6. LR(k) Parsing

LR(k) *parsing:*
The most general deterministic parsing method in which the input string is parsed

(1) in a single ***L****eft-to-right scan,*
(2) producing a ***R****ight parse, and*
(3) using lookahead of length ***k****.*

Generalization of

(1) nondeterministic shift-reduce parser
(2) the simple precedence parser

stack symbols:

grammar symbols are divided up into
one or more "context dependent" symbols

Two stack strings $\gamma_1 X$ *and* $\gamma_2 X$ *are equivalent, if*
exactly same set of parsing actions are valid
in the context of $\gamma_1 X$ *and* $\gamma_2 X$*.*

Replacing X by equivalent class $[\gamma X]$
refinement of stack symbol

$$X \in V \qquad [\gamma X] \in 2^{V^*}$$

6.1 Viable Prefixes

G_{ab}:

$S \to aA \mid bB$

$A \to c \mid dAd$

$B \to c \mid dBd$

$L(G_{ab}) = \{a, b\}\{d^n c d^n \mid n \geq 0\}$.

$\$\alpha c \mid y\$$, where $\$\alpha{:}1 \in \{a, b, d\}$ and $1{:}y\$ \in \{\$, d\}$

reduce-reduce conflict for $A \to c$ *and* $B \to c$.

$(1{:}\alpha = a)$ $(1{:}\alpha = b)$

Extending lookahead and lookback into length k.

$\alpha c \mid x \to \alpha A \mid x$, $\beta c \mid y \to \beta B \mid y$

$\alpha, \beta \in V^*{:}k$, $x, y \in k{:}\Sigma^*\$$.

but

$ad^k c \mid d^k \to ad^k A \mid d^k$, $bd^k c \mid d^k \to bd^k B \mid d^k$

reduce-reduce conflict for any k!

A string γ *is a* ***viable stack string*** *of pda M, if*

$\$\gamma_s \mid w\$ \Rightarrow^* \$\gamma \mid y\$ \Rightarrow^* \$\gamma_f \mid \$$ in M.

stack string in some accepting computation M.

Not *arbitrary string is a* ***viable*** *stack string.*

2^{V^*} vs. 2^{VS} where $VS \subseteq V^*$.

Viable stack strings of G_{ab}:

$$\{\varepsilon\} \cup \{ad^n \mid n\geq 0\} \cup \{ad^n c \mid n\geq 0\}$$
$$\cup \{ad^n A \mid n\geq 0\} \cup \{ad^n Ad \mid n\geq 1\}$$
$$\cup \{bd^n \mid n\geq 0\} \cup \{bd^n c \mid n\geq 0\}$$
$$\cup \{bd^n B \mid n\geq 0\} \cup \{bd^n Bd \mid n\geq 1\}$$
$$\cup \{S\}$$

*Not every **action** is valid, for viable stack string*

$$ad^n c \mid \Rightarrow_{valid} ad^n A \mid, \quad bd^n c \mid \Rightarrow_{valid} bd^n B \mid; \text{ but}$$
$$ad^n c \mid \not\Rightarrow_{valid} ad^n B \mid, \quad bd^n c \mid \not\Rightarrow_{valid} bd^n A \mid.$$

*An action r is **valid** for viable stack string γ of M if*

$$\$\gamma \mid y\$ \Rightarrow^{r} \$\gamma' \mid y'\$ \Rightarrow^{*} \$\gamma_f \mid \$ \text{ in } M$$

*The set of <u>viable stack strings</u> are **infinite**. But we can divide the set of viable stack strings in to a **finite** number of <u>**equivalent classes**</u>.*

*Two <u>viable stack string</u> belongs to the **same equivalent** class if they have same set of **valid actions**.*

Since for any $G = (N, \Sigma, P, S)$ in shift-reduce parser

number of distinct actions = $|\Sigma| + |P| \leq |G|$

number of equivalent classes $\leq 2^{|G|}$.

$\therefore$ number of equivalent classes is finite.

equivalent classes:	*valid actions:*
$\{\varepsilon\}$	*shift a, shift b*
$\{ad^n \mid n \geq 0\} \cup \{bd^n \mid n \geq 0\}$	*shift c, shift d*
$\{ad^nc \mid n \geq 0\}$	*reduce by* $A \rightarrow c$
$\{bd^nc \mid n \geq 0\}$	*reduce by* $B \rightarrow c$
$\{aA\}$	*reduce by* $S \rightarrow aA$
$\{bB\}$	*reduce by* $S \rightarrow bB$
$\{ad^nA \mid n\geq 1\} \cup \{bd^nB \mid n\geq 1\}$	*shift d*
$\{ad^nAd \mid n \geq 1\}$	*reduce by* $A \rightarrow dAd$
$\{bd^nBd \mid n \geq 1\}$	*reduce by* $B \rightarrow dBd$
$\{S\}$	—

stack symbols: equivalent classes (grammar symbol)

$X \Rightarrow [\delta X]$: δX: *viable stack string*

$[\delta X]$: *equivalent class of* δX

shift *a*

$[\delta] \mid a \rightarrow [\delta][\delta a] \mid$

reduce *by* $A \rightarrow X_1 \ldots X_n$

$[\delta][\delta X_1] \ldots [\delta X_1 \ldots X_n] \mid \rightarrow [\delta][\delta A] \mid$

$\gamma_s = [\varepsilon]$ *and* $\gamma_f = \{[\varepsilon][S]\}$

$\therefore [\varepsilon] \mid yz \Rightarrow^* [\varepsilon][Y_1] \ldots [Y_1 \ldots Y_k] \mid z \Rightarrow^* [\varepsilon][S] \mid .$

$Y_1 \ldots Y_i$ *are viable stack string for* $0 \leq {}^{\forall} i \leq k$.

***Regular expression** for valid viable stack strings*

ε, $ad^* \mid bd^*$, ad^*c, aA, $ad^+A \mid bd^+B$, ad^+Ad, bd^*c, bB, bd^+Bd, S

For regular expression E, we define

$$[E] \equiv \cup_{w \in L(E)} [w].$$

$$\therefore L(E) \subseteq [E], \text{ in fact usually } L(E) = [E].$$

equivalent classes:	valid actions:
$[\varepsilon]$	shift a, shift b
$[S]$	—
$[ad^* \mid bd^*]$	shift c, shift d
$[ad^*c]$	reduce by $A \rightarrow c$
$[bd^*c]$	reduce by $B \rightarrow c$
$[aA]$	reduce by $S \rightarrow aA$
$[bB]$	reduce by $S \rightarrow bB$
$[ad^+A \mid bd^+B]$	shift d
$[ad^+Ad]$	reduce by $A \rightarrow dAd$
$[bd^+Bd]$	reduce by $B \rightarrow dBd$

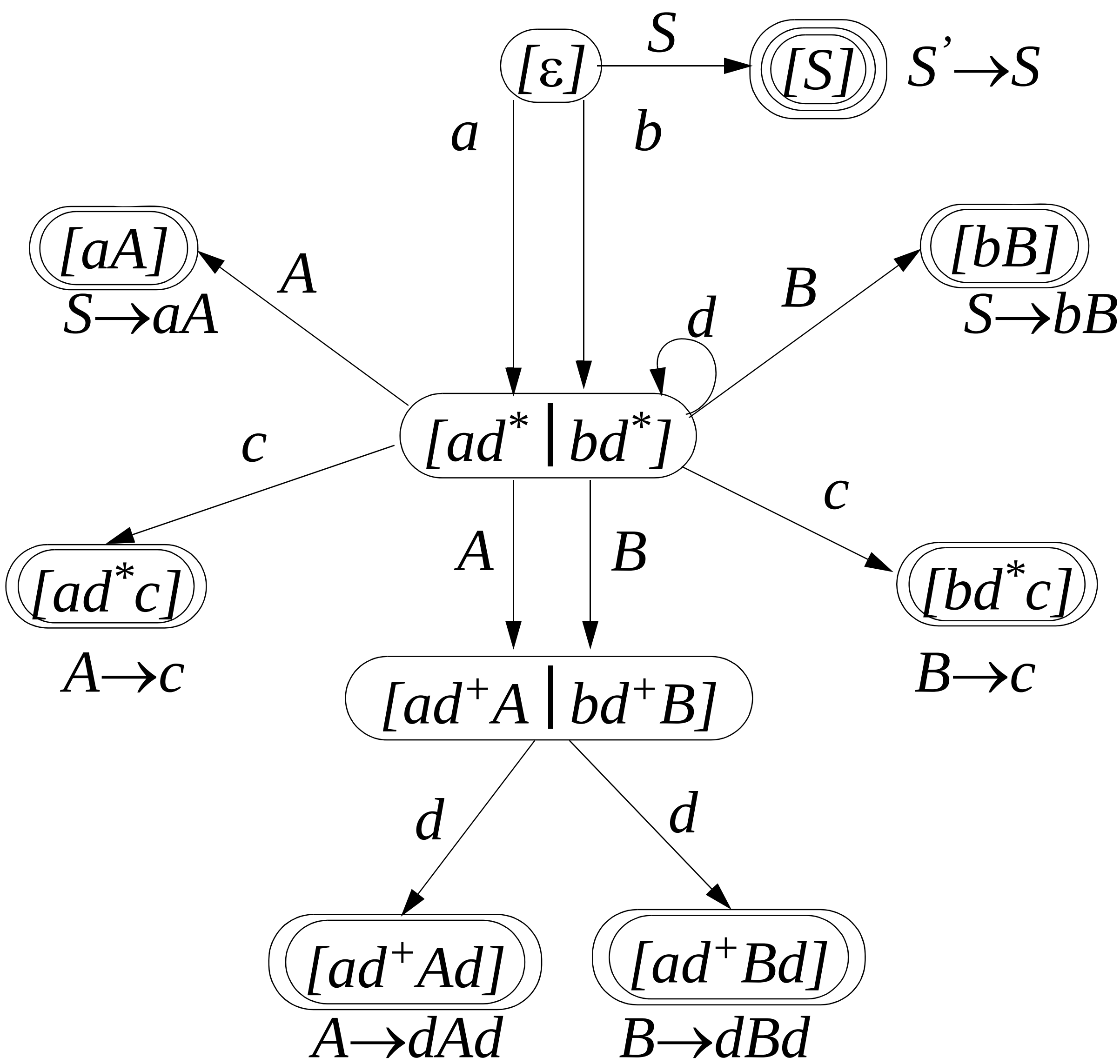
Regular expressions $\Rightarrow$ finite automata

regular expression over $N \cup \Sigma$.

