_k [\gamma a_1]_k \dots [\gamma a_1 \dots a_n]_k \mid y \k

where θ is a sequence of shift actions.

valid k -item \Rightarrow valid stack string

Proof

$S' \Rightarrow^* \delta A z' \Rightarrow \delta \alpha \beta z' = \gamma a_1 \dots a_n \beta z'$, and

$k:z' = z. \quad \forall i, 1 \leq i \leq n,$

(i) if $\alpha = \alpha' a_i \dots a_n$,

$[A \rightarrow \alpha' a_i \dots a_n \cdot \beta, z] \in [\gamma a_1 \dots a_{i-1}]_k$

(ii) if $\delta = \gamma a_1 \dots a_{j-1} A z'$. By lemma 6.2,

$S' \Rightarrow^+ \delta' A' y' \Rightarrow \delta' \alpha'' a_i \beta' y' = \gamma a_1 \dots a_i \beta' y'$, and

$\beta' y' \Rightarrow^* a_{i+1} \dots a_{j-1} A z'$.

$\therefore [A' \rightarrow \alpha'' \cdot a_i \beta, k:y'] \in \langle \gamma a_1 \dots a_{i-1} \rangle_k$,

and $\beta' y' \Rightarrow^* a_{i+1} \dots a_{j-1} A z'$

$\Rightarrow^* a_{i+1} \dots a_n \beta z'$.

Lemma 6.32 Let M be a LR(k) parser.

If $X_1 \dots X_m \Rightarrow_{rm}^{\pi R} Y_1 \dots Y_n x$ in G ,

$[A \rightarrow \alpha \bullet \beta, z] \in \langle X_1 \dots X_m \rangle_k$

$k: y \$^k \in \text{First}_k(\beta z)$, and

either $Y_1 \dots Y_n = \varepsilon$ or $Y_n \in N$.

Then $\tau(\theta) = \pi$, $|\theta| = |\pi| + |x|$, and

$[\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid xy \k

$\Rightarrow^\theta [\varepsilon]_k [X_1]_k \dots [X_1 \dots X_m]_k \mid y \k , $\theta \in \Gamma^*$ in M .

Proof Induction on $|\pi|$.

i) $\pi = \varepsilon$. $X_1 \dots X_m = Y_1 \dots Y_n x$.

$[\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k \mid xy \k

$\Rightarrow^\theta [\varepsilon]_k [Y_1]_k \dots [Y_1 \dots Y_n]_k [Y_1 \dots Y_n 1 : x]_k \dots$
 $[Y_1 \dots Y_n x]_k \mid y \$$,

θ is a $|x|$ -length shift action string (**L6.31**)

$|\theta| = |x|$, $\tau(\theta) = \pi = \varepsilon$.

ii) $\pi = B \rightarrow \omega \cdot \pi_1$.

$X_1 \dots X_m \Rightarrow_{rm}^{\pi_1 R} Z_1 \dots Z_p B x_1$

$\Rightarrow_{rm}^r Z_1 \dots Z_p \omega x_1 = Y_1 \dots Y_n x$ in G .

where $x = vx_1$.

$\exists \theta_1 . \exists . \tau(\theta_1) = \varepsilon_1$, $|\theta_1| = |\pi_1| + |x_1|$, and

$[\varepsilon] [Z_1] \dots [Z_1 \dots Z_p] [Z_1 \dots Z_p B] \mid x_1 y \$$

$\Rightarrow^{\theta_1} [\varepsilon] [X_1] \dots [X_1 \dots X_m] \mid y \$$.

And because $[A \rightarrow \alpha \bullet \beta, z] \in \langle X_1 \dots X_m \rangle_k$

$\exists \delta, z' . \exists$.

$$S' \xrightarrow{rm}^* \delta A z' \xrightarrow{rm} \delta \alpha \beta z' = X_1 \dots X_m \beta z' \\ \xrightarrow{rm} X_1 \dots X_m u z' \text{ in } G',$$

and $k:z' = z, k:uz = k:y\$$.

$$\therefore S' \xrightarrow{rm}^* Z_1 \dots Z_p B_1 u z' \xrightarrow{rm} Z_1 \dots Z_p \omega x_1 u z' \text{ in } G'.$$

Here $k:x_1 u z' = k:x_1 u z = k:x_1 y \$$, so

$$[B \rightarrow \omega \cdot, k:x_1 y \$] \in \langle Z_1 \dots Z_p \omega \rangle_k$$

Then $[\varepsilon][Y_1] \dots [Y_1 \dots Y_n] \mid xy \$$

$$= [\varepsilon][Y_1] \dots [Y_1 \dots Y_n] \mid vx_1 y \$$$

$$\xrightarrow{rm}^{\theta_2} [\varepsilon][Y_1] \dots [Y_1 \dots Y_n v] \mid x_1 y \$$$

$$= [\varepsilon][Z_1] \dots [Z_1 \dots Z_p] \dots [Z_1 \dots Z_p \omega] \mid x_1 y \$ \text{ in } M,$$

for some $|v|$ -length shift action string θ_2 .

Then \exists an action $r' . \exists$.

$$r' = [Z_1 \dots Z_p] \dots [Z_1 \dots Z_p \omega] \mid y' \rightarrow$$

$$[Z_1 \dots Z_p] \dots [Z_1 \dots Z_p B] \mid y',$$

where $y' = k:x_1 y \$$.

$$\therefore [\varepsilon][Y_1] \dots [Y_1 \dots Y_n] \mid xy \$$$

$$\xrightarrow{rm}^{\theta_1} [\varepsilon][X_1] \dots [X_1 \dots X_m] \mid y \$ \text{ in } M,$$

where $\theta = \theta_2 r' \theta_1$.

And $\tau(\theta) = \tau(\theta_2) \tau(r') \tau(\theta_1) = r \pi_1 = \pi$,

$$|\theta| = |\pi| + |x|.$$

Lemma 6.33 *Let M be a canonical LR(k) parser for G . Then*

- (1) $L(G) \subseteq L(M)$,
- (2) $\forall \pi$: *right parse* of w in G , $\tau(\theta) = \pi$ in M ,
- (3) $\text{Time}_G(w) \leq \text{Time}_M(w) + |w|$.

Proof

$$Y_1 \dots Y_m = \varepsilon, X_1 \dots X_m = S,$$

$$[A \rightarrow \alpha \bullet \beta, z] = [S' \rightarrow \bullet S, \$^l], y = \varepsilon.$$

Theorem 6.34 *Let M be a canonical LR(k) parser for G . Then*

- (1) M is a *right parser* for G .
- (2) $\forall w \in L(G)$, M produces *all right parses* of w .
- (3) $\text{Time}_M(w) = \text{Time}_G(w) + |w|$.

6.4 LR(k) Grammar

G is **LR(k) grammar** if its canonical LR(k) parser is **deterministic** and $S \Rightarrow^+ S$ in G .

Theorem 6.35 Any LR(k) grammar is **unambiguous**.

“**reduce-reduce conflicts**”

$$[A_1 \rightarrow \omega_1 \cdot, y_1], [A_2 \rightarrow \omega_2 \cdot, y_2],$$

$$\text{if } y_1 = y_2, A_1 \rightarrow \omega_1 \neq A_2 \rightarrow \omega_2.$$

“**shift-reduce conflicts**”

$$[A \rightarrow \alpha \cdot a \beta, z], [B \rightarrow \omega \cdot, y]$$

$$\text{if } y \in \text{First}_k(a\beta z).$$

Lemma 6.36 Let M be the CLR(k) parser for G .

Then M is nondeterministic iff

\exists a state which contains a pair of items exhibiting a reduce-reduce or shift-reduce conflict.

Proof

(\Leftarrow) $I_1, I_2 \in \langle \gamma \rangle_k$ and either

(i) $I_1 = [A \rightarrow X_1 \dots X_n \cdot, y]$, $I_2 = [B \rightarrow Y_1 \dots Y_m \cdot, y]$, or

(ii) $I_1 = [A \rightarrow X_1 \dots X_n \cdot a \beta, z]$, $I_2 = [B \rightarrow Y_1 \dots Y_m \cdot, y]$,
 $y \in \text{First}_k(a\beta z)$.

Assume $m \leq n$, then for $[\delta][\delta X_1] \dots [\delta X_1 \dots X_n] | y$,

(i) $[\delta][\delta X_1] \dots [\delta X_1 \dots X_n] | y \rightarrow [\delta][\delta A] | y$ and

$$[\delta X_1 \dots X_{i-1}] [\delta X_1 \dots X_i] \dots [\delta X_1 \dots X_n] | y \\ \rightarrow [\delta X_1 \dots X_{i-1}] [\delta X_1 \dots X_{i-1} B] | y \text{ in } M,$$

$$Y_1 \dots Y_m = X_i \dots X_n (i = n - m + 1; \mathbf{L6.24})$$

(ii) $[\delta X_1 \dots X_n] | ay' \rightarrow [\delta X_1 \dots X_n] [\delta X_1 \dots X_n a] | y'$

$$[\delta X_1 \dots X_{i-1}] [\delta X_1 \dots X_i] \dots [\delta X_1 \dots X_n] | y \\ \rightarrow [\delta X_1 \dots X_{i-1}] [\delta X_1 \dots X_{i-1} B] | y, k: ay' = y \text{ in } M.$$

\therefore parser is nondeterministic.

(\Rightarrow) let r_1, r_2 are conflicting actions in $[\gamma]$ of M , then

(i) $(r_1) [\gamma] | ay \rightarrow [\gamma] [\gamma a] | y$,

$(r_2) [\gamma] | ayv \rightarrow [\gamma] [\gamma a] | yv$,

where $[A_1 \rightarrow \alpha_1 \cdot a \beta_1, z_1], [A_2 \rightarrow \alpha_2 \cdot a \beta_2, z_2] \in \langle \gamma \rangle_k$
 $y \in \text{First}_{\max\{k-1, 0\}}(\beta_1 z_1), yv \in \text{First}_{\max\{k-1, 0\}}(\beta_2 z_2)$.

Here $\beta_1, \beta_2 \not\Rightarrow * x \$$, and if $y \neq yv$, then $y:1 = \$$,

$\therefore v = \varepsilon$. \therefore no shift-shift conflict.

(ii) $(r_1) [\delta] \dots [\delta X_1 \dots X_n] | y \rightarrow [\delta] [\delta A] | y$,

$(r_2) [\gamma] \dots [\gamma Y_1 \dots Y_m] | y \rightarrow [\gamma] [\gamma B] | y$,

then $[\delta X_1 \dots X_n] = [\gamma Y_1 \dots Y_m]$, and

$[A \rightarrow X_1 \dots X_n \cdot, y] \in \langle \delta X_1 \dots X_n \rangle_k$ and

$[B \rightarrow Y_1 \dots Y_m \cdot, y] \in \langle \gamma Y_1 \dots Y_m \rangle_k$

\therefore reduce-reduce conflict.

Lemma 6.37 $\langle \gamma \rangle_k$ contains a pair of items exhibiting a reduce-reduce conflict iff

$$(a) S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta}_1 \underline{\omega}_1 y_1 = \gamma y_1,$$

$$(b) S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta}_2 \underline{\omega}_2 y_2 = \gamma y_2,$$

$$(c) k:y_1 = k:y_2, \text{ and}$$

$$(d) A_1 \rightarrow \omega_1 \neq A_2 \rightarrow \omega_2$$

hold in G' .

Proof $[A \rightarrow \omega_1, k:y_1], [A \rightarrow \omega_2, k:y_2] \in \langle \gamma \rangle_k$

Lemma 6.38 $\langle \gamma \rangle_k$ contains a pair of items exhibiting a shift-reduce conflict iff

$$(a) S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta}_1 \underline{\omega}_1 y_1 = \gamma y_1,$$

$$(b) S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta}_2 \underline{\omega}_2 y_2 = \gamma v y_2,$$

$$(c) k:y_1 = k:vy_2, v \neq \varepsilon$$

hold in G' .

Proof

(\Leftarrow) $[A \rightarrow \alpha \cdot a \beta, z]$ and $[B \rightarrow \omega \cdot, y]$ are in $\langle \gamma \rangle_k$

(\Rightarrow) $\omega_2 = \alpha v, [A_2 \rightarrow \alpha \cdot v, k:y_2]$, or

$$v = az\omega_2, S' \Rightarrow^* \delta_2 A_2 y_2 = \gamma az A_2 y_2.$$

By lemma 6.2, $\exists A' \rightarrow \alpha'' \cdot a \beta' \in P$,

$[A' \rightarrow \alpha'' \cdot a \beta', k:y'] \in \langle \gamma \rangle_k$

$$a\beta'y' \Rightarrow^* az A_2 y_2 \Rightarrow az\omega_2 y_2 = v y_2.$$

By (c), there exists a shift reduce conflict.