$\therefore$ finite automaton with input alphabet $N \cup \Sigma$.

Characteristic finite state machine

Deterministic parsing of context-free languages?



No "reduce-reduce conflicts" by $A \to c$ and $B \to c$.

$[ad^* | bd^*][ad^*c] \ | \!\!\to [ad^* | bd^*][aA] \ |$ (reduce by $A \to c$),

$[ad^* | bd^*][bd^*c] \ | \!\!\to [ad^* | bd^*][bB]$ (reduce by $B \to c$).

note that $[aA] \neq [ad^+A|bd^+B] \neq [bB]$.

But "shift-shift conflict"

$[ad^*|bd^*] \mid c \rightarrow [ad^*|bd^*][ad^*c] \mid$ (shift c),

$[ad^*|bd^*] \mid c \rightarrow [ad^*|bd^*][bd^*c] \mid$ (shift c),

and

$[ad^+A|bd^+B] \mid d \rightarrow [ad^+A|bd^+B][ad^+Ad] \mid$,

$[ad^+A|bd^+B] \mid d \rightarrow [ad^+A|bd^+B][bd^+Bd] \mid$.

Consider ad^n, ad^nA, *and* ad^nB *for* $n \geq 0$.

$ad^n \in [ad^*|bd^*]$. *But*

$ad^nA \in [aA]$ *and* $[ad^+A|bd^+B]$.

$bd^nA \in [aB]$ *and* $[ad^+A|bd^+B]$.

$[ad^*|bd^*]$ *is split into* $[a]$, $[ad^+]$, $[b]$, *and* $[bd^+]$

Since $[ad^*c] \neq [bd^*c]$, $[ad^*] \neq [bd^*]$.

Since $[aA] \neq [ad^+A]$, $[a] \neq [ad^+]$.

Since $[bA] \neq [bd^+A]$, $[b] \neq [bd^+]$.

$[ad^+A|\ bd^+B]$ *is split into* $[ad^+A]$ *and* $[bd^+B]$

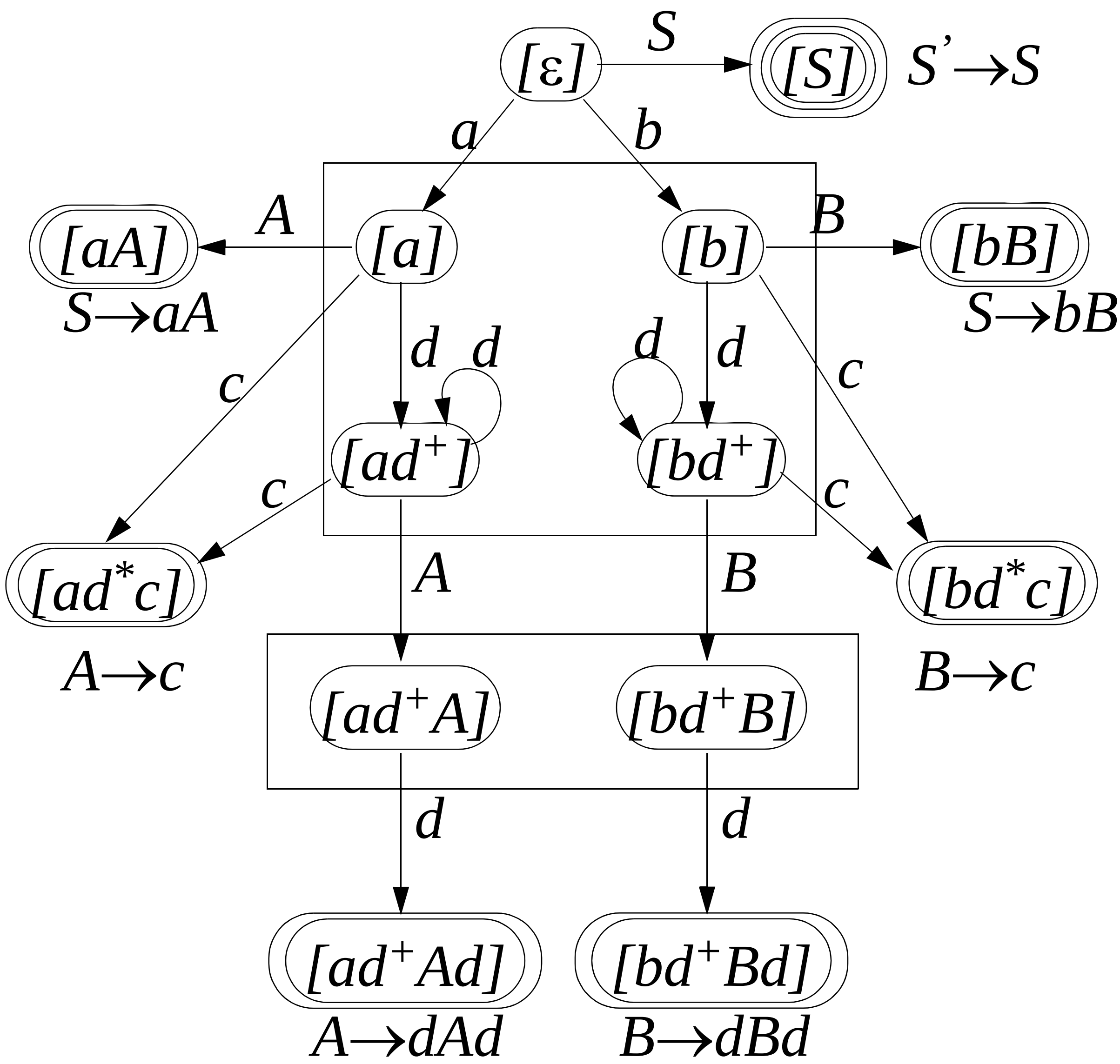
Since $[ad^+Ad] \neq [bd^+Bd]$, $[ad^+A] \neq [bd^+B]$.

$[ad^* \mid bd^*]$

$[a], [b], [ad^+], [bd^+]$

$[ad^+A \mid bd^+B]$

$[ad^+A], [bd^+B]$

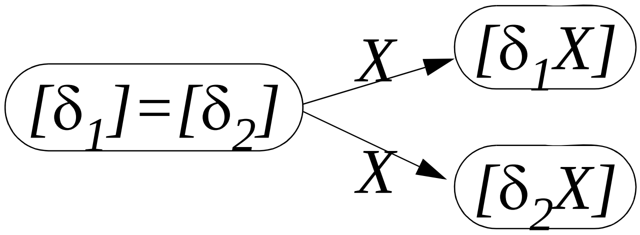


Right invariant(unique outgoing symbol)

If two stack string δ_1 and δ_2 are equivalent, they remain equivalent when they are lengthened.

$$\text{If } [\delta_1] = [\delta_2],\ [\delta_1 X] = [\delta_2 X].$$

Otherwise "shift-shift" conflict.



Unique entry symbol

Two equivalent stack string should end with same symbols. If $[\gamma_1] = [\gamma_2]$, $\gamma_1{:}1 = \gamma_2{:}1$.

Otherwise, reduce action is not uniquely defined.

Consider $[\delta][\delta X_1] \ldots [\delta X_1 \ldots X_n]\ | \rightarrow [\delta A]\ |$.

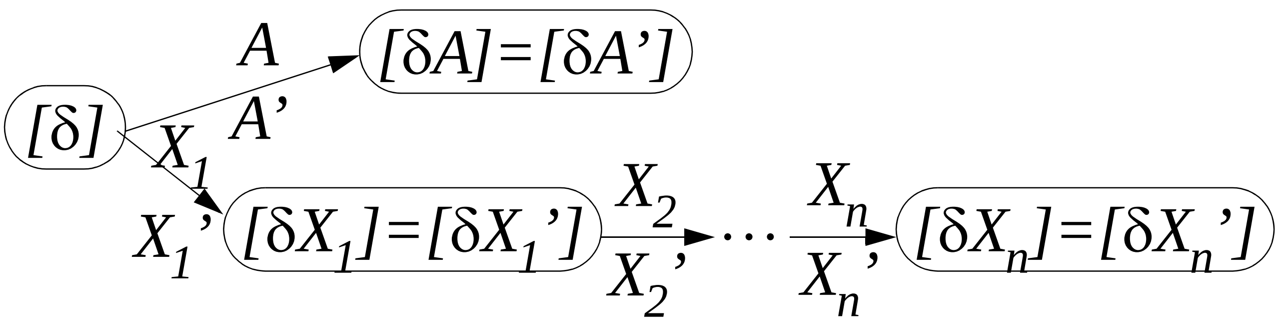
the rule $A \rightarrow X_1 \ldots X_n$ is uniquely defined, if

$$[\delta][\delta X_1'] \ldots [\delta X_1' \ldots X_n']\ | \rightarrow [\delta A']\ |,$$

$$\exists\ A' \rightarrow X_1' \ldots X_n' \in P$$

$$.\text{э}.\ [\delta A] = [\delta A'],\ [\delta X_1] = [\delta X_1'],\ \ldots,$$

$$[\delta X_1 \ldots X_n] = [\delta X_1' \ldots X_n'].$$



Let G = (N, Σ, P, S) be a grammar. String $\gamma \in V^*$ is a ***viable prefix*** *of G, if*

$$S \Rightarrow_{rm}^{*} \delta A y \Rightarrow_{rm} \delta\alpha\beta y \ (=\gamma\beta y)$$

where $\delta \in V^*$, $y \in \Sigma^*$, *and* $A \to \alpha\beta \in P$.

γ *is a* ***complete viable prefix***, *if* $\beta = \varepsilon$.

Fact 6.1 *Any viable prefix is a* ***prefix*** *of some* *complete viable prefix.*

Lemma 6.4 *Any* ***prefix*** *of a* *viable prefix* *is a* ***viable prefix***.

Proof $S \Rightarrow_{rm}^{n} \delta A y \Rightarrow_{rm} \underline{\delta\alpha}\beta y = \underline{\gamma_1\gamma_2}\beta y$

i) δ *is a prefix of* γ_1.

$\gamma_1 = \delta\alpha'$ *where* $\alpha = \alpha'\gamma_2$. $\therefore$ γ_1 *is a viable prefix.*

ii) γ_1 *is a prefix of* δ.($\delta \neq \varepsilon$)

$\delta A = \gamma_1\eta$. $n > 0$, γ_1 *is a viable prefix. (**L6.2**)*

Lemma 6.2 *Let G = (N, Σ, P, S) be a grammar,*

$\pi \in P^+$, $\gamma, \eta, \delta \in V^*$, $A \in N$, *and* $y \in \Sigma^*$.

If $S \Rightarrow_{rm}^{\pi} \gamma\eta y = \delta A y$ *in G, and* $\pi \neq \varepsilon$. *Then*

$$S \Rightarrow_{rm}^{\pi'} \delta' A' y'$$

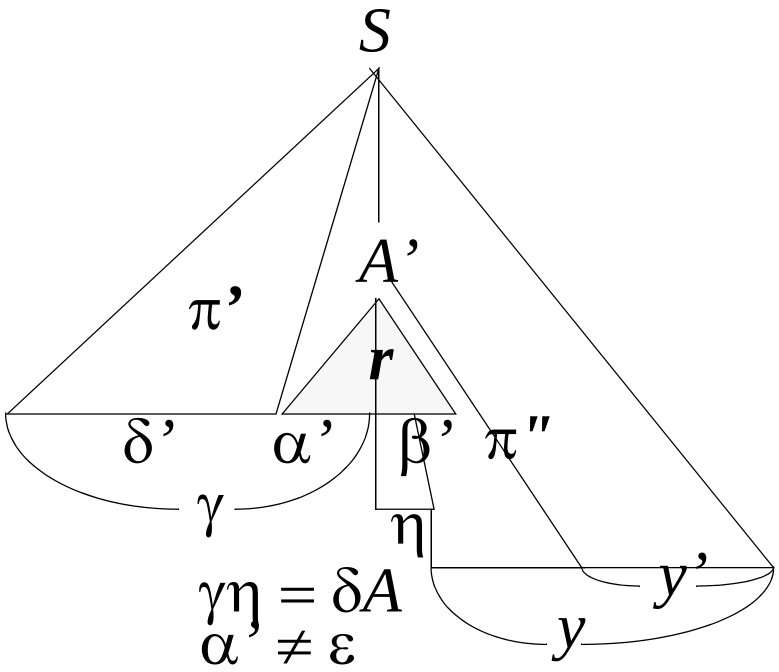
$$\Rightarrow_{rm}^{r} \underline{\delta'\alpha'}\beta' y' = \underline{\gamma}\beta' y'$$

$$\Rightarrow_{rm}^{\pi''} \gamma\eta y = \delta A y,$$

$\pi' r \pi'' = \pi$, *and* $\underline{\alpha' \neq \varepsilon}$ ($\alpha':1 = \gamma:1$).

If γ is a ***prefix*** *of some nontrivially derived right sentential form(not extending over the last nonterminal), the derivation contains a segment rule(r) that proves γ to be a* ***viable prefix****, even so that the right-hand side of the rule r cuts γ properly.*

Any prefix of nontrivially derived right sentential form(not extending over the last nonterminal) is a ***viable prefix****.*



Proof *induction on the length of* π.

i) $|\pi| = 1$. $\pi = S \rightarrow \gamma\eta y = A' \rightarrow \alpha'\beta' y$.

$\delta' = y' = \varepsilon$, $(\gamma = \alpha', \eta y = \beta')$

ii) $|\pi| > 1$. *Assume that IH holds for* π_1 *where* $\pi = \pi_1 r_1$.

$S \Rightarrow_{rm}^{\pi_1} \gamma_1\eta_1 y_1 = \delta_1 A_1 y_1$

$\Rightarrow_{rm}^{r_1} \delta_1\omega y_1 = \gamma\eta y = \delta A y$, $\pi = r\pi_1$.

Then $S \Rightarrow_{rm}^{\pi_1'} \delta_1' A_1' y_1'$

$\Rightarrow_{rm}^{r_1} \underline{\delta_1'\alpha_1'}\beta_1' y_1' = \underline{\gamma_1}\beta_1' y_1'$

$\Rightarrow_{rm}^{\pi_1''} \gamma_1\eta_1 y_1 = \delta_1 A_1 y_1$,

$\pi_1' r_1 \pi_1'' = \pi_1$, *and* $\delta_1'\alpha_1' = \gamma_1$.

Note that $\gamma = \delta_1\alpha''$ *where* $\alpha'' \neq \varepsilon$ *or* $\gamma\alpha = \delta_1$

a) $\gamma = \delta_1\alpha''$, *it is trivial, since*

$\delta' = \delta_1$, $y' = y_1$, $\pi' = \pi_1$, $\pi'' = \varepsilon$, $r = r_1$.

b) $\gamma\alpha = \delta_1$

$S \Rightarrow_{rm}^{\pi_1} \gamma\eta_1 y_1 = \delta_1 A_1 y_1$.

Lemma 6.3

$S \Rightarrow_{rm}^{+} \delta Ay$. *Then* δ *is a* ***viable prefix***.

Proof. $\eta = \varepsilon$.

Lemma 6.5 *Let* $A \rightarrow \alpha\beta \in P$. *Then if* γA *is a viable prefix of G, then so is* $\gamma\alpha$.

Lemma 6.6 *If*

$\$ \mid w\$ \Rightarrow^{\pi} \$\gamma\eta \mid y\$$. *Then*

$\$ \mid w\$ \Rightarrow^{\pi'} \$\gamma \mid z\$ \Rightarrow^{\pi''} \$\gamma\eta \mid y\$$, *and* $\pi = \pi'\pi''$.

Proof *induction on* $|\pi|$.

i) $\pi = \varepsilon, \gamma = \eta = \varepsilon$.

ii) $\pi \neq \varepsilon$ *and* $\eta \neq \varepsilon$, $\pi = \pi_1 r_1$.

(1) $\$ \mid w\$ \Rightarrow^{\pi_1} \$\psi \mid ay\$ \Rightarrow^{r_1} \$\psi a \mid y\$$, *or*

(2) $\$ \mid w\$ \Rightarrow^{\pi_1} \$\delta\omega \mid y\$ \Rightarrow^{r_1} \$\delta A \mid y\$$.

γ *is a prefix of* ψ *in (1), and a prefix of* δ *in (2).*

Theorem 6.7

Let G = (N, Σ, P, S), M be a shift-reduce parser of G.
Any viable stack string of M is
either S or ***viable prefix*** *of G.*
Conversely, any viable prefix of G is
a ***viable stack string*** *of M,*
provided that G is reduced.

Proof *from lemma 5.17, 5.19.*
(shift-reduce parser = right parser)

Given a grammar $G = (N, \Sigma, P, S)$,
Let $G_{VP} = (N_{VP}, \Sigma_{VP}, P_{VP}, [S])$ where

$N_{VP} = \{[A] \mid A \in N\}$,

$\Sigma_{VP} = N \cup \Sigma$, *and*

$P_{VP} = \{[A] \rightarrow \alpha \mid A \rightarrow \alpha\beta \in P\} \cup$
$\{[A] \rightarrow \alpha[B] \mid A \rightarrow \alpha B\beta \in P, B \in N\}$.

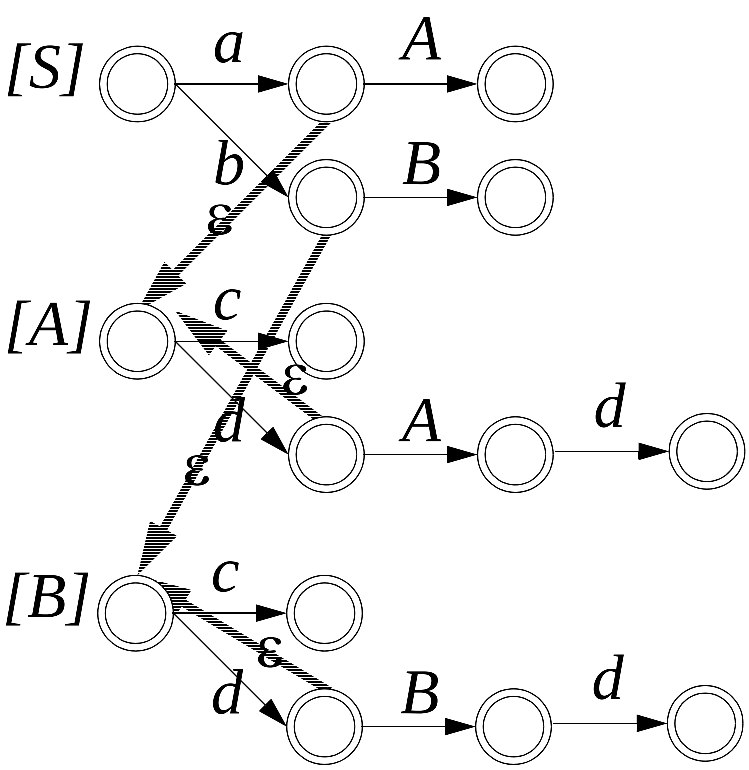
Example)

$(G_{ab})_{VP}$:

$[S] \rightarrow \varepsilon|a|aA|b|bB|a[A]|b[B]$
$[A] \rightarrow \varepsilon|c|d|dA|dAd|d[A]$
$[B] \rightarrow \varepsilon|c|d|dB|dBd|d[B]$

G_{ab}:

$S \rightarrow aA \mid bB$
$A \rightarrow c \mid dAd$
$B \rightarrow c \mid dBd$



Rule automaton

Lemma 6.8 *Let* $S \Rightarrow_{rm}^{n} \delta A y$ *in G. Then*

$[S] \Rightarrow^{*} \delta[A]$ *in* G_{VP}.

Proof

i) $n=0$: *it is clear* ($A=S$, $\delta=y=\varepsilon$).

ii) $0 \leq {}^{\forall}m < n$:

$S \Rightarrow_{rm}^{m} \delta' A' y' \Rightarrow_{rm} \delta' \alpha A \beta y' = \delta A \beta y'$ *in G.* (**L6.2**)

$[A'] \rightarrow \alpha[A] \in P_{VP}$, *since* $A' \rightarrow \alpha A \beta \in P$

$[S] \Rightarrow^{*} \delta'[A'] \Rightarrow \delta' \alpha [A] = \delta[A]$ *in* G_{VP}.

Lemma 6.9 *Let* $[S] \Rightarrow^{n} \delta[A]$ *in* G_{VP}. *Then*

$S \Rightarrow^{*} \delta A y$ *in G.*

Proof

i) $n=0$: $\delta=\varepsilon$, $A=S$, $y=\varepsilon$.

ii) $0 \leq {}^{\forall}m < n$:

$S \Rightarrow^{m} \delta'[A'] \Rightarrow \delta' \alpha [A] = \delta[A]$ *in* G_{VP},

$A' \rightarrow \alpha A \beta \in P$, $\beta \Rightarrow^{*} x \in \Sigma^{*}$, *since* $[A'] \rightarrow \alpha[A] \in P_{VP}$.