Lemma 6.39 *The following statements are logically equivalent for all G and $k \geq 0$.*

(a) *The canonical LR(k) parser of G is **deterministic**.*

(b) *In the canonical LR(k) machine of the $\$$ -augmented grammar G' no states contains a pair of items exhibiting a **reduce-reduce** or **shift-reduce** conflicts.*

(c) *The conditions*

$$S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta}_1 \underline{\omega}_1 y_1 = \forall y_1,$$

$$S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta}_2 \underline{\omega}_2 y_2 = \forall v y_2,$$

$$\text{and } k:y_1 = k:vy_2, v \neq \varepsilon$$

always implies that

$$\delta_1 = \delta_2, A_1 = A_2, \text{ and } \omega_1 = \omega_2.$$

Theorem 6.40 *For all $k \geq 0$, the class of LR(k) grammars is **properly contained** in the class of LR(k+1) grammars.*

Proposition 6.41 *Any pushdown automaton M with input alphabet Σ can be transformed into an equivalent grammar G with terminal alphabet Σ such that M is **deterministic** if and only if G is **LR(k)** for some $k \geq 0$.*

LR(k) languages = LR(1) languages

deterministic languages = LR(1) languages

Lemma 6.42 Let G be LR(k) grammar and M be a LR(k) parser for G .

Further let $x, y \in \Sigma^*$ and

$$\psi \in [G']^* .\exists. [\varepsilon] | xy\$ \Rightarrow^* \psi | y\$.$$

If $\forall y' .\exists. \text{ the condition } xy' \in L(G), k:y \neq k:y',$
then $\psi|y\$$ is an error configuration.

Proof by contradiction

$$\psi = [\varepsilon][X_1] \dots [X_1 \dots X_n], X_1 \dots X_n \Rightarrow^* x.$$

If $\psi|y\$$ were not an error configuration, then

$$[A \rightarrow \alpha \cdot \beta, z] \in \langle X_1 \dots X_n \rangle_k$$

$$k:y\$ \in \text{First}_k(\beta z).$$

$$S' \Rightarrow^* \delta A z' \$ \Rightarrow \delta \alpha \beta z' \$ = X_1 \dots X_n \beta z' \$,$$

$k:z' \$ = z, \beta \Rightarrow^* v,$ then $k:y\$ = k:vz = k:vz' = k:y',$
and $xvz' \in L(G),$ a contradiction.

Lemma 6.43 Let G be a LR(k) grammar and M be a LR(k) parser for G , $k \geq 1$.

Then M detects an error in any input string in $\Sigma^* \setminus L(G)$.

Proof

(i) $k:w \neq k:w'$, for all $w' \in L(G)$, by lemma 6.42.

(ii) $k:w = k:w'$ for some $w' \in L(G)$.

Then $\exists x, y, y' . \exists$.

(a) $w = xy$,

(b) $k:y = k:y'$ and $xy' \in L(G)$.

(c) $\forall y'' \in \Sigma^*$, $xy'' \in L(G)$ implies $k+1:y \neq k+1:y''$.

Let $y = ay_1$, $y' = ay_1'$, and there exist $\psi, \psi' . \exists$.

$[\epsilon] | xay_1 \$ \Rightarrow \psi | ay_1 \$ \Rightarrow \psi' | y_1 \$$ in M .

Then

$[A \rightarrow \alpha \bullet a \beta, z] \in \langle X_1 \dots X_n \rangle_k$

if $\psi = [\epsilon] \dots [X_1 \dots X_n]$, where $k:ay_1' \in \text{First}(a\beta z)$.

By lemma 6.32,

$$\begin{aligned} [\epsilon] | xay_1 \$ &\Rightarrow^* [\epsilon] [X_1] \dots [X_1 \dots X_n] | ay_1 \$ \\ &= \psi | ay_1 \$ \Rightarrow^* \psi' | y_1 \$. \end{aligned}$$

By (c), $xay_1'' \in L(G)$ always implies $k:y_1 \neq k:y_1''$.

\therefore By lemma 6.42, $\psi | y_1 \$$ is an error configuration.

The parser loops forever when

(1) $S \rightarrow S|a$: LR(0) parser is deterministic,

(2) $S \rightarrow a^{k+1}|ASb^k$, $A \rightarrow \varepsilon$: not LR(k).

Theorem 6.44 Let $k \geq 0$. Then M does not loop forever on any input string.

Proof Assume that M loops forever for $w = xy \in \Sigma^*$.

Then $\exists \psi_i, r_i \cdot \exists$.

$$[\varepsilon]|w\$ \Rightarrow^* \psi_1|y$,$$

$$\psi_i|y\$ \Rightarrow^{r_i} \psi_{i+1}|y\$ \text{ in } M \forall i \geq 1.$$

Let $\psi_i = [\varepsilon] \dots [\gamma_i]$, $\gamma_1 \Rightarrow^* x$, $\gamma_{i+1} \Rightarrow_{rm} \gamma_i$.

Let r_i be reduce action by $A_i \rightarrow \omega_i$ and

$$[A_i \rightarrow \omega_i, k:z_i\$] \in \langle \gamma_i \rangle \forall i.$$

$$[\varepsilon]|xz_i\$ \Rightarrow^* \psi_1|z_i\$ \Rightarrow \psi_2|z_i$,$$

and more generally,

$$\psi_i|z_i\$ \Rightarrow^{r_n} \psi_{n+1}|z_i$.$$

M loops forever on all xz_i , $i \geq 1$.

But $z_i = \varepsilon$ because $xz_i \in L(M) \forall i$ and

M is deterministic. Then

$$S \Rightarrow^* \delta_{n+1} A_{n+1} \Rightarrow \delta_{n+1} w_{n+1} = \gamma_{n+1} \Rightarrow^n \gamma_1 \Rightarrow^* x.$$

G is ambiguous, a contradiction.

6.5 LALR(k) parsing

Theorem 6.45 *The size of the canonical LR(k) parser for grammar G is $O(2^{(|\Sigma|+1)^k|G|} + k \log|\Sigma| + \log|G|)$.*

Proof

$2^{(|\Sigma|+1)^k|G|}$: # of distinct LR(k)-equivalent classes.

$2^{(|\Sigma|+1)^k|G|} \cdot |G| \cdot (|\Sigma|+1)^k$
: sum of the lengths of all reduce actions

Whether does the grammar exist with this upper bound?

$k=0$

Proposition 6.46 *For each $n \geq 0$, let $G_n = (\{A_0, A_1, \dots, A_n\}, \{0, 1, a, a_0, a_1, \dots, a_n\}, P, A_0)$ where P is*

$$A_i \rightarrow 1A_{i+1}a_i, \quad 0 \leq i \leq n-1$$

$$A_n \rightarrow 1A_0a_n$$

$$A_i \rightarrow 0A_i a_i, \quad 1 \leq i \leq n$$

$$A_i \rightarrow 0A_0 a_i, \quad 1 \leq i \leq n$$

$$A_0 \rightarrow a.$$

Then the size of the canonical LR(0) collection for G_n is at least $2^{c|G_n|}$ for all $n \geq 0$, $c > 0$.

Let $G = (N, \Sigma, P, S)$ be a cfg. Then a rule automaton,

$$M = (Q, N \cup \Sigma, R, \langle S', \varepsilon \rangle, Q)$$

$$Q = \{ \langle A, \alpha \rangle \mid A \rightarrow \alpha\beta \in P \} \cup \{ \langle S', \varepsilon \rangle, \langle S', S \rangle \}$$

$$R = \{ \langle S', \varepsilon \rangle S \rightarrow \langle S', S \rangle \}$$

$$\cup \{ \langle S', \varepsilon \rangle \rightarrow \langle B, \varepsilon \rangle \mid S \rightarrow B\beta \in P \}$$

$$\cup \{ \langle A, \alpha \rangle X \rightarrow \langle A, \alpha X \rangle \mid A \rightarrow \alpha X\beta \in P \}$$

$$\cup \{ \langle A, \alpha \rangle \rightarrow \langle B, \varepsilon \rangle \mid A \rightarrow \alpha B\beta \in P \}$$

M is a dfa but ε -moves.

$$M' = (K, N \cup \Sigma, R, \langle S', \varepsilon \rangle, K)$$

$$K = \{ \langle A, \alpha \rangle \mid A \rightarrow \alpha\beta \in P, \alpha \neq \varepsilon \} \cup \{ \langle S', \varepsilon \rangle, \langle S', S \rangle \}$$

$$R' = \{ \langle S', \varepsilon \rangle S \rightarrow \langle S', S \rangle \}$$

$$\cup \{ \langle S', \varepsilon \rangle \rightarrow \langle B, X \rangle \mid S \rightarrow B\beta \in P, B \rightarrow X\gamma \in P \}$$

$$\cup \{ \langle A, \alpha \rangle X \rightarrow \langle A, \alpha X \rangle \mid A \rightarrow \alpha X\beta \in P, \alpha \neq \varepsilon \}$$

$$\cup \{ \langle A, \alpha \rangle X \rightarrow \langle B, X \rangle \mid A \rightarrow \alpha B\beta \in P, \alpha \neq \varepsilon, \\ B \rightarrow X\gamma \in P \}$$

$$|K| = |Q| - |M|$$

$$R' = \{ \langle S', \varepsilon \rangle S \rightarrow \langle S', S \rangle \}$$

$$\cup \{ \langle S', \varepsilon \rangle \rightarrow \langle B, X \rangle \mid S \rightarrow B\beta \in P, B \rightarrow X\gamma \in P \}$$

$$\cup \{ \langle A, \alpha \rangle a \rightarrow \langle A, \alpha a \rangle \mid A \rightarrow \alpha a\beta \in P, \alpha \neq \varepsilon \}$$

$$\cup \{ \langle A, \alpha \rangle B \rightarrow \langle A, \alpha B \rangle, \langle A, \alpha \rangle X \rightarrow \langle B, X \rangle \\ \mid A \rightarrow \alpha B\beta \in P, \alpha \neq \varepsilon, B \rightarrow X\gamma \in P \}$$

Let $G = (N, \Sigma, P, S)$ be a cfg. Then LR(k) automaton,

$$M_k = (Q, N \cup \Sigma, R, \langle S', \varepsilon, \$^k \rangle, Q)$$

$$Q = \{ \langle A, \alpha, x \rangle \mid A \rightarrow \alpha\beta \in P, x \in \text{Follow}_k(A) \}$$

$$\cup \{ \langle S', \varepsilon, \$^k \rangle, \langle S', S, \$^k \rangle \}$$

$$R = \{ \langle S', \varepsilon, \$^k \rangle S \rightarrow \langle S', S, \$^k \rangle \}$$

$$\cup \{ \langle S', \varepsilon, \$^k \rangle \rightarrow \langle B, \varepsilon, y \rangle \mid$$

$$S \rightarrow B\beta \in P, y \in \text{First}_k(\beta \$^k) \}$$

$$\cup \{ \langle A, \alpha, x \rangle X \rightarrow \langle A, \alpha X, x \rangle \mid A \rightarrow \alpha X\beta \in P \}$$

$$\cup \{ \langle A, \alpha, x \rangle \rightarrow \langle B, \varepsilon, y \rangle \mid$$

$$A \rightarrow \alpha B\beta \in P, y \in \text{First}_k(\beta x) \}$$

M is a dfa but ε -moves.

$$M' = (K, N \cup \Sigma, R, \langle S', \varepsilon, \$^k \rangle, K)$$

$$K = \{ \langle A, \alpha, x \rangle \mid A \rightarrow \alpha\beta \in P, \alpha \neq \varepsilon, x \in \text{Follow}_k(A) \}$$

$$\cup \{ \langle S', \varepsilon, \$^k \rangle, \langle S', S, \$^k \rangle \}$$

$$R' = \{ \langle S', \varepsilon, \$^k \rangle S \rightarrow \langle S', S, \$^k \rangle \}$$

$$\cup \{ \langle S', \varepsilon, \$^k \rangle \rightarrow \langle B, X, y \rangle \mid$$

$$S \rightarrow B\beta \in P, B \rightarrow X\gamma \in P, y \in \text{First}_k(\beta \$^k) \}$$