$S \Rightarrow^{*} \delta' A' y' \Rightarrow \delta' \alpha A \beta \Rightarrow^{*} \delta' \alpha A x y' = \delta A y$ *in G.*

Theorem 6.10 *The grammar G_{VP} generates the set of* ***viable prefixes*** *of G. And G_{VP} is* ***right linear****.*

Proof.

If $S \Rightarrow_{rm}^{*} \delta Ay \Rightarrow_{rm} \delta\alpha\beta y$ $(=\gamma\beta y)$ *in G* $(A \rightarrow \alpha\beta \in P)$,

$[S] \Rightarrow^{*} \delta[A] \Rightarrow \delta\alpha$ $(=\gamma)$.

If $[S] \Rightarrow^{*} \delta[A] \Rightarrow \delta\alpha \in V^{*}$ *and* $A \rightarrow \alpha\beta \in P$, *then*

$S \Rightarrow_{rm}^{*} \delta Ay \Rightarrow_{rm} \delta\alpha\beta y$.

Theorem 6.11 *For any grammar $G = (N, \Sigma, P, S)$, the set of all viable prefixes is a* ***regular expression*** *over V.*

viable prefices = valid stack strings
= regular expression

G_{VP} is a regular grammar generating the set of viable prefixes of G.

C_0 in G is the ***dfa*** *for G_{VP}.*

6.2 Valid LR(k) Items

Let $A \rightarrow \alpha\beta \in P$. *Then* $[A \rightarrow \alpha\bullet\beta, y]$ *is a* ***k-item***, *if*

$A \rightarrow \alpha\bullet\beta$ *is a* <u>*position*</u> *of G and* $y \in \Sigma^k$.

0-item $[A \rightarrow \alpha\bullet\beta, \varepsilon] \equiv [A \rightarrow \alpha\bullet\beta]$

$A \rightarrow \alpha\bullet\beta$ *is* ***core*** *of the item,*

y is the ***lookahead*** *of the item.*

A k-item $[A \rightarrow \alpha\bullet\beta, y]$ *is* ***LR(k)-valid*** *(or* ***valid****) for string* $\gamma(=\delta\alpha) \in V^*$ *if*

$S \Rightarrow_{rm}^* \delta Az \Rightarrow_{rm} \delta\alpha\beta z\ (=\gamma\beta z)$ *and* $y = k{:}z\$^k$.

Let R_k *denotes the set of whole valid LR(k) items.*

Fact 6.12 *If* $[A \rightarrow \alpha\bullet\beta, y]$ *is a LR(k) valid item for string* $\gamma(=\delta\alpha)$, *then* γ *is a* ***viable prefix*** *and*

$y \in Follow_k(\delta\alpha\beta) = Follow_k(\delta A) \subseteq Follow_k(A)$.

Conversely, if a string γ *is a viable prefix, then some item is LR(k)-valid for* γ.

Define $Valid_{LR(k)}^{G}: V^* \rightarrow 2^{R_k}$.

Let $\gamma \in V^*$. *Then*

$Valid_k(\gamma)_{LR(k)}^{G} =$

$$\{[A\rightarrow\alpha.\beta, x] \mid S \underset{rm}{\overset{*}{\Rightarrow}} \delta Az \underset{rm}{\Rightarrow} \delta\alpha\beta z = \gamma\beta z,\ x = k{:}z\$^k\}$$

Valid LR(k) items for the ***viable prefix*** γ

$Valid_{LR(k)}^{G} \equiv Valid_{LR(k)} \equiv Valid_k \equiv Valid$

$Valid_k: V^* \rightarrow 2^{R_k}$.

Define $\rho_{LR(k)} \subseteq V^* \times V^*$

γ_1 *is* ***LR(k)-equivalent*** *to* γ_2,

written $\gamma_1\ \rho_{LR(k)}\ \gamma_2$ (*or* $\gamma_1\ \rho_k\ \gamma_2$),

if $Valid_k(\gamma_1) = Valid_k(\gamma_2)$.

The relation ρ_k *is called the* ***LR(k)-equivalence*** *for G.*

ρ_k *is an* ***equivalent*** *relation.*

$[\gamma]_{\rho_k}$ *denotes an* ***equivalent class*** *of* γ *under* ρ_k

$$[\gamma]_{\rho_k} = \{\delta \mid \gamma\ \rho_k\ \delta\}$$

$[\gamma]_{\rho_k} \equiv [\gamma]_k \equiv [\gamma]$.

We denote $[\gamma]_{\rho_k}$ *by* $[\gamma]_k$ (*or even* $[\gamma]$).

We extend the domain of $Valid_k$ *from* V^* *to* 2^{V^*}:

$$Valid_k(L) = \{I \in R_k \mid I \in Valid_k(\alpha), \alpha \in L \subseteq V^*\}$$

$$Valid_k([\gamma]_k) = \{I \mid I \in Valid_k(\delta), \delta \in [\gamma]_k\}$$

Since $Valid_k(\gamma_1) = Valid_k(\gamma_2)$, *if* $\gamma_1,\gamma_2 \in [\gamma]_k$ *or* $\gamma_1\ \rho_k\ \gamma_2$

We may write $Valid_k(\gamma)$ *to denote* $Valid_k([\gamma]_k)$.

$$Valid_k(\gamma) = Valid_k([\gamma]_k) = \{I \in R_k \mid I \in Valid_k(\delta), \delta \in [\gamma]_k\}$$

$[\gamma]_k$: *denotes an* ***equivalent class*** *of* γ*(viable prefices) under* ρ_k.

may be ***infinite***$([\gamma]_k \subseteq V^*)$

$Valid_k(\gamma)$: *denotes a set of* $[A\rightarrow\alpha.\beta, x]$*(LR(k) items) under* ρ_k.

always be ***finite***$(Valid_k(\gamma) \subseteq R_k)$

$[\gamma_1]_k = [\gamma_2]_k$ *iff* $Valid_k(\gamma_1) = Valid_k(\gamma_2)$,

bijective correspondence between $[\gamma]_k$ *and* $Valid_k(\gamma)$.

We may write $\langle\gamma\rangle_k$ *instead of* $Valid_k(\gamma)$

Is it possible that $\gamma = \delta$ *implies* $[\gamma, x]_k = [\delta, y]_k$?

$Valid_k(\varepsilon) = \{[S\rightarrow.aA, \$^k], [S\rightarrow.bB, \$^k]\}$

$Valid_k(a) = \{[S\rightarrow a.A, \$^k],[A\rightarrow.c, \$^k],[A\rightarrow.dAd, \$^k]\}$

$Valid_k(aA) = \{[S\rightarrow aA., \$^k]\}$

$Valid_k(ad^{n+1}) = \{[A\rightarrow d.Ad,k{:}d^n\$^k],[A\rightarrow.c,k{:}d^{n+1}\$^k],$
$[A\rightarrow.dAd, k{:}d^{n+1}\$^k]\}$

$Valid_k(ad^nc) = \{[A\rightarrow c., k{:}d^n\$^k]\}$

$Valid_k(ad^{n+1}A) = \{[A\rightarrow dA.d, k{:}d^n\$^k]\}$

$Valid_k(ad^{n+1}Ad) = \{[A\rightarrow dAd., k{:}d^n\$^k]\}$

…

$1+ 2\cdot(2 + 4(k+1)) = 8k + 13$ (LR(k) states)

$[ad^+]_0 = [ad]_k \cup \ldots \cup [ad^k]_k \cup [ad^{k+1}d^*]_k.$

$[ad^+]_0 = [add^*]_0$
$= [ad]_1 \cup [addd^*]_1$
$= [ad]_2 \cup [add]_2 \cup [adddd^*]_2$
$=[ad]_3\cup [add]_3 \cup [addd]_3 \cup [addddd^*]_3$

$[ad^*c]_0 = [ac]_k \cup \ldots \cup [ad^{k-1}c]_k \cup [ad^kcd^*]_k.$

$[ad^+A]_0 = [adA]_k \cup \ldots \cup [ad^kA]_k \cup [ad^{k+1}d^*A]_k.$

$[ad^+Ad]_0 = [adAd]_k \cup \ldots \cup [ad^kd]_k \cup [ad^{k+1}d^*d]_k.$

…

$[\gamma, \varepsilon]_{\rho_0}$ *denotes the an equivalent class of* γ,
under ρ_0.
$[\gamma]_{\rho_0} \equiv [\gamma]_0 \equiv [\gamma]$ *denotes an equivalent class of valid prefixes under LR(0) equivalence.*

$Valid_k(\langle\gamma, x\rangle)$ *denotes an equivalent class of* $[A\rightarrow\alpha.\beta, x]$ *under* ρ_k.
$Valid_k(\gamma)$ *can denotes an equivalent class of valid LR(k)-items under LR(k) equivalence.*

ρ_k: $V^* \times V^*$, ι_k: $I \times I$. *equivalent relation*

$[\gamma]_{\rho k}$: 2^{V^*}, $[I]_{\iota k}$: 2^I. *equivalent class*

$Valid_k$: $V^* \rightarrow 2^I$, or $2^{V*} \rightarrow 2^I$.

$Valid^{-1}{}_k$: $I \rightarrow 2^{V*}$, or $2^I \rightarrow 2^{V*}$.

$Valid_k(\{[\gamma]_{\rho_k}\}) = Valid^{-1}{}_k(\{[I]_{\iota_k}\})$
iff $I \in Valid_k(\gamma)$ *and/or* $\gamma \in Valid^{-1}{}_k(I)$

Theorem 6.13 *The LR(k)-equivalence* ρ_k *for G is an* ***equivalence relation*** *on* V^*, ρ_k *is a* ***finite index****, and the index of* ρ_k *is at most* $2^{|G|\cdot(|\Sigma|+1)^k}$.

One of the equivalent class under ρ_k *is*

$\{\gamma \mid \gamma$ *is* ***not*** *a viable prefix of G*$\}$.

Proof.

As $[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}$ *iff* $\langle\gamma_1\rangle_{\rho_k} = \langle\gamma_2\rangle_{\rho_k}$

<u>*bijective*</u> *correspondence:* $[\gamma]_{\rho_k}$ *and* $\langle\gamma\rangle_{\rho_k}$.

$\therefore$ *index of* ρ_k *= number of distinct sets* $\langle\gamma\rangle_{\rho_k}$

At most distinct |G| item cores in G and

$|\Sigma|^k + |\Sigma|^{k-1} + \ldots + |\Sigma|^1 + 1 \leq (|\Sigma| + 1)^k$.

A string γ *is a viable prefix iff* $[\gamma]_{\rho_k} \neq \emptyset$

$\therefore$ *set of non viable prefixes forms a single equivalent class under* ρ_k.

Lemma 6.14 *Let $k \le l$. Then*

$$\langle\gamma\rangle_k = \{[A\to\alpha.\beta, k{:}y] \mid [A\to\alpha.\beta, y] \in \langle\gamma\rangle_l\}.$$

Lemma 6.15 *Let $k \le l$. Then LR(l)-equivalence is a* ***refinement*** *of LR(k)-equivalence. More specifically*

$$[\gamma]_k = \cup\ [\delta]_l.$$

$[\gamma]_k$ are ***bijective*** *correspondence with $\langle\gamma\rangle_k$.*

$\langle\gamma\rangle_k$: finite representation of the class $[\gamma]_k$.

collection *of all sets $[\gamma]_k$.*

finite representation of the entire LR(k) equivalence

canonical collection *of* ***set*** *of* ***LR(k)-valid items*** *for G*

canonical LR(k) collection *for G: C_k.*

canonical LR(k) machine *M*

(or ***deterministic LR(k) machine****)*

$M = (C_k, V, \{[\gamma]_k\cdot X \to [\gamma\cdot X]_k\}, [\varepsilon]_k, \emptyset)$

ε-free, normal-form, completely specified, and deterministic fa

(1) ***right-invariance*** *of the LR(k)-equivalence.*

Since dfa, if $[\gamma_1]_k = [\gamma_2]_k$, $[\gamma_1\cdot X]_k = [\gamma_2\cdot X]_k$.

(2) $[\gamma]_k$ has a ***unique entry symbol****.*

Since $[\gamma]_k\cdot X \to [\gamma\cdot X]_k \in P$,

if $[\gamma_1] = [\gamma_2]$, $\gamma_1{:}1 = \gamma_2{:}1$

$[A \to \alpha\cdot B\beta, y]\ \partial_{LR(k)}\ [B \to \cdot\omega, z],\ z \in First_k(\beta y)$

$\partial_{LR(k)} \equiv \partial_k \equiv \partial.$

I_2 *is an* ***immediate*** *LR(k)-**descendant** of* I_1, *if* $I_1\ \partial\ I_2$.

I_2 *is an LR(k)-**descendant** of* I_1, *if* $I_1\ \partial^*\ I_2$.

I_1 *is an (**immediate**) LR(k)-**ancestor** of* I_2,

if I_2 *is an (**immediate**) LR(k)-**descendant** of* I_1.

$[B \to \cdot\omega, z]$ *is* ***immediate*** *LR(k)-**descendant** of*

$[A \to \alpha\cdot B\beta, y]$, *if* $z \in First_k(\beta y)$

$\langle\gamma\rangle_k^n = \{[A\to\alpha\cdot\beta, y] \mid$

$S \Rightarrow_{rm}^n \delta Az \Rightarrow_{rm} \delta\alpha\beta z\ (=\gamma\beta z),\ y = k{:}z\}$

Fact 6.16 $\langle\gamma\rangle_k = \cup_{n=0}^{\infty}\ \langle\gamma\rangle_k^n = \langle\gamma\rangle_k^*$

Lemma 6.17 *If*

$[A \to \alpha\cdot B\beta, y] \in \langle\gamma\rangle_k^n$ *and* $\beta \Rightarrow^m v \in \Sigma^*$. *Then*

$[B \to \cdot\omega, k{:}vy] \in \langle\gamma\rangle_k^{n+m+1}(\gamma)$.

Lemma 6.18 $\langle\gamma\rangle_k$ *is* ***closed under*** ∂_k, *i.e.,*

$\partial_k^*(\langle\gamma\rangle_k) = \langle\gamma\rangle_k$.

Lemma 6.19 *If*

$[B \rightarrow \cdot\omega, z] \in \langle\gamma\rangle_k^n$ *and* $n > 0$. *Then*

$[A \rightarrow \alpha\cdot B\beta, y] \in \langle\gamma\rangle_k^m$, $\beta \Rightarrow^{n-m-1} v$, $k{:}vy = z$.

Fact 6.20 $\langle\gamma\rangle_k^0 = \{[S \rightarrow \gamma\cdot\omega, \varepsilon] \mid S \rightarrow \gamma\omega \in P\}$

$[A \rightarrow \alpha\cdot\beta]$ *is* ***LR-essential*** *(or* ***essential****), if* $\alpha \neq \varepsilon$
inessential*, otherwise.*

$Ess_{LR}(q) = \{I \in q \mid I \text{ is LR-essential}\}$.

Lemma 6.21 *Let* $I \in \langle\gamma\rangle_k^n$, $k, n \geq 0$.

(1) $n = 0$, $\gamma = \varepsilon$, $I = [S \rightarrow \cdot\omega, \varepsilon]$.
(2) $\gamma \neq \varepsilon$ *and* I *is* ***essential****.*
(3) $n > 0$, I *is* ***inessential***
and $\exists J$, $J \,\partial_k\, I$, $J \in \langle\gamma\rangle_k^m$, $m < n$.

Lemma 6.22

$\langle\varepsilon\rangle_k^n \subseteq \partial_k^*(\{S \rightarrow \cdot\omega, \varepsilon] \mid S \rightarrow \omega \in P\})$

$\langle\gamma\rangle_k^n \subseteq \partial_k^*(Ess(\langle\gamma\rangle_k))$, *if* $\gamma \neq \varepsilon$.

Lemma 6.23 (F.6.16, L6.18*, and* ***L6.22)***

$\langle\varepsilon\rangle_k = \partial_k^*(\{[S \rightarrow \cdot\omega, \varepsilon] \mid S \rightarrow \omega \in P\})$

$\langle\gamma\rangle_k = \partial_k^*(Ess(\langle\gamma\rangle_k))$, *if* $\gamma \neq \varepsilon$.

χ_k^X: *relation on set of LR(k) items.*

$[A \to \alpha{\cdot}X\beta, y]\ \chi_k^X\ [A \to \alpha X{\cdot}\beta, y]$,

pass-X, *or* χ^X *for short*

$Basis_{LR}(q, X) = \{[A \to \alpha X{\cdot}\beta, y] \mid [A \to \alpha{\cdot}X\beta, y] \in q\}$

$\equiv \chi_k^X(q)$.

δ_k^X: *relation on set of LR(k) items.*

$Goto_{LR}(q, X) = \partial_k^*(Basis_{LR}(q, X)) = \partial_k^*(\chi_k^X(q))$

$= \chi_k^X{\cdot}\partial_k^* \equiv \delta_k^X(q)$.

X-**successor**, δ_k^X *for short*

Fact 6.24

If $[A \to \alpha{\cdot}\omega\beta, y] \in \langle\gamma\rangle_k^n$, $[A \to \alpha\omega.\beta, y] \in \langle\gamma\omega\rangle_k^n$.

If $[A \to \alpha\omega{\cdot}\beta, y] \in \langle\gamma\rangle_k^n$. $[A \to \alpha{\cdot}\omega\beta, y] \in \langle\delta\rangle_k^n, \gamma=\delta\omega$.

Lemma 6.25 $Ess(\langle\gamma X\rangle_k) = Basis(\langle\gamma\rangle_k, X)$

$Ess(\langle\gamma X\rangle_k) = \chi_k^X(\langle\gamma\rangle_k)$.

Lemma 6.26 $\langle\gamma X\rangle_k = Goto(\langle\gamma\rangle_k, X))$

$\langle\gamma X\rangle_k = \partial_k^*(\chi_k^X(\langle\gamma\rangle_k)) = \delta_k^X(\langle\gamma\rangle_k)$.

$\delta_k^{\varepsilon}(q) = \partial_k^{*}(q)$

$\delta_k^{\gamma\cdot X}(q) = \partial_k^{*}(\chi_k^{X}(\delta_k^{\gamma}(q))),\ \ \gamma \neq \varepsilon.$

$\therefore\ \ \delta_k^{\gamma}(q_s) = \delta_k^{X_1}(\delta_k^{X_2}(\ \ldots\ (\delta_k^{X_n}(\delta_k^{\varepsilon}(q_s)))\ \ldots\ ))$

$= \partial_k^{*}(\chi_k^{X_1}(\partial_k^{*}(\chi_k^{X_2}(\ \ldots\ (\partial_k^{*}(\chi_k^{X_n}(\partial_k^{*}(q_s)))\ \ldots\ )))).$

$[\varepsilon]_k = \delta_k^{\varepsilon}(q_s) = \partial_k^{*}(\{S' \to .S,\ \$^k]\})$

$[X]_k = \partial_k^{*}(\chi_k^{X}([\varepsilon]_k))$

$[\gamma X]_k = \partial_k^{*}(\chi_k^{X}([\gamma]_k))$

Algorithm *Compute* $M = (C_k, V, P, q_s, \emptyset)$

$q_s := \partial_k^{*}([S' \to .S,\ \$^k]);$

$C_k := \{q_s\};$

$P := \emptyset;$

repeat

for $q \in C_k$ ***and*** $X \in V$ ***do***

$p := \partial_k^{*}(\chi_k^{X}(q));$

$C_k := C_k \cup \{p\};$

$P := P \cup \{q\cdot X \to p\}$

od

until *nothing is added to* C_k.

Lemma 6.27 *Let $M = (Q_M, V, P_M, q_s, F)$ be a canonical LR(k) machine for $G = (V, \Sigma, P, S)$. Then*

(a) M is deterministic.

(b) $q \in Q_M$, $Goto(p, X) = q$, unique X.

(c) $q_s \mid \gamma \Rightarrow^* \Phi q \mid$, iff $q = \langle\gamma\rangle_k$.

(d) If $F = \{\langle\gamma\rangle_k\}$ for some γ, $L_M = [\gamma]_k$.

(e) If $F = \{\langle\gamma\rangle_k \mid \langle\gamma\rangle_k \neq \emptyset\}$,

L_M = Set of viable prefixes of X.

(f) If $F = \{\langle\gamma\rangle_k\}$ for all γ, $L_M = V^*$.