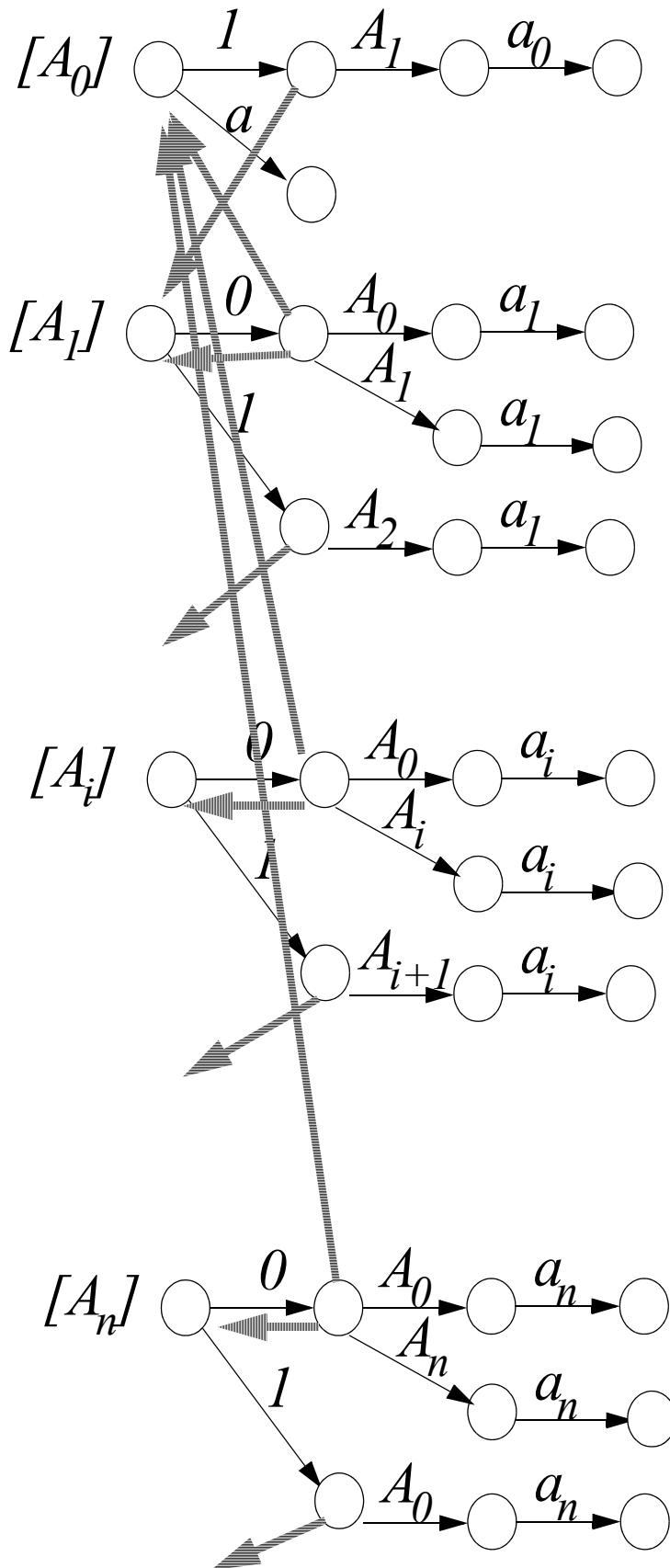
$$\cup \{ \langle A, \alpha, x \rangle X \rightarrow \langle A, \alpha X, x \rangle \mid A \rightarrow \alpha X\beta \in P, \alpha \neq \varepsilon \}$$

$$\cup \{ \langle A, \alpha, x \rangle X \rightarrow \langle B, X, y \rangle \mid A \rightarrow \alpha B\beta \in P, \alpha \neq \varepsilon,$$

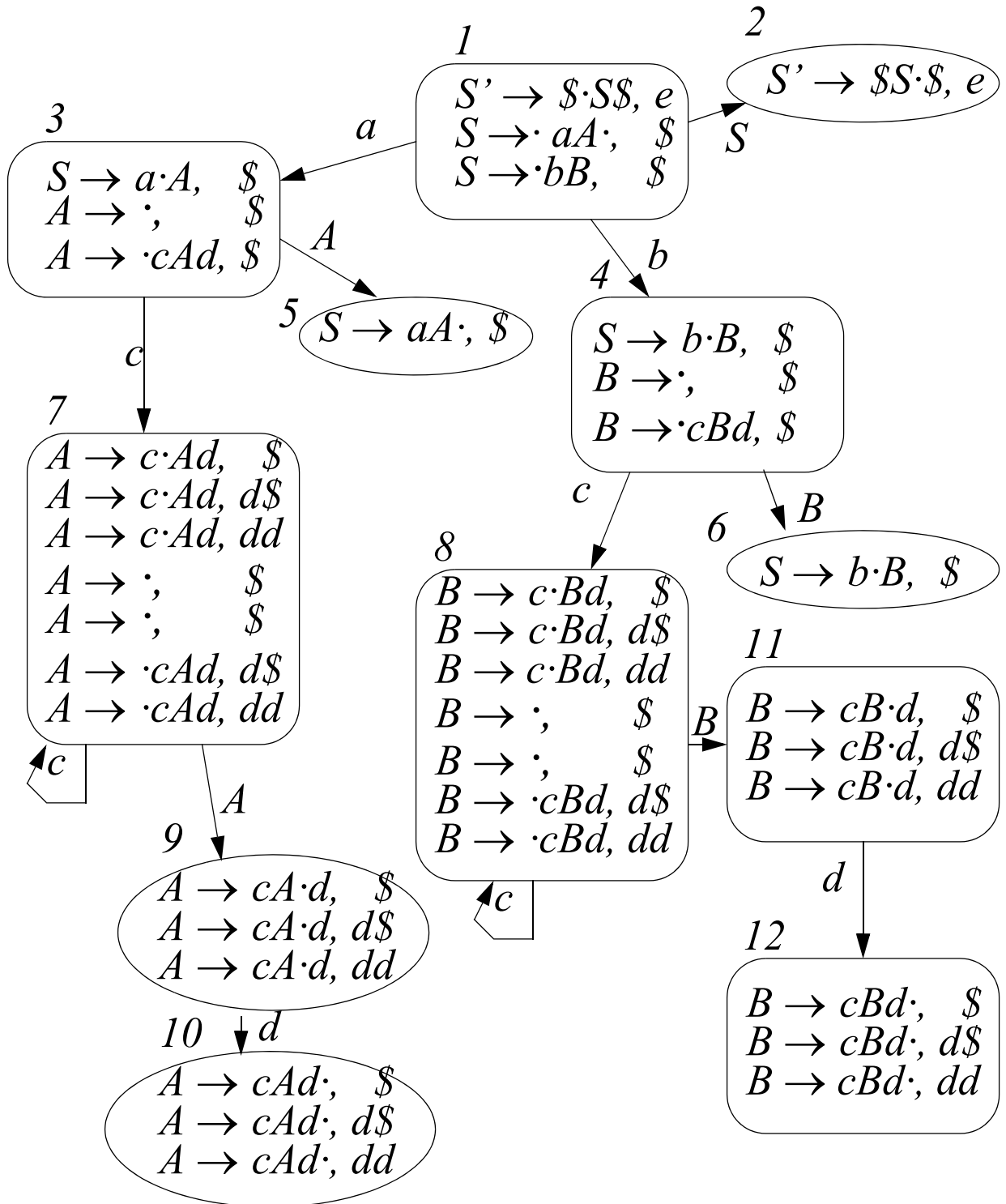
$$B \rightarrow X\gamma \in P, y \in \text{First}_k(\beta x) \}$$

$$|K| = |Q| - |N|$$

$$\begin{aligned}
R' = & \{ \langle S', \varepsilon \rangle S \rightarrow \langle S', S \rangle \} \\
& \cup \{ \langle S', \varepsilon \rangle \rightarrow \langle B, X \rangle \mid S \rightarrow B\beta \in P, B \rightarrow X\gamma \in P \} \\
& \cup \{ \langle A, \alpha \rangle a \rightarrow \langle A, \alpha a \rangle \mid A \rightarrow \alpha a\beta \in P, \alpha \neq \varepsilon \} \\
& \cup \{ \langle A, \alpha \rangle B \rightarrow \langle A, \alpha B \rangle, \langle A, \alpha \rangle X \rightarrow \langle B, X \rangle \\
& \quad \mid A \rightarrow \alpha B\beta \in P, \alpha \neq \varepsilon, B \rightarrow X\gamma \in P \}
\end{aligned}$$



$G_{ab\epsilon}: S \rightarrow aA|bB, A \rightarrow \epsilon|cAd, B \rightarrow \epsilon|cBd$



the partition of the LR(0)-equivalent classes into LR(k) equivalent classes are:

$$[\]_0 = [\]_k$$

$$[S]_0 = [S]_k$$

$$[a]_0 = [a]_k$$

$$[a]_0 = [a]_k$$

$$[aA]_0 = [aA]_k$$

$$[ac^+]_0 = [ac]_k \sqcup \dots \sqcup [ac^k]_k \sqcup [ac^k c^+]_k$$

$$[ac^+ A]_0 = [acA]_k \sqcup \dots \sqcup [ac^k A]_k \sqcup [ac^k c^+ A]_k$$

$$[ac^+ Ad]_0 = [acAd]_k \sqcup \dots \sqcup [ac^k Ad]_k \sqcup$$

$$[ac^k c^+ Ad]_k$$

$$[b]_0 = [b]_k$$

$$[bB]_0 = [bB]_k$$

$$[bc^+]_0 = [bc]_k \sqcup \dots \sqcup [bc^k]_k \sqcup [bdc^k c^+]_k$$

$$[bc^+ B]_0 = [bc]_k \sqcup \dots \sqcup [bc^k]_k \sqcup [bdc^k c^+]_k$$

$$[bc^+ Bd]_0 = [bcd]_k \cup \dots \cup [bc^k d]_k \cup [bdc^k c^+ d]_k$$

$$6k + 12 \quad \text{linear in } k$$

Theorem 6.47

Let q be the state of LALR(k) machine for G , and q is accessible upon reading string δ .

Then δ is a viable prefix of G , and

$$\exists \gamma \in V^* . \exists . \forall I \in q, I \in \langle \gamma \rangle_k \text{ where } \gamma \rho_0 \delta.$$

Conversely,

If $I \in \langle \gamma \rangle_k$

Then $\exists q$ s.t. $I \in q$ and

q is accessible upon reading any viable prefix δ

$$. \exists . \delta \rho_0 \gamma.$$

$$\begin{aligned} \text{LR}(k) \text{ states: } \langle [\gamma]_k \rangle_k &= \langle \gamma \rangle_k \leftrightarrow [\gamma]_k \\ &= \{ [A \rightarrow \alpha . \beta, x] \in \langle \delta \rangle_k \mid \delta \in [\gamma]_k \} \end{aligned}$$

$$\begin{aligned} \text{LR}(0) \text{ states: } \langle [\gamma]_0 \rangle_0 &= \langle \gamma \rangle_k \leftrightarrow [\gamma]_0 \\ &= \{ [A \rightarrow \alpha . \beta] \in \langle \delta \rangle_0 \mid \delta \in [\gamma]_0 \} \end{aligned}$$

$$\begin{aligned} \text{LALR}(k) \text{ states: } \langle [\gamma]_0 \rangle_k &= \langle \gamma \rangle_{k, 0} \leftrightarrow [\gamma]_0 \\ &= \{ [A \rightarrow \alpha . \beta, x] \in \langle \delta \rangle_k \mid \delta \in [\gamma]_0 \} \end{aligned}$$

Since $[\gamma]_k \subseteq [\gamma]_0$

$$\langle \gamma \rangle_k \subseteq \langle \gamma \rangle_{k, 0}$$

$$\text{core}(\langle \gamma \rangle_k) = \text{core}(\langle \gamma \rangle_{k, 0}) = \text{core}(\langle \gamma \rangle_0)$$

Let $G = (N, \Sigma, P, S)$. The **LALR(k) parser** for G is a pushdown transducer $M = ([G]_0, \Sigma, \Gamma, P, \tau, [\varepsilon]_0, \{[\varepsilon]_0[S]_0\}, \$, \mid)$ where

$$\Gamma = \{[\delta]_0[\delta X_1]_0 \dots [\delta X_1 \dots X_n]_0 \mid y \rightarrow [\delta]_0[\delta A]_0 \mid y$$

$$\mid [A \rightarrow X_1 \dots X_n; y] \in \langle [\delta X_1 \dots X_n]_0 \rangle_k\}$$

(ra)

$$\cup \{[\delta]_0 \mid ay \rightarrow [\delta]_0[\delta a]_0 \mid y$$

$$\mid a \in \Sigma, [A \rightarrow \alpha \cdot a \beta, z] \in \langle [\gamma]_0 \rangle_k$$

$$y \in \text{First}_{\max\{k-1, 0\}}(\beta z)\}$$

(sa)

Theorem 6.48 The size of LALR(k) parser for G is $O(2^{|G|} + k \log |\Sigma| + \log |G|)$.

Correctness of LALR(k) parser as a right parser.

L6.49: *right parser \Rightarrow LALR(k) parser*

LR(0) parser(L6.29)

L6.50: *LALR(k) parser \Rightarrow right parser*

LR(k) parser(L6.32)

Theorem 6.51 *For the LALR(k) parser M for G ,*

(1) M is a right parser for G

(2) $\forall w \in L(G)$, M produces all right parses of w in G

(3) $\text{TIME}_G(w) = \text{TIME}_M(w) + |w|$.

making LALR(k) parser

from LR(k) parser \Rightarrow uniting all states with the same set of item cores

from LR(0) parser \Rightarrow add suitable k -length lookahead strings to 0-items

*LALR(k) lookahead set is sufficient and **minimal***

Theorem 6.52

Let $[A \rightarrow \alpha \cdot \beta, z] \in q$. Then $\exists x, y$ and $X_1 \dots X_m \cdot \exists$.

$[\epsilon] | xy \Rightarrow^ [\epsilon] [X_1] \dots [X_1 \dots X_m] | y \$$ in M ,*

where the set of cores in $\langle X_1 \dots X_m \rangle_0$ is same as in q and $k:y \$ = \text{First}_k(\beta z)$.