Proof

Assume $\langle\gamma_1\rangle_k \cdot X \rightarrow \langle\gamma_1 X\rangle_k$, *and*

$\langle\gamma_2\rangle_k \cdot X \rightarrow \langle\gamma_2 X\rangle_k$.

where $\langle\gamma_1\rangle_k = \langle\gamma_2\rangle_k$.

Then $\langle\gamma_1 X\rangle_k = Goto(\langle\gamma_1\rangle_k, X)$

$= Goto(\langle\gamma_1\rangle_k, X) = \langle\gamma_2 X\rangle_k$.

$\therefore \langle\gamma_1 X\rangle_k = \langle\gamma_2 X\rangle_k$. $\therefore$ *M is deterministic.(a)*

Assume $\langle\gamma_1\rangle_k \cdot X_1 \rightarrow \langle\gamma\rangle_k$, *and* $\langle\gamma_2\rangle_k \cdot X_1 \rightarrow \langle\gamma\rangle_k$. *Then*

$\langle\gamma\rangle_k = \langle\gamma_1 X_1\rangle_k = \partial_k^*(Basis(\langle\gamma_1\rangle_k, X_1))$,

$= \langle\gamma_2 X_2\rangle_k = \partial_k^*(Basis(\langle\gamma_2\rangle_k, X_2))$.

$\therefore Basis(\langle\gamma_1\rangle_k, X_1) = Basis(\langle\gamma_2\rangle_k, X_2) \neq \emptyset$.

$\therefore X_1 = X_2$.*(b)*

$[\varepsilon]_k \cdot \gamma_1 \gamma_2 \Rightarrow^* [\gamma_1]_k \gamma_2$ *in M.*
Since M is deterministic, $[\gamma]_k$ *is the only state.(c)*

Theorem 6.28
(a) The LR(k) equivalence of G is the equivalence induced by the canonical LR(k) machine of G.

$[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}$,

iff $q_s | \gamma_1 \Rightarrow^* q_s \ldots q |$ *and* $q_s | \gamma_2 \Rightarrow^* q_s \ldots q |$.

(b) The LR(k) equivalence of G is ***right invariance****.*

If $[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}$, $[\gamma_1 \cdot X]_{\rho_k} = [\gamma_2 \cdot X]_{\rho_k}$.

(c) The LR(k) equivalence of G is ***ends with same symbols****.*

If $[\gamma_1]_{\rho_k} = [\gamma_2]_{\rho_k}$, $\gamma_1{:}1 = \gamma_2{:}1$.

$[\gamma]_k = \delta_M^{\gamma}(q_s) = \delta_M^{\gamma}([\varepsilon]_k) \mathrel{?=} (\partial_k^{*}\chi_k^{\gamma})^{*}(\partial_k^{*}([\varepsilon]_k)).$

$[\varepsilon]_k = \partial_k^{*}([S' \to .S, \$^k]).$

$[\gamma\cdot X]_k = \partial_k^{*}(\chi_k^{X}([\gamma]_k)).$

$\equiv \delta_k^{X}([\gamma]_k).$

$\therefore [\gamma]_k = \delta_k^{\gamma}([\varepsilon]_k).$

$\delta_k^{\varepsilon}(q) = \{q\}.$

$\delta_k^{\gamma\cdot X}(q) = \partial_k^{*}(\chi_k^{X}(\delta_k^{\gamma}(q))).$

$[\varepsilon]_k = \partial_k^{*}([S' \to .S, \$^k]).$

$[\gamma\cdot X]_k = \delta_k^{X}(\delta_k^{\gamma}(q)).$

6.3 Canonical LR(k) Parser

Let $G = (N, \Sigma, P, S)$. *The* ***canonical LR(k) parser*** *for* G *is a pushdown transducer* $M = ([G]_k, \Sigma, \Gamma, P, \tau, [\varepsilon]_k, \{[\varepsilon]_k[S]_k\}, \$, |)$ *where*

$$[G]_k = \{[\delta]_k \mid \delta \in V^*\}$$

$$\Gamma = \{[\delta]_k[\delta X_1]_k\ldots[\delta X_1\ldots X_n]_k | y \to [\delta]_k[\delta A]_k | y \mid [A \to X_1\ldots X_n\bullet, y] \in \langle\delta X_1\ldots X_n\rangle_k\} \quad (ra)$$

$$\cup \{[\delta]_k | ay \to [\delta]_k[\delta a]_k | y \mid a \in \Sigma, [A \to \alpha\bullet a\beta, z] \in \langle\delta\rangle_k, y \in First_{max\{k-1, 0\}}(\beta z)\} \quad (sa)$$

$$\tau([\delta]_k[\delta X_1]_k\ldots[\delta X_1\ldots X_n]_k | y \to [\delta]_k[\delta A]_k | y) = A \to X_1\ldots X_n,$$
$$\tau([\delta]_k | ay \to [\delta]_k[\delta a]_k | y) = \varepsilon.$$

$[B \to \alpha\bullet A\beta, x] \in \langle\delta\rangle_k$ $\qquad$ $[B\to\alpha A\bullet\beta, x] \in \langle\delta A\rangle_k$

$[A \to \bullet X_1\ldots X_n, y] \in \langle\delta\rangle_k$

$[A \to X_1\bullet X_2\ldots X_n, y] \in \langle\delta X_1\rangle_k$

…

$[A \to X_1\ldots X_n\bullet, y] \in \langle\delta X_1\ldots X_n\rangle_k$

Canonical LR(k) Parser
right parser

LR(k) parser $\Leftrightarrow$ *right parser*

T6.34 *(**T5.21**: s/r par., **T5.65**: simple-prec. par)*

LR(k) parser $\Rightarrow$ *right parser*

L6.29, **L6.30** *(**L5.17**, **5.18** and **L5.60**, **L5.61**)*

homomorphism h:

action in LR(k) parser

$\rightarrow$ *action in shift-reduce parser*

$$h([\delta]_k[\delta X_1]_k\ldots[\delta X_1\ldots X_n]_k \mid y \rightarrow [\delta]_k[\delta A]_k \mid y)$$
$$= X_1 \ldots X_n \mid \rightarrow A \mid,$$
$$h([\delta]_k \mid a \rightarrow [\delta]_k[\delta a]_k \mid) = \mid a \rightarrow a \mid.$$

Furthermore, h:

configuration in LR(k) parser

$\rightarrow$ *configuration in shift-reduce parser*

$$h([\varepsilon]_k[X_1]_k \ldots [X_1\ldots X_n]_k \mid w\$^k) = \$X_1 \ldots X_n \mid w\$^k.$$

$$\therefore \quad [\varepsilon]_k[X_1]_k \ldots [X_1\ldots X_n]_k \mid x\$^k$$
$$\Rightarrow^{\theta} [\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_m]_k \mid y\$^k \text{ implies}$$
$$\$X_1 \ldots X_n \mid x\$^k \Rightarrow^{h(\theta)} \$Y_1 \ldots Y_m \mid y\$^k.$$

LR(k) parser $\Leftarrow$ *right parser*

L6.31, **L6.32**, **L6.33**

*(**L5.19**, **5.20** and **L5.63**, **L5.64**)*

Lemma 6.29 *Let M be a LR(k) parser for G. If*
$[\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_n]_k \,|\, xy\$^k$
$\Rightarrow^{\theta} [\varepsilon]_k[X_1]_k \ldots [X_1\ldots X_m]_k \,|\, y\$,\ \theta \in \Gamma^*$ *in M,*
then

$$X_1\ldots X_m \Rightarrow_{rm}^{\tau(\theta)^R} Y_1\ldots Y_n x \text{ in } G,$$
$$\text{and } |\theta| = |\tau(\theta)| + |x|.$$

Proof *Induction on the length of action string* θ.
i) $\theta = \varepsilon$. $x = \varepsilon$, $Y_1\ldots Y_n = X_1\ldots X_m$, *and* $\tau(\varepsilon) = \varepsilon$.
ii) $\theta = r\theta'$.
ii.1) r is reduce by $A \to Y_p\ldots Y_n$, $1 \le p \le n$.
$[\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_n]_k \,|\, xy\$^k$
$\Rightarrow^{r} [\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_{p-1}]_k[Y_1\ldots Y_{p-1}A]_k \,|\, xy\$^k$
$\Rightarrow^{\theta'} [\varepsilon]_k[X_1]_k \ldots [X_1\ldots X_m]_k \,|\, y\$^k.$
$X_1\ldots X_m \Rightarrow_{rm}^{\tau(\theta')R} Y_1 \ldots Y_{p-1}Ax \Rightarrow_{rm}^{A\to\omega} Y_1 \ldots Y_n x$,
and

$$|\theta'| = |\tau(\theta')| + |x|.$$

$\therefore X_1\ldots X_m \Rightarrow_{rm}^{(\tau(\theta')\cdot A\to\omega)R} Y_1 \ldots Y_n x$, *and*
$|\theta| = 1+|\theta'| = 1+|\tau(\theta')| + |x| = |\tau(\theta)| + |x|.$
ii.2) $r = [\delta]_k \,|\, a \to [\delta]_k[\delta a]_k \,| \in \Gamma$,
$[\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_n]_k \,|\, xy\$$
$= [\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_n]_k \,|\, ax'y\$$
$\Rightarrow^{r} [\varepsilon]_k[Y_1]_k \ldots [Y_1\ldots Y_n]_k[Y_1\ldots Y_n a]_k \,|\, x'y\$$

$\Rightarrow^{\theta'}$ $[\varepsilon]_k[X_1]_k \ldots [X_1 \ldots X_m]_k \mid y\$$

and $|\theta'| = |\tau(\theta')| + |x'|$.

$X_1 \ldots X_m a x' y \Rightarrow_{rm}^{\tau(\theta')R} Y_1 \ldots Y_n a x' y$

$\therefore\ \delta \Rightarrow_{rm}^{\tau(\theta')\cdot\varepsilon} \gamma x$ *in G, and*

$|\theta| = 1 + |\theta'| = |\tau(\theta')| + 1 + |x'| = |\tau(\theta)| + |x|$

Lemma 6.30 *Let M be a* ***canonical LR(k) parser*** *for G. Then*

(1) $L(M) \subseteq L(G)$,

(2) $\forall\theta$: *actions in M,* $\tau(\theta)$ *is a* ***right parse*** *of w,*

(3) $Time_G(w) \le Time_M(w) - |w|$.