In CLR(k) parser

every item $[A \rightarrow \alpha \cdot \beta, z]$ in any state q

can be "used" in the parsing of

all terminal strings of the form xy ,

where $k:y \in \text{First}_k(\beta z)$ and $q = \langle \$ \gamma \rangle_0$.

sentence LALR(k) is same as LR(k)

*no sentence additional **reduce** actions
in LALR(k)*

Immediate Error Detection Property in LR(k)

reduce stack for error recovery in LALR(k)

$G = (N, \Sigma, P, S)$ is **LALR(k)** if

its LALR(k) parser is deterministic and

$S \Rightarrow^+ S$ is impossible in G .

A language over alphabet Σ is LALR(k) if

it is generated by an LALR(k) grammar.

Theorem 6.53 (Characterization of LALR(k) Grammars) Let G' be an augmented grammar.

The LALR(k) parser of G is deterministic iff
in the LALR(k) machine of G'

no state contains a pair of items
exhibiting a reduce-reduce or
a shift-reduce conflict.

Theorem 6.54 The class of LALR(0) grammars coincides with the class of LR(0) grammars. For $k \geq 1$ the class of LALR(k) grammars is properly contained in the class of LR(k) grammars.

Proof

(i) $LALR(k) \subseteq LR(k)$:

uniting of states in CLR(k) machine
can only increase # of reduce-reduce conflicts.

(ii) $LALR(k) \neq LR(k)$

counter example:

$$S \rightarrow aAa|bAb|aBb|bBa,$$

$$A \rightarrow c,$$

$$B \rightarrow c.$$

This grammar is LR(1)

but not LALR(k) for any k .

Generalize the LALR concepts:

LA(k)LR(l) machine

→ unite q_1 and q_2

whenever the truncating of the k -length
lookahead strings to length $l \leq k$,
yields the same set of l -items.

- unite q_1, q_2 if $\text{Trunc}_l(q_1) = \text{Trunc}_l(q_2)$,

$$\text{Trunc}_l(q) = \{[A \rightarrow \alpha \cdot \beta, l:y] \mid [A \rightarrow \alpha \cdot \beta, y] \in q\}$$

Fact 6.55

The LA(k)LR(k) machine is same to LR(k) machine.

The LA(k)LR(0) machine

is same to LALR(k) machine.

Theorem 6.56 Let q be a state in LA(k)LR(l) machine
accessible upon reading string δ .

Then δ is a viable prefix of G , and

$$\exists \gamma \in V^* . \exists . \forall I \in q, I \in \langle \gamma \rangle_k \text{ where } \delta \rho_l \gamma.$$

Conversely,

If $I \in \langle \gamma \rangle_k$

then $\exists q$ s.t. $I \in q$ and

q is accessible upon reading any viable prefix δ

$$. \exists . \gamma \rho_l \delta.$$

States in LA(k)LR(l) machine

$$\langle [\gamma]_l \rangle_k \leftrightarrow [\gamma]_l$$

6.6. SLR(k) Parsing

SLR(k) stands for Simple LR(k).

adding k-lookaheads in a crude, simple way.

seldom minimal lookaheads.

SLR(k) parser for G is the pushdown transducer $M = ([G]_0, \Sigma, \Gamma, P, \tau, [\epsilon]_0, \{[\epsilon]_0[S]_0\}, \$, |)$ where

$$\Gamma = \{[\delta]_0[\delta X_1]_0 \dots [\delta X_1 \dots X_n]_0 \mid y \rightarrow [\delta]_0[\delta A]_0 \mid y \mid [A \rightarrow X_1 \dots X_n \cdot] \in [\delta X_1 \dots X_n]_0 \text{ and } y \in \text{Follow}_k(A)\} \quad (ra)$$

$$\cup \{[\delta]_0 \mid ay \rightarrow [\delta]_0[\delta a]_0 \mid y$$

$$\mid a \in \Sigma, [A \rightarrow \alpha \cdot a \beta] \in [\delta]_0,$$

$$y \in \text{First}_{\max\{k-1, 0\}}(\beta \text{Follow}_k(A))\} \quad (sa)$$

Theorem 6.57

The SLR(k) parser M for G is a right parser for G.

Moreover, $\forall w \in L(G)$,

M produces all right parses of w in G, and

$$\text{TIME}_G(w) = \text{TIME}_M(w) + |w|.$$

Theorem 6.58 (Characterization of SLR(k) Grammars) *The SLR(k) parser of G is deterministic iff for all state q in SLR(k) machine,*

(1) *Whenever $[A_1 \rightarrow \omega_1 \cdot], [A_2 \rightarrow \omega_2 \cdot] \in q$, then*

$$\text{Follow}_k(A_1) \cap \text{Follow}_k(A_2) = \emptyset.$$

(2) *Whenever $[A \rightarrow \alpha \cdot a \beta], [B \rightarrow \omega \cdot] \in q$, then*

$$\text{First}_k(a\beta\text{Follow}_k(A)) \cap \text{Follow}_k(B) = \emptyset.$$

Theorem 6.59

*The class of SLR(0) grammars
coincides with the class of LR(0) grammars.*

For $k \geq 1$,

*the class of SLR(k) grammars is properly
contained in the class of LALR(k) grammars.*

$$\begin{aligned} \text{(eg)} \quad S &\rightarrow Ac|bA|bc, \\ A &\rightarrow \varepsilon \end{aligned}$$

Time to test SLR(k) property for G : polynomial to $|G|$

Transformation of G into $T_k(G)$,

which is $SLR(k)$ if and only if G is $LR(k)$.

Idea : replace A by $([\gamma]_k A) \dots$

$$G = (N, \Sigma, S, P)$$

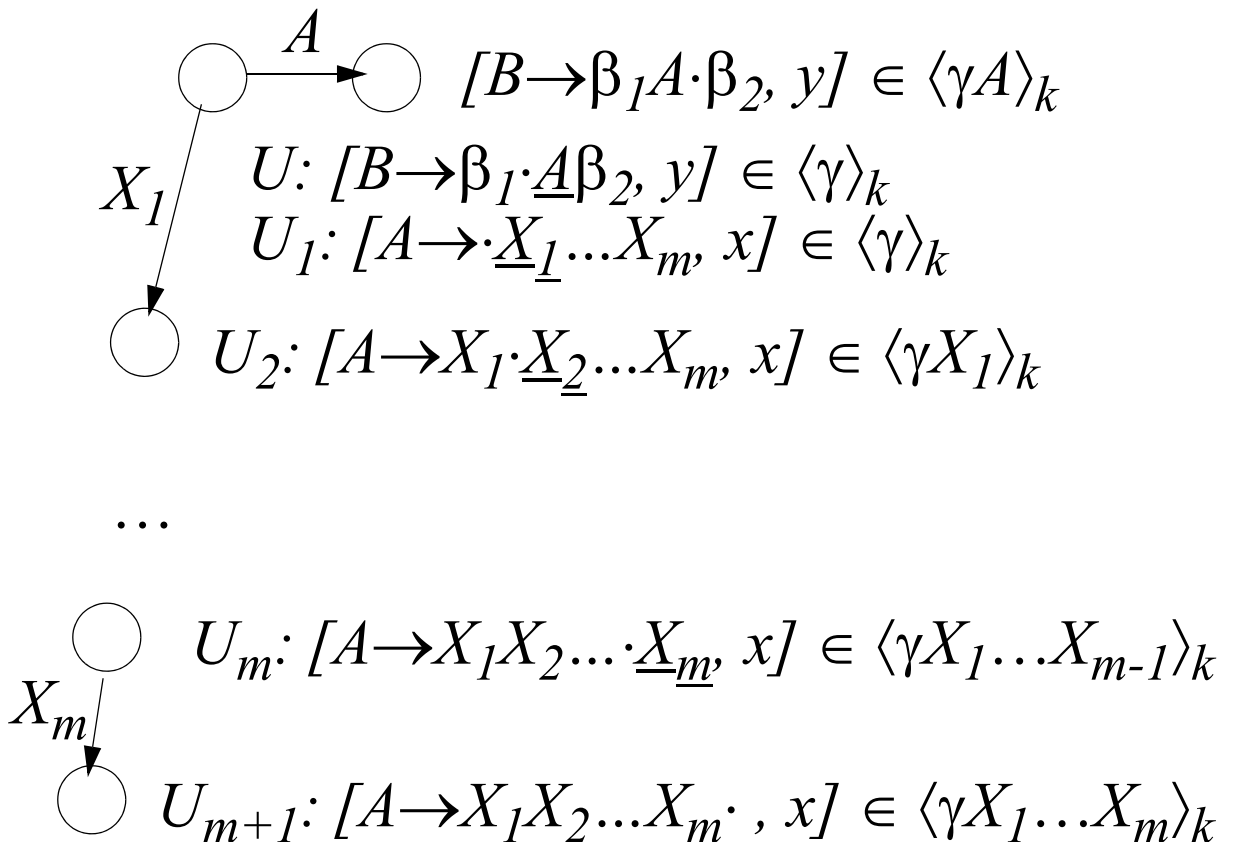
$T_k(G) = ([G]_k \times N, \Sigma, ([\varepsilon]_k S), \hat{P})$, where

$$\hat{P} = \{([\gamma]_k A) \rightarrow U_1 \dots U_m$$

$$| [B \rightarrow \beta_1 \cdot A \beta_2, y]_k \in \langle \gamma \rangle_k \quad A \rightarrow X_1 \dots X_m \in P,$$

$$1 \leq \forall i \leq m \quad U_i = ([\gamma X_1 \dots X_{i-1}]_k X_i) \text{ if } X_i \in N,$$

$$= X_i \quad \text{if } X_i \in \Sigma \cup \{\$ \}.$$



$([\gamma]_k A)$ is a **useful** nonterminal in $T_k(G)$, iff
 $[\gamma]_k \in [G]_k$, $A \in N$, and $[B \rightarrow \alpha.A\beta] \in [\gamma]_k$

$$S \Rightarrow_{rm}^* \gamma Ay$$

$$([\varepsilon]_k S) \Rightarrow_{rm}^* \Phi([\gamma]_k A)y$$

$$\text{Follow}_k([\gamma]_k A) = \{k:y \mid S \Rightarrow_{rm}^* \gamma Ay\}$$

$T_k(G)$ **right-to-right covers** G

\equiv right parses in $T_k(G)$ are mapped into
 right parses in G by a homomorphism h .

Furthermore,

$T_k(G)$ is **structurally equivalent** to G .

\equiv parse trees in $T_k(G)$ and G have **same structure**
 parse trees are same except for
 the labeling of the nonterminal nodes.
 parse trees are **isomorphic**

Cover relations between Grammars

Let $x, y \in \{\text{"left"}, \text{"right"}\}$. Then

an **x -to- y cover** of G is a pair (\hat{G}, h) where

$$\hat{G} = (\hat{N}, \Sigma, \hat{P}, \hat{S}) \text{ and } h: \hat{P}^* \rightarrow P^* .\exists.$$

i) $\forall w \in L(\hat{G})$ and x -parses $\hat{\pi}$ of w in \hat{G} ,

$h(\hat{\pi})$ is a y -parse of w in G .

ii) $\forall w \in L(G)$ and y -parses π of w in G ,

$\exists \hat{\pi} \in \hat{P}^*$, $\hat{\pi}$ is a x -parse of w in \hat{G} and $h(\hat{\pi}) = \pi$.

h maps x -parses of \hat{G} into y -parses of G

If $\exists h$, (\hat{G}, h) is x -to- y covers of G ,

\hat{G} x -to- y covers G with respect to h

If \hat{G} x -to- y covers G with respect to h ,

(\hat{G}, h) is x -to- y covers of G ,

Fact 6.60 If \hat{G} x -to- y covers G , then $L(\hat{G}) = L(G)$.

Fact 6.61 If (M, τ) is an x parser of \hat{G} and if \hat{G} x -to- y covers G w.r.t. h , then $(M, \tau \circ h)$ is a **y parser** of G .

$(T_k(G), h_k)$ right-to-right covers G , if

$h_k(U \rightarrow U_1 \dots U_m) = A \rightarrow X_1 \dots X_m$, where

$$U = ([\gamma], A), U_i = ([\gamma X_1 \dots X_{i-1}], X_i) \text{ if } X_i \in N, \\ = X_i \text{ if } X_i \in \Sigma.$$

(M, τ) is a x -parser of G , if $\tau(\theta) = \pi_x$.

(\hat{G}, h) is a x -to- y cover of G , if $h(\hat{\pi}_x) = \pi_y$.

$(M, \tau \circ h)$ is a x -parser of G .

G

\hat{G}

(\hat{M}, τ)

$(M, \tau \circ h)$

Consider a function $M: \Sigma^* \rightarrow \{\Gamma^*\}$

$$M(w) = \theta.$$

M is deterministic if $M: \Sigma^* \rightarrow \Gamma^*$.

Consider a function $G_x: \Sigma^* \rightarrow \{P^*\}$

$$G_x(w) = \pi.$$

G is unambiguous, if $G_x: \Sigma^* \rightarrow P^*$.

$M \circ \tau$ is a x -parser of G , if $M \circ \tau: \Sigma^* \rightarrow \{P^*\}$

$$M \circ \tau = G_x.$$

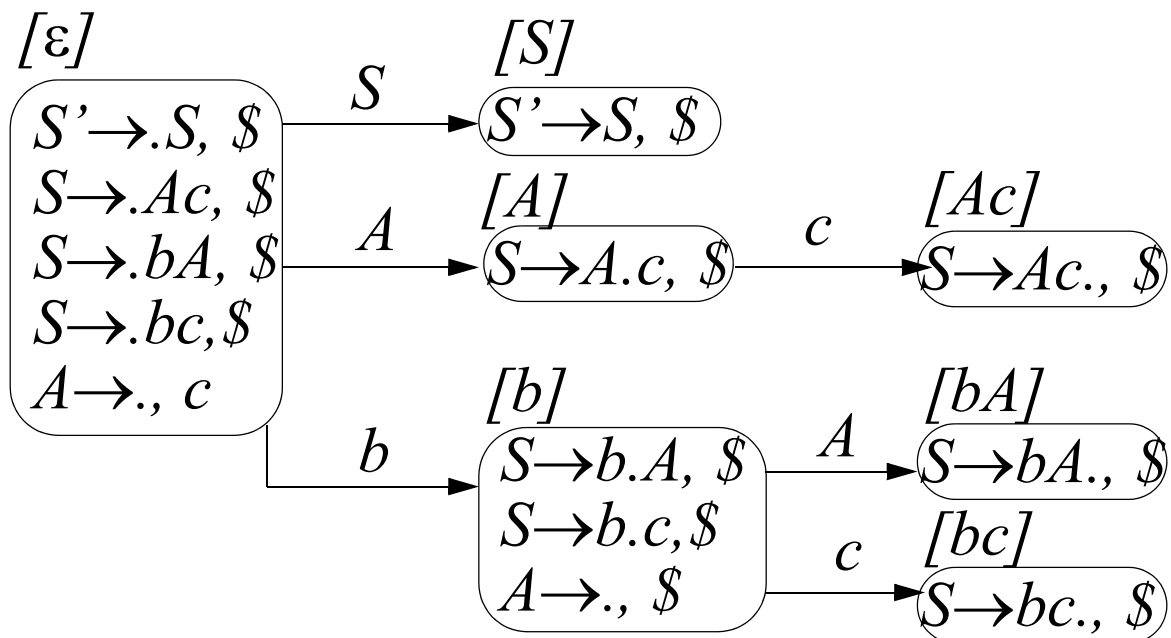
If G is ambiguous,

Example

$$G: \quad S \rightarrow Ac \mid bA \mid bc$$

$$A \rightarrow \varepsilon$$

$$\text{Follow}(A) = \{\$, c\}$$

$$G \text{ is not SLR}(1) \text{ in state } [b]$$


$$T_k(G): \quad ([\varepsilon], S) \rightarrow ([\varepsilon], A) c \mid b ([b], A) \mid b c$$

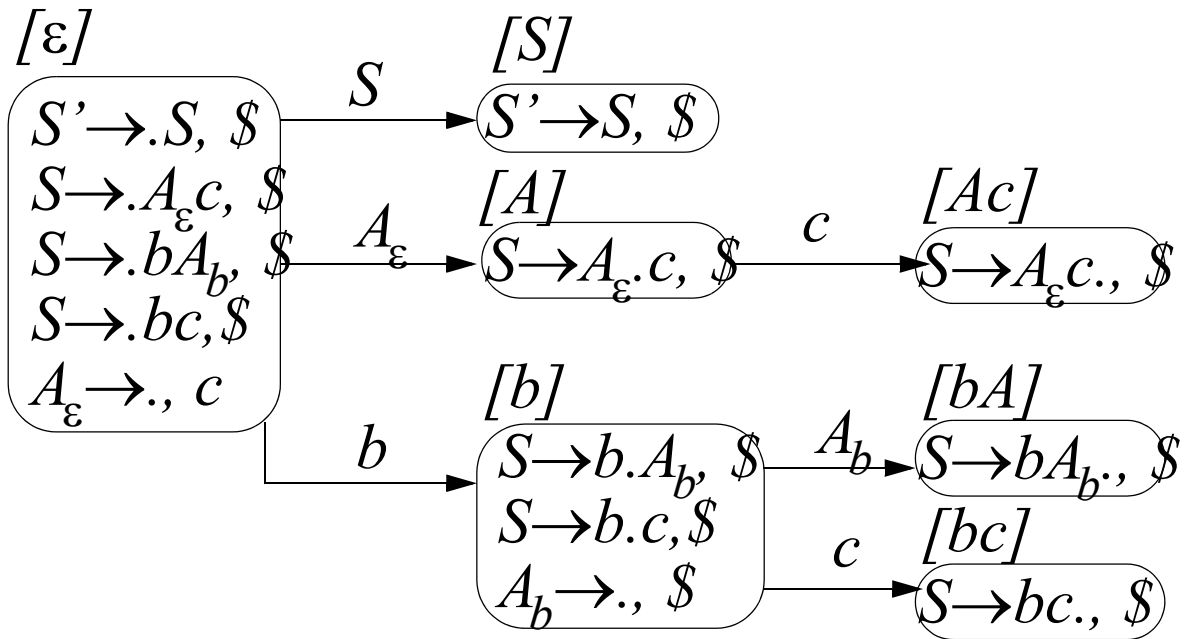
$$([\varepsilon], A) \rightarrow \varepsilon$$

$$([b], A) \rightarrow \varepsilon$$

$$S_\varepsilon \rightarrow A_\varepsilon c \mid b A_b \mid b c$$

$$A_\varepsilon \rightarrow \varepsilon$$

$$A_b \rightarrow \varepsilon$$



$$\text{Follow}(A_\epsilon) = \{c\}$$

$$\text{Follow}(A_b) = \{\$, \}$$

$T_k(G)$ is SLR(1)

$T_k(G)$ right-to-right covers G w.r.t. h_k (T6.64)

$\forall w \in L(\hat{G})$ where $\hat{\pi}$ is a right parse of w in $T_k(G)$,

$h_k(\hat{\pi})$ is a right parse of w in G . (L6.62)

$\forall w \in L(G)$, where π is a right parse of w in G

$\exists \hat{\pi} \text{ s.t. } h_k(\hat{\pi}) = \pi$, $\hat{\pi}$ is a right parse of w in

$T_k(G)$.

(L6.63)

Lemma 6.62 If $[\gamma A] \in [G]_k$ $([\gamma], A) \Rightarrow^{\hat{\pi}} \Phi$ in $T_k(G)$.

Then let $\Phi = U_1 \dots U_m y$,

$\gamma X_1 \dots X_m \in [G]_k$ and

$A \Rightarrow^{hk(\hat{\pi})} X_1 \dots X_m y$ in G , where

$$\begin{aligned} U_i &= ([\gamma X_1 \dots X_{i-1}], X_i) \text{ if } X_i \in N, \\ &= X_i \text{ if } X_i \in \Sigma \cup \{\$\}. \end{aligned}$$

Proof by induction on $|\hat{\pi}|$.

IB: $\hat{\pi} = \varepsilon$

IH: $\hat{\pi} = \hat{\pi}_1 \hat{r}$, $\hat{r} = W \rightarrow W_1 \dots W_p \in \hat{P}$.

$([\gamma], A) \Rightarrow^{\hat{\pi}_1} U_1 \dots U_n W y$

$$\Rightarrow^{\hat{r}} U_1 \dots U_n W_1 \dots W_p y = \Phi.$$

By definition of $T_k(G)$,

$W = ([\delta], B)$, $\delta = \gamma h_k(U_1 \dots U_n)$, and

$$\begin{aligned} W_i &= ([\delta Z_1 \dots Z_{i-1}], Z_i), \text{ if } Z_i \in N, \\ &= Z_i, \text{ if } Z_i \in \Sigma \cup \{\$\}. \end{aligned}$$

By IH, $\exists \gamma X_1 \dots X_n B \in [G]_k$ and,

$A \Rightarrow^{hk(\hat{\pi}_1)} X_1 \dots X_n B y$ in G ,

$[\gamma X_1 \dots X_n] = [\delta]$, and by right invariance,

$$[\delta Z_1 \dots Z_{i-1}] = [\gamma X_1 \dots X_n Z_1 \dots Z_{i-1}].$$

If choose $m = n+p$, $1 \leq i \leq p$,

$U_{n+i} = W_i$, $X_{n+i} = Z_i$, then the lemma is proved.

Lemma 6.63 $\gamma A \in [G]_k$ $A \Rightarrow^\pi X_1 \dots X_m \gamma$ in G , and either $X_1 \dots X_m = \varepsilon$ or $X_m \in N$.

Then $\exists \hat{\pi} \in \hat{P}^*$. \exists . $h_k(\hat{\pi}) = \pi$ and

$$([\gamma], A) \Rightarrow^{\hat{\pi}} U_1 \dots U_m \gamma \text{ in } T_k(G).$$

Proof by induction on $|\pi|$.

Base: $\pi = \varepsilon$. Then $\hat{\pi} = \varepsilon$ and $U_1 = ([\gamma], A)$.

Induction Step: $\pi = \pi_1 r$, $r = B \rightarrow Z_1 \dots Z_p$.

$$\begin{aligned} A &\xRightarrow{rm}^{\pi_1} Y_1 \dots Y_n B y_1 \xRightarrow{rm}^r Y_1 \dots Y_n Z_1 \dots Z_p y_1 \\ &= X_1 \dots X_m \gamma \text{ in } G. (m=n+p) \end{aligned}$$

$\exists \bar{\pi}_1$ of $T_k(G)$. \exists .

$$h_k(\bar{\pi}_1) = \pi_1 \text{ and}$$

$$([\gamma], A) \xRightarrow{rm}^{\bar{\pi}_1} U_1 \dots U_n ([\gamma Y_1 \dots Y_n], B) y_1$$

$$\begin{aligned} \text{where } U_i &= ([\gamma Y_1 \dots Y_{i-1}], Y_i) \text{ if } Y_i \in N, \\ &= Y_i \text{ if } Y_i \in \Sigma. \end{aligned}$$

Then $\exists \bar{r} = ([\gamma Y_1 \dots Y_n], B) \rightarrow U_{n+1} \dots U_{n+p}$ in $T_k(G)$

$$\begin{aligned} \text{where } U_i &= ([\gamma Y_1 \dots Y_n Z_1 \dots Z_{i-1}], Z_i) \text{ if } Z_i \in N, \\ &= Z_i \text{ if } Z_i \in \Sigma. \end{aligned}$$

$\therefore h_k(\bar{\pi}_1 \bar{r}) = h_k(\bar{\pi}_1) h_k(\bar{r}) = \pi_1 r = \pi$, and

$$\begin{aligned} ([\gamma], A) &\xRightarrow{rm}^{\pi_1 r} U_1 \dots U_{n+p} y_1 = U_1 \dots U_m z y_1 \\ &= U_1 \dots U_m \gamma. \end{aligned}$$

Theorem 6.64 $T_k(G)$ right-to-right covers G w.r.t. h_k .

Corollary 6.65 If (M, τ) is a right parser of $T_k(G)$, then $(M, \tau h_k)$ is a right parser of G .

Lemma 6.66 $y \in \text{Follow}_k([\gamma]_k A)$ in $T_k(G)$ iff $S \Rightarrow^* \delta Az$ in G , $[\delta]_k = [\gamma]_k$ $l:z = y$.

In other words,

y is a follower of $([\gamma]_k A)$ in $T_k(G)$

iff y is a follower of A in G

in some context LR(k)-equivalent to γ .

Lemma 6.67

$[U \rightarrow U_m \dots U_i U_{i+1} \dots U_p, y] \in \langle U_1 \dots U_i \rangle_l$

in $T_k(G)$ iff

$[A \rightarrow X_m \dots X_i X_{i+1} \dots X_p, y] \in \langle X_1 \dots X_i \rangle_b$

$U = ([X_1 \dots X_{m-1}], A)$, and $1 \leq j \leq p$,

$U_j = ([X_1 \dots X_{j-1}], X_j)$ if $X_j \in N$,

$= X_j$ if $X_j \in \Sigma \cup \{\$ \}$.

Lemma 6.68 *If G is non-LR(k), then so is $T_k(G)$.*

Proof

(i) $S \xRightarrow{rm}^+ S$, then $([\varepsilon]_k S) \xRightarrow{rm}^+ ([\varepsilon]_k S)$,

$\therefore T_k(G)$ is non-LR(k).

(ii) $\langle X_1 \dots X_i \rangle_k$ contains a conflict. Then

$[A \rightarrow X_m \dots X_i, y] \in \langle X_1 \dots X_i \rangle_k$

$[B \rightarrow X_n \dots X_i X_{i+1} \dots X_p, u] \in \langle X_1 \dots X_i \rangle_k$

$y \in \text{First}_k(X_{i+1} \dots X_p u)$.

Then by lemma 6.67,

$[U \rightarrow U_m \dots U_i, y] \in \langle U_1 \dots U_i \rangle_k$ and

$[W \rightarrow U_n \dots U_i U_{i+1} \dots U_p, u] \in \langle U_1 \dots U_i \rangle_k$

where $U = ([X_1 \dots X_{m-1}]_k A)$,

$W = ([X_1 \dots X_{n-1}]_k B)$,

$U_j = ([X_1 \dots X_{j-1}]_k X_j)$ if $X_j \in N$,

$= X_j$ if $X_j \in \Sigma \cup \{\$\}$.