Lemma 6.31 *Let M be a LR(k) parser.*

If $[A \to \alpha\cdot\beta, z] \in \langle\gamma a_1 \ldots a_n\rangle_k$ *and* $k{:}y\$^k \in First_k(\beta z)$,

then $[\varepsilon]_k \ldots [\gamma]_k \mid a_1 \ldots a_n y\$^k$

$\Rightarrow^\theta [\varepsilon]_k \ldots [\gamma]_k [\gamma a_1]_k \ldots [\gamma a_1 \ldots a_n]_k \mid y\$^k$

where θ *is a sequence of shift actions.*

valid k-item $\Rightarrow$ *valid stack string*

Proof

$S' \Rightarrow^* \delta A z' \Rightarrow \delta\alpha\beta z' = \gamma a_1 \ldots a_n \beta z'$, *and*

$k{:}z' = z.\ \forall i,\ 1 \le i \le n,$

(i) if $\alpha = \alpha' a_i \ldots a_n$,

$[A \to \alpha' a_i \ldots a_n \cdot \beta, z] \in [\gamma a_1 \ldots a_{i-1}]_k$

(ii) if $\delta = \gamma a_1 \ldots a_{j-1} A z'$. *By lemma 6.2,*

$S' \Rightarrow^+ \delta' A' y' \Rightarrow \delta' \alpha'' a_i \beta' y' = \gamma a_1 \ldots a_i \beta' y'$, *and*

$\beta' y' \Rightarrow^* a_{i+1} \ldots a_{j-1} A z'$.

$\therefore\ [A' \to \alpha'' \cdot a_i \beta, k{:}y'] \in \langle\gamma a_1 \ldots a_{i-1}\rangle_k$,

and $\beta' y' \Rightarrow^* a_{i+1} \ldots a_{j-1} A z'$

$\Rightarrow^* a_{i+1} \ldots a_n \beta z'$.

Lemma 6.32 *Let M be a LR(k) parser.*

If $X_1 \ldots X_m \Rightarrow_{rm}^{\pi^R} Y_1 \ldots Y_n x$ *in G,*

$[A \to \alpha \bullet \beta, z] \in \langle X_1 \ldots X_m \rangle_k$,

$k{:}y\$^k \in First_k(\beta z)$, *and*

either $Y_1 \ldots Y_n = \varepsilon$ *or* $Y_n \in N$.

Then $\tau(\theta) = \pi$, $|\theta| = |\pi| + |x|$, *and*

$$[\varepsilon]_k[Y_1]_k \ldots [Y_1 \ldots Y_n]_k \mid xy\$^k$$
$$\Rightarrow^{\theta} [\varepsilon]_k[X_1]_k \ldots [X_1 \ldots X_m]_k \mid y\$^k, \theta \in \Gamma^* \text{ in } M.$$

Proof *Induction on* $|\pi|$.

i) $\pi = \varepsilon$. $X_1 \ldots X_m = Y_1 \ldots Y_n x$.

$$[\varepsilon]_k[Y_1]_k \ldots [Y_1 \ldots Y_n]_k \mid xy\$^k$$
$$\Rightarrow^{\theta} [\varepsilon]_k[Y_1]_k \ldots [Y_1 \ldots Y_n]_k [Y_1 \ldots Y_n 1{:}x]_k \ldots [Y_1 \ldots Y_n x]_k \mid y\$,$$

θ *is a* $|x|$*-length shift action string(**L6.31**)*

$|\theta| = |x|$, $\tau(\theta) = \pi = \varepsilon$.

ii) $\pi = B \to \omega \cdot \pi_1$.

$X_1 \ldots X_m \Rightarrow_{rm}^{\pi_1^R} Z_1 \ldots Z_p B x_1$

$\Rightarrow_{rm}^{r} Z_1 \ldots Z_p \omega x_1 = Y_1 \ldots Y_n x$ *in G.*

where $x = vx_1$.

$\exists\theta_1$.э. $\tau(\theta_1) = \varepsilon_1$, $|\theta_1| = |\pi_1| + |x_1|$, *and*

$[\varepsilon][Z_1] \ldots [Z_1 \ldots Z_p][Z_1 \ldots Z_p B] \mid x_1 y\$$

$\Rightarrow^{\theta_1} [\varepsilon][X_1] \ldots [X_1 \ldots X_m] \mid y\$$.

And because $[A \rightarrow \alpha \bullet \beta, z] \in \langle X_1 \ldots X_m \rangle_k$

$\exists \delta, z'$.э.

$S' \underset{rm}{\Rightarrow}^* \delta A z' \underset{rm}{\Rightarrow} \delta\alpha\beta z' = X_1 \ldots X_m \beta z'$

$\underset{rm}{\Rightarrow} X_1 \ldots X_m u z'$ *in* G',

and $k{:}z' = z$, $k{:}uz = k{:}y\$$.

$\therefore S' \underset{rm}{\Rightarrow}^* Z_1 \ldots Z_p B_1 u z' \underset{rm}{\Rightarrow} Z_1 \ldots Z_p \omega x_1 u z'$ *in* G'.

Here $k{:}x_1uz' = k{:}x_1uz = k{:}x_1y\$$, *so*

$[B \rightarrow \omega\cdot, k{:}x_1y\$] \in \langle Z_1 \ldots Z_p \omega \rangle_k$.

Then $[\varepsilon][Y_1]\ldots[Y_1\ldots Y_n] \mid xy\$$

$= [\varepsilon][Y_1]\ldots[Y_1\ldots Y_n] \mid vx_1y\$$

$\overset{\theta_2}{\underset{rm}{\Rightarrow}} [\varepsilon][Y_1]\ldots[Y_1\ldots Y_n v] \mid x_1y\$$

$= [\varepsilon][Z_1]\ldots[Z_1\ldots Z_p]\ldots[Z_1\ldots Z_p\omega] \mid x_1y\$$ *in M*,

for some $|v|$*-length shift action string* θ_2.

Then $\exists$ *an action* r' .э.

$r' = [Z_1\ldots Z_p]\ldots[Z_1\ldots Z_p\omega] \mid y' \rightarrow$

$[Z_1\ldots Z_p]\ldots[Z_1\ldots Z_p B] \mid y'$,

where $y' = k{:}x_1y\$$.

$\therefore [\varepsilon][Y_1]\ldots[Y_1\ldots Y_n] \mid xy\$$

$\overset{\theta_1}{\underset{rm}{\Rightarrow}} [\varepsilon][X_1]\ldots[X_1\ldots X_m] \mid y\$$ *in M*,

where $\theta = \theta_2 r' \theta_1$.

And $\tau(\theta) = \tau(\theta_2)\tau(r')\tau(\theta_1) = r\pi_1 = \pi$,

$|\theta| = |\pi| + |x|$.

Lemma 6.33 *Let M be a* ***canonical LR(k) parser*** *for G. Then*

(1) $L(G) \subseteq L(M)$,

(2) $^{\forall}\pi$: ***right parse*** *of w in G,* $\tau(\theta) = \pi$ *in M,*

(3) $Time_G(w) \le Time_M(w) + |w|$.

Proof

$Y_1 \ldots Y_m = \varepsilon$, $X_1 \ldots X_m = S$,

$[A \to \alpha \bullet \beta, z] = [S' \to \bullet S, \$^1]$, $y = \varepsilon$.

Theorem 6.34 *Let M be a* ***canonical LR(k) parser*** *for G. Then*

(1) M is a ***right parser*** *for G.*

(2) $^{\forall}w \in L(G)$, *M produces* ***all right parses*** *of w.*

(3) $Time_M(w) = Time_G(w) + |w|$.

6.4 LR(k) Grammar

*G is **LR(k) grammar** if its canonical LR(k) parser is **deterministic** and $S \not\Rightarrow^{+} S$ in G.*

Theorem 6.35 *Any LR(k) grammar is **unambiguous**.*

*"reduce-reduce **conflicts**"*

$[A_1 \to \omega_1\cdot, y_1], [A_2 \to \omega_2\cdot, y_2],$

if $y_1 = y_2$, $A_1 \to \omega_1 \neq A_2 \to \omega_2$.

*"shift-reduce **conflicts**"*

$[A \to \alpha\cdot a\beta, z], [B \to \omega\cdot, y]$

if $y \in First_k(a\beta z)$.

Lemma 6.36 *Let M be the CLR(k) parser for G. Then M is nondeterministic iff*

$\exists$ *a state which contains a pair of items exhibiting a reduce-reduce or shift-reduce conflict.*

Proof

($\Leftarrow$) $I_1, I_2 \in \langle\gamma\rangle_k$, and either

(i) $I_1 = [A \to X_1 \ldots X_n\cdot, y]$, $I_2 = [B \to Y_1 \ldots Y_m\cdot, y]$, or

(ii) $I_1 = [A \to X_1 \ldots X_n\cdot a\beta, z]$, $I_2 = [B \to Y_1 \ldots Y_m\cdot, y]$,

$y \in First_k(a\beta z)$.

Assume $m \le n$, *then for* $[\delta][\delta X_1]\ldots[\delta X_1\ldots X_n]|y$,

(i) $[\delta][\delta X_1]\ldots[\delta X_1\ldots X_n]|y \to [\delta][\delta A]|y$ *and*

$[\delta X_1\ldots X_{i-1}][\delta X_1\ldots X_i]\ldots[\delta X_1\ldots X_n]|y$

$\to [\delta X_1\ldots X_{i-1}][\delta X_1\ldots X_{i-1}B]|y$ *in M,*

$Y_1\ldots Y_m = X_i\ldots X_n (i=n-m+1;$ **L6.24**)

(ii) $[\delta X_1\ldots X_n]|ay' \to [\delta X_1\ldots X_n][\delta X_1\ldots X_n a]|y'$

$[\delta X_1\ldots X_{i-1}][\delta X_1\ldots X_i]\ldots[\delta X_1\ldots X_n]|y$

$\to [\delta X_1\ldots X_{i-1}][\delta X_1\ldots X_{i-1}B]|y$, $k{:}ay' = y$ *in M.*

$\therefore$ *parser is nondeterministic.*

($\Rightarrow$) *let* r_1, r_2 *are conflicting actions in* $[\gamma]$ *of M, then*

(i) (r_1) $[\gamma]|ay \to [\gamma][\gamma a]|y$,

(r_2) $[\gamma]|ayv \to [\gamma][\gamma a]|yv$,

where $[A_1 \to \alpha_1 \cdot a\beta_1, z_1]$, $[A_2 \to \alpha_2 \cdot a\beta_2, z_2] \in \langle\gamma\rangle_k$,

$y \in First_{max\{k-1,\, 0\}}(\beta_1 z_1)$, $yv \in First_{max\{k-1,\, 0\}}(\beta_2 z_2)$.