$U \neq W$ or $m \neq n$ or $i+1 \leq p$,

$y \in \text{First}_k(U_{i+1} \dots U_p u)$.

$\therefore T_k(G)$ is non-LR(k).

Lemma 6.69 *If $T_k(G)$ is non-SLR(k), then G is non-LR(k).*

Proof *If $T_k(G)$ is non-LR(k), we have*

$$[U \rightarrow U_m \dots U_i \cdot] \in \langle U_1 \dots U_i \rangle_0,$$

$$[W \rightarrow U_n \dots U_i \cdot U_{i+1} \dots U_p] \in \langle U_1 \dots U_i \rangle_0,$$

$$y \in \text{Follow}_k(U) \cap \text{First}_k(U_{i+1} \dots U_p \text{Follow}_k(W)),$$

$U_{i+1} \in \Sigma$ whenever $i+1 \leq p$. Then there exist

$$U = ([X_1 \dots X_{m-1}]_k A),$$

$$W = ([X_1 \dots X_{n-1}]_k B), \text{ and}$$

$$U_j = ([X_1 \dots X_{j-1}]_k X_j) \text{ if } X_j \in N,$$

$$= X_j \text{ if } X_j \in \Sigma \cup \{\$, \}, 1 \leq \forall j \leq p.$$

By lemma 6.66,

$$S \Rightarrow^* \gamma Az, [\gamma]_k = [X_1 \dots X_{m-1}]_k \text{ k:z} = y,$$

$$S \Rightarrow^* \delta Bu, [\delta]_k = [X_1 \dots X_{n-1}]_k \text{ k:xu} = y, \text{ and}$$

$$x \in \text{First}_k(U_{i+1} \dots U_p).$$

(i) *If $U_{i+1} \in \Sigma$, $X_{i+1} \in \Sigma$, too. Then*

$$[A \rightarrow X_1 \dots X_i \cdot, \text{ k:z\$}], [B \rightarrow X_1 \dots X_i \cdot X_{i+1} \dots X_p, \text{ k:u\$}]$$

exhibit a conflict.

(ii) *$i \leq p$, by the right invariance, $i = p$, then reduce-reduce conflict.*

Theorem 6.70 *Any grammar G can be transformed into a structurally equivalent grammar which is SLR(k) iff G is LR(k).*

Theorem 6.71 *For any $k \geq 0$,
the families of LR(k) languages,
LALR(k) languages, and
SLR(k) languages
are all equal.*

6.7. Covering LR(k) Grammars by LR(1) Grammars

- *LR(k) language \equiv LR(1) language*
LR(k) grammar \Rightarrow LR(1) grammar
right-to-right cover
- *deterministic language \Rightarrow SLR(1) parsing*

*$T_{k, 1}(G)$ right-to-right covers G , and
 $T_{k, 1}(G)$ is LR(1), iff G is LR(k+1)*

Idea

- *shift the derivation trees in G
 k symbols to the right*
- *reduce actions are postponed until 1 symbol
lookahead is sufficient to resolve uniquely.*

*A is replaced by set of (x, A, y) 's, where
 $y \in \text{Follow}_k(A)$, $x \in \text{First}_k(Ay)$.
 $|y| \leq k$ and $|x| = k$.*

*$L((x, A, y)) =$
 $\{z \mid S \Rightarrow^* uAw \Rightarrow^* uvw, y = k:w, x = k:vw, xz =$
 $vy, |x| = k\}$*

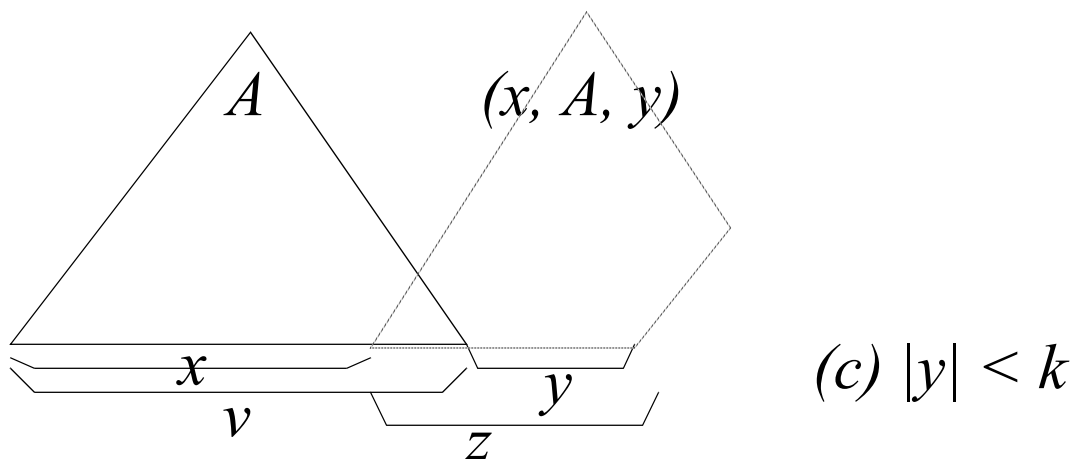
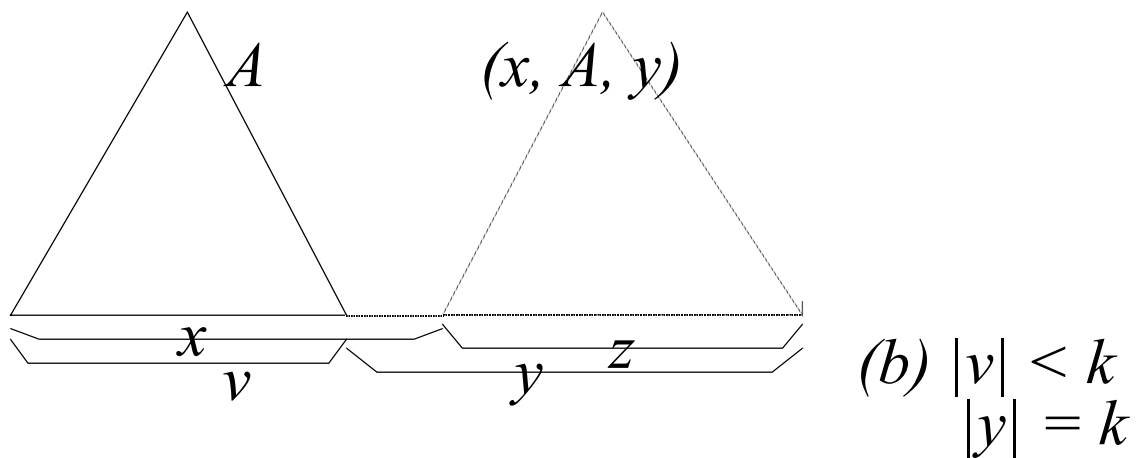
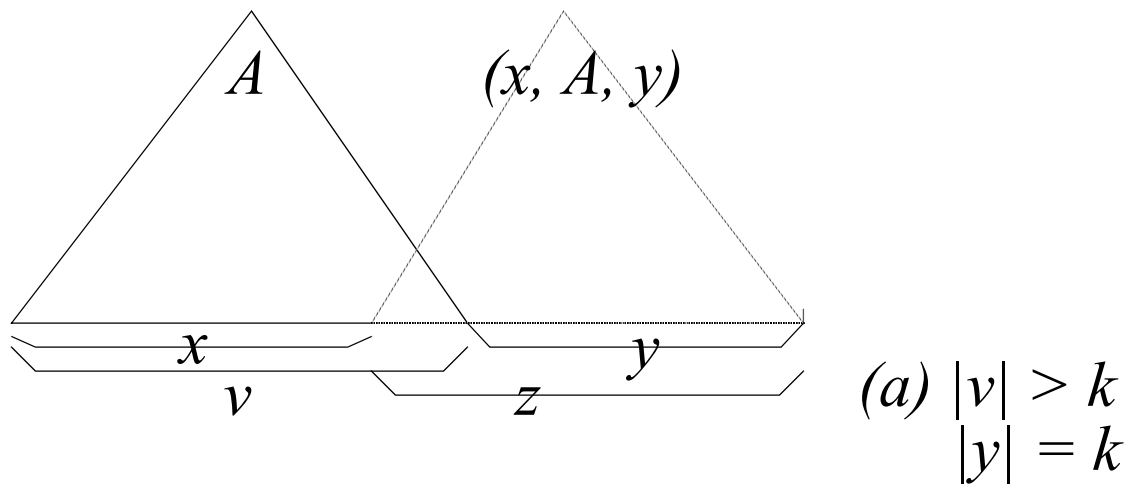


Figure 6.20 (p85)

$vy = xz$, where $A \Rightarrow^* v$, $(x, A, y) \Rightarrow^* z$,
and $|x| = k$.

Let $G = (N, \Sigma, P, S)$ be a grammar.

$T_{k,1}(G) = (N', \Sigma, P', S')$, where

$$N' = \{S'\}$$

$$\cup \{(x, X, y) \mid y \in \text{Follow}_k(X), x \in \text{First}_k(Xy)\}$$

$$P' = \{S' \rightarrow x(x, S, \varepsilon) \mid x \in \text{First}_k(S)\}$$

$$\cup \{(y_0, A, y_m) \rightarrow$$

$$(y_0, X_1, y_1)(y_1, X_2, y_2) \dots (y_{m-1}, X_m, y_m)$$

$$\mid m \geq 0, A \rightarrow X_1 \dots X_m \in P, y_m \in \text{Follow}_k(A),$$

$$0 \leq \forall i < m, y_i \in \text{First}_k(X_{i+1}y_{i+1}),$$

$$\text{Follow}_k \text{ First}_k \text{ in the context of } A \Rightarrow^* y_0 \dots \}$$

$$\cup \{(ax, a, xb) \rightarrow b \mid xb \in \text{Follow}_k(a), |xb| = k\}$$

$$\cup \{(ax, a, x) \rightarrow \varepsilon \mid x \in \text{Follow}_k(a), |x| < k\}.$$

$$y_i \in \text{Follow}_k(X_i)$$

$$h_{k,1}: P' \rightarrow P \cup \{\varepsilon\}$$

$$h_{k,1}(S' \rightarrow x(x, S, \varepsilon)) = \varepsilon,$$

$$h_{k,1}((y_0, A, y_m) \rightarrow (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m))$$

$$= A \rightarrow X_1 \dots X_m,$$

$$h_{k,1}((ax, a, xb) \rightarrow b) = \varepsilon, \text{ and}$$

$$h_{k,1}((ax, a, x) \rightarrow \varepsilon) = \varepsilon.$$

Two problems

i) $(T_{k,1}(G), h_{k,1})$ is a right-to-right cover of G .

T6.78 (\Rightarrow : L6.72-74, \Leftarrow : L6.75-77)

ii) $T_{k,1}(G)$ is LR(1) if and only if G is LR($k+1$).

T6.85 (\Leftarrow : L6.79-81, \Rightarrow : L6.82-84)

Example

	$First_1$	$Follow_1$
$S \rightarrow Abb \mid Bb$	$\{a\}$	$\{\varepsilon\}$
$A \rightarrow aA \mid a$	$\{a\}$	$\{b\}$
$B \rightarrow aB \mid a$	$\{a\}$	$\{b\}$

$S_0 \rightarrow a (a, S, \varepsilon)$		ε
$(a, S, \varepsilon) \rightarrow (a, A, b)(b, b, b)(b, b, \varepsilon)$		$S \rightarrow Abb$
$(a, B, b) (b, b, \varepsilon)$		$S \rightarrow Bb$
$(a, A, b) \rightarrow (a, a, a) (a, A, b)$		$A \rightarrow aA$
(a, a, b)		$A \rightarrow a$
$(aa, B, b) \rightarrow (a, a, a) (a, B, b)$		$B \rightarrow aB$
(a, a, b)		$B \rightarrow a$
$(a, a, a) \rightarrow a$		ε
$(a, a, b) \rightarrow b$		ε
$(b, a, b) \rightarrow b$		ε
$(b, b, \varepsilon) \rightarrow \varepsilon$		ε

Example

	$First_2$	$Follow_2$
$S \rightarrow Abb \mid Bb$	$\{aa, ab\}$	$\{\varepsilon\}$
$A \rightarrow aA \mid a$	$\{aa, a\}$	$\{bb\}$
$B \rightarrow aB \mid a$	$\{aa, a\}$	$\{b\}$

$S_0 \rightarrow aa (aa, S, \varepsilon)$	ε
$ab (ab, S, \varepsilon)$	ε
$(aa, S, \varepsilon) \rightarrow (aa, A, bb)(bb, b, b)(b, b, \varepsilon)$	$S \rightarrow Abb$
$(aa, B, b) (b, b, \varepsilon)$	$S \rightarrow Bb$
$(ab, S, \varepsilon) \rightarrow (ab, A, bb)(bb, b, b)(b, b, \varepsilon)$	$S \rightarrow Abb$
$(ab, B, b) (b, b, \varepsilon)$	$S \rightarrow Bb$
$(aa, A, bb) \rightarrow (aa, a, aa) (aa, A, bb)$	$A \rightarrow aA$
$(aa, a, ab) (ab, A, bb)$	$A \rightarrow aA$
$(ab, A, bb) \rightarrow (ab, a, bb)$	$A \rightarrow a$
$(aa, B, b) \rightarrow (aa, a, aa) (aa, B, b)$	$B \rightarrow aB$
$(aa, a, ab) (ab, B, b)$	$B \rightarrow aB$
$(ab, B, b) \rightarrow (ab, a, b)$	$B \rightarrow a$
$(aa, a, aa) \rightarrow a$	ε
$(aa, a, ab) \rightarrow b$	ε
$(ab, a, bb) \rightarrow b$	ε
$(ab, a, b) \rightarrow \varepsilon$	ε
$(bb, b, b) \rightarrow \varepsilon$	ε
$(b, b, \varepsilon) \rightarrow \varepsilon$	ε

Lemma 6.72 Consider G and $T_{k, 1}(G)$ be grammar.

If $(x, A, y) \Rightarrow^{\pi'} (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m)z$
 $= \Phi$ in $T_{k, 1}(G)$.

Then $y_0 = x$, $y_m z = vy$; and

$A \Rightarrow^{\pi} X_1 X_2 \dots X_m v$ in G .

Moreover, if $|y_m| < k$, then $z = \varepsilon$.

Proof by induction on $|\pi'|$

IB: $m = 1$, $y_0 = x$, $y_m = y$, $z = v = \varepsilon$, and $X_1 = A$.

IH: $\pi' = \pi_1' r'$

$(x, A, y) \Rightarrow^{\pi_1'} (y_0, X_1, y_1) \dots (y_{n-2}, X_{n-1}, y_{n-1})(y_{n-1}, X,$
 $y_n')z_1 \Rightarrow^{r'} (y_0, X_1, y_1) \dots (y_{n-2}, X_{n-1}, y_{n-1})\omega z_1 = \Phi$

Moreover, $\exists v_1$ s.t.

$\psi_0 = x$, $y_n' z_1 = v_1 y$, and

$A \Rightarrow^{\pi} X_1 X_2 \dots X_{n-1} X v_1$ in G .

Case 1:

$r' = (y_{n-1}, X, y_n') \rightarrow (y_{n-1}, X_n, y_n) \dots (y_{m-1}, X_m, y_m)$,
 $m \geq n - 1$.

By definition, $y_m = y_n'$ and $h(r') = X \rightarrow X_n \dots X_m \in P$.

Then we have:

$y_0 = x$, $y_m z_1 = y_n' z_1 = v_1 y$, and

$A \Rightarrow^{\pi_1'} X_1 X_2 \dots X_{n-1} X v_1 \Rightarrow^{r'} X_1 \dots X_m v_1$ in $T_{k, 1}(G)$.

$z = z_1$, $v = v_1$.

Case 2: $r' = (au, a, ub) \rightarrow b. h(r') = \varepsilon.$

If $m = n - 1, z = bz_1,$ and $v = av_1,$ we then have:

$$\begin{aligned} \Phi &= (y_0, X_1, y_1) \dots (y_{n-2}, X_{n-1}, y_{n-1})bz_1 \\ &= (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m)z, \quad (y_n' = ub) \end{aligned}$$

$$y_0 = x, y_m z = y_{n-1}bz_1 = ay_n'z_1 = av_1y = vy, \text{ and}$$

$$A \Rightarrow^{h(\pi'r')} X_1 \dots X_{n-1}Xv = X_1 \dots X_mav_1 = X_1 \dots X_mv$$

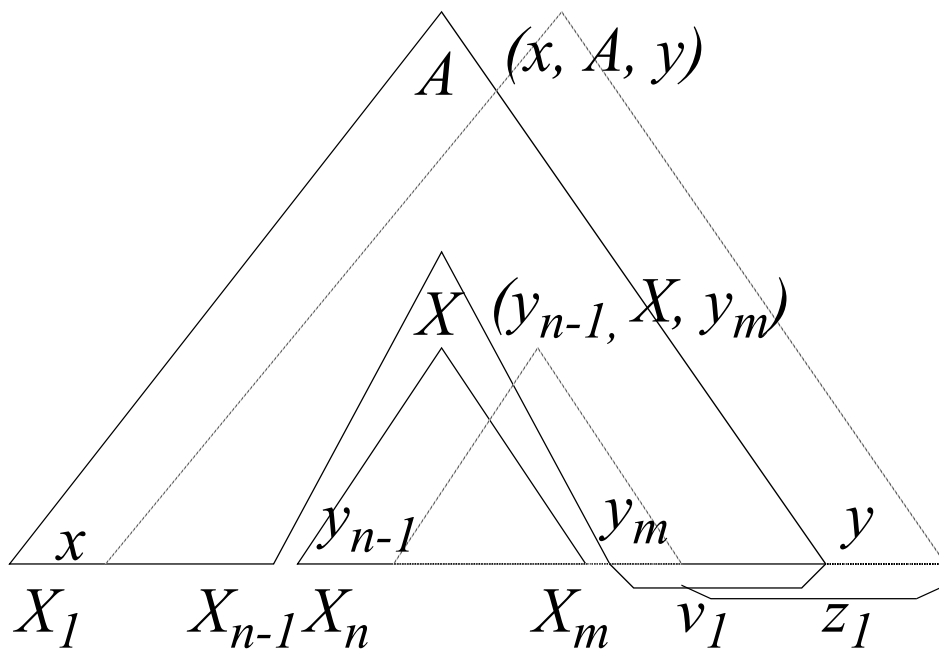
Case 3: $r' = (ay_n', a, y_n') \rightarrow \varepsilon. |y_n'| < k, h(r') = \varepsilon.$

If $m = n - 1, z = z_1,$ and $v = av_1,$ we then have:

$$\begin{aligned} \Phi &= (y_0, X_1, y_1) \dots (y_{n-2}, X_{n-1}, y_{n-1})z_1 \\ &= (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m)z, \end{aligned}$$

$$y_0 = x, y_m z = y_{n-1}z_1 = ay_n'z_1 = av_1y = vy, \text{ and}$$

$$A \Rightarrow^{h(\pi'r')} X_1 \dots X_{n-1}Xv = X_1 \dots X_mv_1 = X_1 \dots X_mv$$



Lemma 6.73 *If*

$(x, A, y) \Rightarrow^{\pi'} z$ in $T_{k, 1}(G)$, then

$A \Rightarrow^{h(\pi')} v$ in G , where $vy = xz$.

Proof

$\Phi = z$ in **L6.72**.

Lemma 6.74 *If π' is a right parse of w in $T_{k, 1}(G)$, then $h(\pi')$ is a right parse of w in G .*

Lemma 6.75 Let $G = (N, \Sigma, P, S)$

$$A \Rightarrow^\pi X_1 X_2 \dots X_m v \text{ in } G,$$

$m = 0$ or X_m is a nonterminal,

$$y \in \text{Follow}_k(A), y_m = k:vy, y_m z = vy, \text{ and}$$

$$y_i \in \text{First}_k(X_{i+1}y_{i+1}) \quad 0 \leq i < m.$$

Then there is a rule string π' of $T_{k,1}(G)$. \exists .

$$h(\pi') = \pi, \text{ and}$$

$$(y_0, A, y) \Rightarrow^{\pi'} (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m) z$$

in $T_{k,1}(G)$.

Proof by induction on $|\pi|$

IB: $m = 1, y_0 = x, y_m = y, z = v = \varepsilon, \text{ and } X_1 = A.$

IH: $\pi = \pi_1 r.$ Then

$$A \Rightarrow^{\pi_1} X_1 X_2 \dots X_n B v_1$$

$$\Rightarrow^r X_1 X_2 \dots X_n X_{n+1} \dots X_p v_1$$

$$= X_1 \dots X_m v \text{ in } G$$

Here, $v = X_{m+1} \dots X_p v_1$, because $r = B \rightarrow X_{n+1} \dots X_p$.

If $p > m$, let $y_p = k:v_1 y$, $\exists z_1$. \exists . $y_p z_1 = v_1 y$.

And let $y_i = k:X_{i+1}y_{i+1}$, $m < i < p$.

Then $y_m = k:vy = k:X_{m+1} \dots X_p v_1 y = k:X_{m+1} y_{m+1}$,

and $y_m \in \text{First}_k(B y_p)$.

By *IH:*

$$h(\pi_1') = \pi_1, \text{ and}$$

$$(y_0, A, y) \Rightarrow^{\pi_1'} (y_0, A, y) \\ \Rightarrow^{\pi_1'} (y_0, X_1, y_1) \dots (y_{n-1}, X, y'_n) z_1 \text{ in } T_{k, 1}(G).$$

And $\exists r' . \exists$.

$$r' = (y_n, X_{n+1}, y_{n+1}) \dots (y_{p-1}, X_p, y_p) \\ h(r') = r.$$

Then

$$(y_0, A, y) \Rightarrow^{\pi_1' r'} (y_0, X_1, y_1) \dots (y_{p-1}, X_p, y_p) z_1 \\ (y_m, X_{m+1}, y_{m+1}) \dots (y_{p-1}, X_p, y_p) \Rightarrow^{\pi_2'} u \in \Sigma^* \\ \text{in } T_{k, 1}(G), \text{ where } \pi_2' = p - m \text{ rules of the form} \\ (ax, a, xb) \rightarrow b \text{ or } (ax, a, x) \rightarrow \varepsilon.$$

Then

$$(y_0, A, y) \Rightarrow^{\pi_1' r' \pi_2'} (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m) u z_1 \\ \text{in } T_{k, 1}(G).$$

Let $\pi' = \pi_1' r' \pi_2'$, then $h(\pi') = \pi$.

And $y_m u z_1 = v y = y_m z$ implying $u z_1 = z$, as claimed.

Lemma 6.76 *If*

$A \Rightarrow^\pi v$ in G , and

$y \in \text{Follow}_k(A)$, $x = k:vy$, and $xz = vy$,

then for some π' of $T_{k, 1}(G)$,

$h(\pi') = \pi$ and

$(x, A, y) \Rightarrow^{\pi'} z$ in $T_{k, 1}(G)$.

Proof

$m=0$ and $y_0 = x$ in **L6.75**.

Lemma 6.77 *If π is a right parse of w in G , then w has in $T_{k, 1}(G)$ a right parse π' . \exists . $h(\pi') = \pi$.*

Theorem 6.78 *For all grammars G and $k > 0$, $T_{k, 1}(G)$ right-to-right covers G w.r.t. the homomorphism h .*

Lemma 6.79

$(y_0, X_1, y_1) \dots (y_{n-1}, X_n, y_n) \xRightarrow{rm}^* z$ in $T_{k, 1}(G)$.

Then $\exists v . \exists . X_1 \dots X_n \xRightarrow{rm}^* v$ in G and

$$vy_n = y_0z.$$

Lemma 6.80

$[U \rightarrow \phi \cdot \psi, d] \in \langle \Phi \rangle_1$.

Then the form of Φ and $[U \rightarrow \phi \cdot \psi, d]$ are

(i) $\Phi = x, [S_0 \rightarrow x \cdot y(xy, S, \varepsilon), \$]$.

(ii) $\Phi = x(x, S, \varepsilon), [S_0 \rightarrow x(x, S, \varepsilon); \$]$.

(iii) $\Phi = y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, y_r),$

$$[(y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, y_r) \cdot \\ (y_r, X_{r+1}, y_{r+1}) \dots (y_{n-1}, X_n, y_n), d],$$

$$\text{where } [A \rightarrow X_{m+1} \dots X_r \cdot X_{r+1} \dots X_n, y_n d] \\ \in \langle X_1 \dots X_r \rangle_{k+1}.$$

(iv) $\Phi = y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, ax)$ and

$$[(ax, a, xb) \rightarrow \cdot b, d]$$

$$\text{where } [A \rightarrow \alpha \cdot a \beta, y'] \in \langle X_1 \dots X_r \rangle_{k+1} \\ \text{and } xbd \in \text{First}_{k+1}(\beta y').$$

(v) $\Phi = y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, ax)b$ and

$$[(ax, a, xb) \rightarrow \cdot b, d] \text{ where... same to (iv).}$$

(vi) $\Phi = y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, ax)b$ and

$$[(ax, a, xb) \rightarrow \cdot, d]$$

$$\text{where } [A \rightarrow \alpha \cdot a \beta, y'] \in \langle X_1 \dots X_r \rangle_{k+1} \\ \text{and } x\$ \in \text{First}_{k+1}(\beta y').$$

Proof cases on the form of U .

Case1: $U = S_0$. Then (i) or (ii) is true.

Case2: $U = (x, A, y)$. Then

$$\begin{aligned} S_0' &\xRightarrow{rm} S_0\$ \xRightarrow{rm} y_0(y_0, S, \varepsilon)\$ \xRightarrow{rm}^* y_0\gamma(x, A, y)z\$ \\ &\xRightarrow{rm} y_0\gamma\phi\psi z\$ = \Phi\psi z\$ \text{ in } T_{k, 1}(G)' \end{aligned}$$

and $1:z\$ = d$.

$$\therefore (y_0, S, \varepsilon) \xRightarrow{rm}^* \gamma(x, A, y)z \text{ in } T_{k, 1}(G).$$

$$\gamma(x, A, y)z = (y_0, X_1, Y_1) \dots (y_{m-1}, X_m, x)(x, A, y)z$$

and $S \xRightarrow{rm}^* X_1 \dots X_m A y z$ in G .

If $y_m = x, y_n = y, A \rightarrow X_{m+1} \dots X_n \in P$,

$$U \rightarrow \phi\psi$$

$$= (y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{n-1}, X_n, y_n).$$

Then

$$S' \xRightarrow{rm} X_1 \dots X_m A y z \$ \xRightarrow{rm} X_1 \dots X_m X_{m+1} \dots X_n y z \$ \text{ in } G'$$

and $[A \rightarrow X_{m+1} \dots X_r \cdot X_{r+1} \dots X_n, k+1:yz\$]$

$$\langle X_1 \dots X_r \rangle_{k+1}.$$

\therefore (iii) is true.

Case3: $U = (ax, a, xb)$.

By lemma 6.72,

$$(y_0, S, \varepsilon)\$ \xRightarrow{rm}^+ y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, ax)(ax, a, xb)z,$$

and $\exists A' \rightarrow X_{i+1} \dots X_n \in P$,

$$(y_0, S, \varepsilon)\$ \xRightarrow{rm}^+ y_0(y_0, X_1, y_1) \dots (y_{i-1}, X_i, y_i)(y_i, A', y_n)u$$

$$\xRightarrow{rm} y_0(y_0, X_1, y_1) \dots (y_{n-1}, X_n, y_n)u, \text{ and}$$

$$(y_{r+1}, X_{r+2}, y_{r+2}) \dots (y_{n-1}, X_n, y_n) u \xRightarrow{rm}^* z.$$

$$\therefore S' \xRightarrow{rm}^* X_1 \dots X_i A' y_n u \$ \xRightarrow{rm} X_1 \dots X_n y_n u \$$$

in G by lemma 6.72.

$$[A' \rightarrow X_{i+1} \dots X_r \cdot a X_{r+2} \dots X_n, k+1 : y_n u \$] \\ \in \langle X_1 \dots X_r \rangle_{k+1},$$

$$\text{and } X_{r+2} \dots X_n \xRightarrow{rm} v \text{ in } G, v y_n u = y_{r+1} z.$$

$$x b d = k+1 : y_{r+1} z \$ \in \text{First}_{k+1}(X_{r+2} \dots X_n y_n u \$) \\ = \text{First}_{k+1}(X_{r+2} \dots X_n (k+1 : y_n u \$)).$$

\therefore one of (iv) and (v) is true.

Case4: $U = (ax, a, x)$, similar to Case3, (vi) is true.

Lemma 6.81 If $T_{k, 1}(G)$ is non-LR(1), then G is non-LR($k+1$).

Proof

Let Φ be a viable prefix $\exists I, J \in \langle \Phi \rangle_1$

which cause a conflict.

$$\text{Case1: } \Phi = x(x, S, \varepsilon), I = [S_0 \rightarrow x(x, S, \varepsilon) \cdot, \$],$$

$$J = [(x, A, \varepsilon) \rightarrow (x, S, \varepsilon) \cdot, \$].$$

Then $[A \rightarrow S \cdot, \$] \in \langle S \rangle_{k+1}$,

$$S' \xRightarrow{rm} S \$ \xRightarrow{rm}^* A \$ \xRightarrow{rm} S \$ \text{ in } G', \text{ and } S \xRightarrow{rm}^+ S \text{ in } G.$$

$$\text{Case2: } \Phi = x(x, S, \varepsilon), I = [S_0 \rightarrow x(x, S, \varepsilon) \cdot, \$],$$

$$J = [(x, A, \varepsilon) \rightarrow \cdot, \$].$$

Then $[A \rightarrow \cdot, \$] \in \langle S \rangle_{k+1}$, and $S' \xRightarrow{rm} S \$ \xRightarrow{rm}^* A \$ \xRightarrow{rm} S \$$ in G .

Case3: $\Phi = y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, y_r)$,

$I = [(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, y_r) \bullet, d]$

$J = [(y_p, B, y_r) \rightarrow (y_p, X_{p+1}, y_{p+1}) \dots (y_{r-1}, X_r, y_r) \bullet, d]$.

Then $[A \rightarrow X_{m+1} \dots X_r \bullet, d]$ and $[B \rightarrow X_{p+1} \dots X_r \bullet, d]$

cause a conflict in $\langle X_1 \dots X_r \rangle_{k+1}$.

Case4: $\Phi = y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, ax)$,

$I = [(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, ax) \bullet, b]$

$J = [(ax, a, xb) \rightarrow \bullet b, d]$.

Then $[A \rightarrow X_{m+1} \dots X_r \bullet, axb]$ and $[B \rightarrow \alpha \bullet a \beta, y']$

cause a s-r conflict in $\langle X_1 \dots X_r \rangle_{k+1}$

because $xbd \in \text{First}_{k+1}(\beta y')$.

Case5: $\Phi = y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, ax)$,

$I = [(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, ax) \bullet, \$]$

$J = [(ax, a, xb) \rightarrow \bullet, \$]$.

Then $[A \rightarrow X_{m+1} \dots X_r \bullet, ax\$]$ and $[B \rightarrow \alpha \bullet a \beta, y']$

cause a s-r conflict in $\langle X_1 \dots X_r \rangle_{k+1}$

because $x\$ \in \text{First}_{k+1}(\beta y')$.

Lemma 6.82 Let

$$X_i \xRightarrow{rm}^* v_i, y_i \in \text{Follow}_k(X_i), \text{ and } y_{i-1} = k:v_i y_i.$$

Then

$$(y_0, X_1, Y_1) \dots (y_{n-1}, X_n, y_n) \xRightarrow{rm}^* z, \\ \text{where } v_1 \dots v_n y_n = y_0 z.$$

Lemma 6.83 Let

$$[A \rightarrow X_{m+1} \dots X_r \cdot X_{r+1} \dots X_n, y_n d] \\ \in \langle X_1 \dots X_r \rangle_{k+1},$$

$$X_i \xRightarrow{rm}^* v_i \text{ and } y_{i-1} = k:v_i y_i, 1 \leq \forall i \leq n, 0 \leq m \leq r \leq n.$$

Then

$$(b) [(y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, y_r) \cdot \\ (y_r, X_{r+1}, y_{r+1}) \dots (y_{n-1}, X_n, y_n), d] \\ \in \langle y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1.$$

Moreover, if $r < n$ and

$$(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb), \text{ then}$$

$$(c) [(ax, a, xb) \rightarrow \cdot b, 1:ud] \\ \in \langle y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1, \\ \text{where } xbu = v_{i+1} \dots v_n y_n.$$

Similarly, if $r < n$ and

$$(y_r, X_{r+1}, y_{r+1}) = (ax, a, x), \text{ then}$$

$$(d) [(ax, a, xb) \rightarrow \cdot, \$] \\ \in \langle y_0(y_0, X_1, Y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1.$$

Proof

(i) By definition, $\exists v \in \Sigma^* . \exists$.

$$S' \Rightarrow S\$ \Rightarrow^* X_1 \dots X_m A v \$ \Rightarrow X_1 \dots X_m X_{m+1} \dots X_n v \$$$

and $k+1:v\$ = y_n d$.

Then

$$(y_0, S, \varepsilon) \Rightarrow^* (y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m) (y_m, A, y_n) z,$$

$$(y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{n-1}, X_n, y_n)$$

in $T_{k,1}(G)$.

Then

$$S_0' \Rightarrow S_0\$ \Rightarrow y_0(y_0, S, \varepsilon)\$$$

$$\Rightarrow^* y_0(y_0, X_1, y_1) \dots (y_{m-1}, X_m, y_m) (y_m, A, y_n) z \$$$

$$\Rightarrow y_0(y_0, X_1, y_1) (y_{n-1}, X_n, y_n) z \$.$$

\therefore (b) is true.

(ii) Assume that $r < n$, $(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb)$.

Then $(ax, a, xb) \rightarrow b$ in $T_{k,1}(G)$ and

$$X_i \Rightarrow^* v_i, y_i = k:v_{i+1} \dots v_n y_n \in \text{Follow}_k(X_i),$$

$$(y_{r+1}, X_{r+2}, y_{r+2}) \dots (y_{n-1}, X_n, y_n) \Rightarrow^* u \text{ by lem 6.82}$$

where $v_{r+2} \dots v_n y_n = y_{r+1} u = xbu$.

Then by (b) and lemma 6.17, (c) is true.

(iii) in a same manner (d) is true.

Lemma 6.84 Let $G = (N, \Sigma, P, S)$ and $k \geq 0$.
If G is non-LR($k+1$), then $T_{k, 1}(G)$ is non-LR(1).

Proof

(i) if $S \Rightarrow^+ S$ in G , then $T_{k, 1}(G)$ is ambiguous.

$$S \Rightarrow^+ A_1 \dots A_m S, A_i \Rightarrow^* \varepsilon \text{ for all } i.$$

then $\exists x$ in $T_{k, 1}(G)$,

$$(x, S, \varepsilon) \Rightarrow^+ (x, A_1, x) \dots (x, A_m, x) (x, S, \varepsilon).$$

By lemma 6.75, $A_i \Rightarrow^* \varepsilon$ implies $(x, A_i, x) \Rightarrow^* \varepsilon$ for all i .

$\therefore (x, S, \varepsilon)$ derives itself and $T_{k, 1}(G)$ is ambiguous.

(ii)

$$[A \rightarrow X_{m+1} \dots X_r; w'], [B \rightarrow X_{p+1} \dots X_r; w'] \\ \in \langle X_1 \dots X_r \rangle_{k+1}.$$

then for $y_r d = w'$, $y_i = k: v_{i+1} y_{i+1}$,

$$[(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, y_r); d], \\ [(y_p, A, y_r) \rightarrow (y_p, X_{p+1}, y_{p+1}) \dots (y_{r-1}, X_r, y_r); d], \\ \in \langle y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, y_r) \rangle_{k+1}.$$

(iii)

$$[A \rightarrow X_{m+1} \dots X_r; w_1], [B \rightarrow X_{p+1} \dots X_r \cdot X_{r+1} \dots X_n, w_2] \\ \in \langle X_1 \dots X_r \rangle_{k+1}, w_1 \in \text{First}_{k+1}(X_{r+1} \dots X_n w_2).$$

then $\exists v_i \dots X_i \Rightarrow^* v_i$ and $k+1: v_{r+1} \dots v_n w_2 = w_1$.

let $y_i = k: v_{i+1} y_{i+1}$, then

$$y_r = k: v_{r+1} y_{r+1} = k: v_{r+1} v_{r+2} y_{r+2} \\ = \dots = k: v_{r+1} \dots v_n y_n,$$

$$\begin{aligned} \text{and } w_1 &= k+1:v_{r+1}\dots v_n w_2 \\ &= k+1:v_{r+1}\dots v_n y_n d_2. \end{aligned}$$

Then by lem6.83,

$$\begin{aligned} [(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \dots (y_{r-1}, X_r, y_r), d_1] \\ \in \langle y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1. \end{aligned}$$

If $(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb)$,

$$\begin{aligned} [(ax, a, xb) \rightarrow \bullet b, 1:ud_2] \\ \in \langle y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1, \\ xbu \in v_{r+1}\dots v_n y_n. \end{aligned}$$

$$\begin{aligned} \text{Then } w_1 &= k+1:v_{r+1}\dots v_n y_n d_2 = k+1:av_{r+2}\dots v_n y_n d_2 \\ &= k+1:axbud_2 = k+1:y_r bud_2. \end{aligned}$$

$\therefore \exists$ a shift-reduce conflict.

If $(y_r, X_{r+1}, y_{r+1}) = (ax, a, x)$, by lem6.83, $d_2 = \$, m$

$$\begin{aligned} [(ax, a, x) \rightarrow \bullet, \$] \\ \in \langle y_0(y_0, X_1, y_1) \dots (y_{r-1}, X_r, y_r) \rangle_1. \end{aligned}$$

$$y_{r+1} = k:v_{r+2}\dots v_n y_n = v_{r+2}\dots v_n y_n$$

$$y_r = ay_{r+1} = v_{r+1}\dots v_n y_n$$

$\therefore d_1 = \$$ and \exists a reduce-reduce conflict.

Theorem 6.85

For any reduced grammar $G = (N, \Sigma, P, S)$ and $k \geq 0$,
 $T_{k, 1}(G)$ is LR(1) iff G is LR($k+1$).

Theorem 6.86 For $k \geq 1$, any reduced grammar G ,
 G can be transformed into $G' .\exists$.

G' is an equivalent grammar,
 G' right-to-right covers G , and
 G' is LR(1) iff G is LR(k).

Theorem 6.87 For any alphabet Σ ,
the family of deterministic languages over Σ
coincides with
the family of SLR(1) languages over Σ .