Here $\beta_1, \beta_2 \not\Rightarrow^* x\$$, *and if* $y \ne yv$, *then* $y{:}1 = \$$,

$\therefore v = \varepsilon$. $\therefore$ *no shift-shift conflict.*

(ii) (r_1) $[\delta]\ldots[\delta X_1\ldots X_n]|y \to [\delta][\delta A]|y$,

(r_2) $[\gamma]\ldots[\gamma Y_1\ldots Y_m]|y \to [\gamma][\gamma B]|y$,

then $[\delta X_1\ldots X_n] = [\gamma Y_1\ldots Y_m]$, *and*

$[A \to X_1\ldots X_n \cdot, y] \in \langle\delta X_1\ldots X_n\rangle_k$, *and*

$[B \to Y_1\ldots Y_m \cdot, y] \in \langle\gamma Y_1\ldots Y_m\rangle_k$,

$\therefore$ *reduce-reduce conflict.*

Lemma 6.37 *$\langle\gamma\rangle_k$ contains a pair of items exibiting a reduce-reduce conflict iff*

(a) $S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta_1 \omega_1} y_1 = \underline{\gamma} y_1$,

(b) $S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta_2 \omega_2} y_2 = \underline{\gamma} y_2$,

(c) $k{:}y_1 = k{:}y_2$, and

(d) $A_1 \rightarrow \omega_1 \neq A_2 \rightarrow \omega_2$

hold in G'.

Proof $[A \rightarrow \omega_1., k{:}y_1], [A \rightarrow \omega_2., k{:}y_2] \in \langle\gamma\rangle_k$.

Lemma 6.38 *$\langle\gamma\rangle_k$ contains a pair of items exibiting a shift-reduce conflict iff*

(a) $S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta_1 \omega_1} y_1 = \underline{\gamma} y_1$,

(b) $S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta_2 \omega_2} y_2 = \underline{\gamma v} y_2$,

(c) $k{:}y_1 = k{:}vy_2,\ v \neq \varepsilon$

hold in G'.

Proof

($\Leftarrow$) $[A \rightarrow \alpha \cdot a\beta, z]$ *and* $[B \rightarrow \omega\cdot, y]$ *are in* $\langle\gamma\rangle_k$.

($\Rightarrow$) $\omega_2 = \alpha v$, $[A_2 \rightarrow \alpha \cdot v, k{:}y_2]$, or

$v = az\omega_2$, $S' \Rightarrow^* \delta_2 A_2 y_2 = \gamma az A_2 y_2$.

By lemma 6.2, $\exists A' \rightarrow \alpha'' \cdot a\beta' \in P$,

$[A' \rightarrow \alpha'' \cdot a\beta', k{:}y'] \in \langle\gamma\rangle_k$.

$a\beta' y' \Rightarrow^* az A_2 y_2 \Rightarrow az\omega_2 y_2 = vy_2$.

By (c) , there exists a shift reduce conflict.

Lemma 6.39 *The following statements are logically equivalent for all G and k ≥ 0.*

(a) The canonical LR(k) parser of G is ***deterministic****.*

(b) In the canonical LR(k) machine of the $-augmented grammar G' no states contains a pair of items exhibiting a ***reduce-reduce*** *or* ***shift-reduce*** *conflicts.*

(c) The conditions

$$S' \Rightarrow^* \delta_1 A_1 y_1 \Rightarrow \underline{\delta_1 \omega_1} y_1 = \underline{\gamma} y_1,$$

$$S' \Rightarrow^* \delta_2 A_2 y_2 \Rightarrow \underline{\delta_2 \omega_2} y_2 = \underline{\gamma} v y_2,$$

$$\text{and } k{:}y_1 = k{:}vy_2,\ v \neq \varepsilon$$

always implies that

$$\delta_1 = \delta_2,\ A_1 = A_2, \text{ and } \omega_1 = \omega_2.$$

Theorem 6.40 *For all k ≥ 0, the class of LR(k) grammars is* ***properly contained*** *in the class of LR(k+1) grammars.*

Proposition 6.41 *Any pushdown automaton M with input alphabet Σ can be transformed into an equivalent grammar G with terminal alphabet Σ such that M is* ***deterministic*** *if and only if G is* ***LR(k)*** *for some k ≥ 0.*

LR(k) languages = LR(1) languages
deterministic languages = LR(1) languages

Lemma6.42 *Let G be LR(k) grammar and M be a LR(k) parser for G.*
Further let $x, y \in \Sigma^*$ *and*

$$\psi \in [G']^* \text{ .э. } [\varepsilon] \mid xy\$ \Rightarrow^* \psi \mid y\$.$$

If $^{\forall}y'$ *.э. the condition* $xy' \in L(G)$, $k{:}y \neq k{:}y'$,
then $\psi|y\$$ *is an error configuration.*

Proof *by contradiction*

$$\psi = [\varepsilon][X_1]...[X_1...X_n],\ X_1...X_n \Rightarrow^* x.$$

If $\psi|y\$$ *were not an error configuration, then*
$[A \rightarrow \alpha\cdot\beta, z] \in \langle X_1...X_n\rangle_k$,
$k{:}y\$ \in First_k(\beta z)$.

$$S' \Rightarrow^* \delta A z'\$ \Rightarrow \delta\alpha\beta z'\$ = X_1...X_n\beta z'\$,$$

$k{:}z'\$ = z$, $\beta \Rightarrow^* v$, *then* $k{:}y\$ = k{:}vz = k{:}vz' = k{:}y'$, *and* $xvz' \in L(G)$, *a contradiction.*

Lemma6.43 *Let G be a LR(k) grammar and M be a LR(k) parser for G, $k \geq 1$.*
Then M detects an error in any input string in $\Sigma^* \backslash L(G)$.

Proof

(i) $k{:}w \neq k{:}w'$, for all $w' \in L(G)$, by lemma 6.42.

(ii) $k{:}w = k{:}w'$ for some $w' \in L(G)$.

Then $\exists\ x, y, y'$.э.

(a) $w = xy$,

(b) $k{:}y = k{:}y'$ and $xy' \in L(G)$.

(c) $\forall y'' \in \Sigma^*$, $xy'' \in L(G)$ implies $k+1{:}y \neq k+1{:}y''$.

Let $y = ay_1$, $y' = ay_1'$, and there exist ψ, ψ' .э.

$[\varepsilon]|xay_1'\$ \Rightarrow \psi|ay_1'\$ \Rightarrow \psi'|y_1'\$$ in M.

Then

$[A \rightarrow \alpha \bullet a\beta, z] \in \langle X_1 \ldots X_n \rangle_k$

if $\psi = [\varepsilon]\ldots[X_1 \ldots X_n]$, where $k{:}ay_1' \in First(a\beta z)$.

By lemma 6.32,

$[\varepsilon]|xay_1\$ \Rightarrow^* [\varepsilon][X_1]\ldots[X_1 \ldots X_n]|ay_1\$$

$= \psi|ay_1\$ \Rightarrow^* \psi'|y_1\$$.

By (c), $xay_1'' \in L(G)$ always implies $k{:}y_1 \neq k{:}y_1''$.

$\therefore$ By lemma 6.42, $\psi|y_1\$$ is an error configuration.

The parser loops forever when
(1) $S \rightarrow S|a$: LR(0) parser is deterministic,
(2) $S \rightarrow a^{k+1}|ASb^k$, $A \rightarrow \varepsilon$: not LR(k).

Theorem6.44 *Let $k \geq 0$. Then M deos not loop forever on any input string.*
Proof *Assume that M loops forever for $w = xy \in \Sigma^*$.*
Then $\exists$ ψ_i, r_i .э.

$$[\varepsilon]|w\$ \Rightarrow^* \psi_1|y\$,$$

$$\psi_i|y\$ \Rightarrow^{r_i} \psi_{i+1}|y\$ \text{ in } M \ \forall\ i \geq 1.$$

Let $\psi_i = [\varepsilon]...[\gamma_i]$, $\gamma_1 \Rightarrow^* x$, $\gamma_{i+1} \Rightarrow_{rm} \gamma_i$.
Let r_i be reduce action by $A_i \rightarrow \omega_i$ and

$$[A_i \rightarrow \omega_i., k{:}z_i\$] \in \langle\gamma_1\rangle\ \forall i.$$

$$[\varepsilon]|xz_i\$ \Rightarrow^* \psi_1|z_i\$ \Rightarrow \psi_2|z_i\$,$$

and more generally,

$$\psi_i|z_i\$ \Rightarrow^{r_n} \psi_{n+1}|z_i\$.$$

M loops forever on all xz_i, $i \geq 1$.
But $z_i = \varepsilon$ because $xz_i \in L(M)$ $\forall$ i and
M is deterministic. Then

$$S \Rightarrow^* \delta_{n+1}A_{n+1} \Rightarrow \delta_{n+1}w_{n+1} = \gamma_{n+1} \Rightarrow^n \gamma_1 \Rightarrow^* x.$$

G is ambiguous, a contradiction.

6.5 LALR(k) parsing

Theorem 6.45 *The size of the canonical LR(k) parser for grammar G is* $O(2^{|\Sigma|^k|G| + k\log|\Sigma| + \log|G|})$.

Proof

$2^{(|\Sigma|+1)^k|G'|}$ *: # of distinct LR(k)-equivalent classes.*

$2^{(|\Sigma|+1)^k|G'|} \cdot |G| \cdot (|\Sigma|+1)^k$

: sum of the lengths of all reduce actions

Whether does the grammar exist with this upper bound?

k=0

Proposition 6.46 *For each* $n \geq 0$, *let* $G_n = (\{A_0, A_1, \ldots, A_n\}, \{0, 1, a, a_0, a_1, \ldots, a_n\}, P, A_0)$ *where P is*

$A_i \rightarrow 1A_{i+1}a_i$, $0 \leq i \leq n-1$

$A_n \rightarrow 1A_0a_n$,

$A_i \rightarrow 0A_ia_i$, $1 \leq i \leq n$

$A_i \rightarrow 0A_0a_i$, $1 \leq i \leq n$

$A_0 \rightarrow a$.

Then the size of the canonical LR(0) collection for G_n *is at least* $2^{c|Gn|}$ *for all* $n \geq 0$, $c > 0$.

Let $G = (N, \Sigma, P, S)$ *be a cfg. Then a rule automaton,*

$M = (Q, N \cup \Sigma, R, \langle S', \varepsilon\rangle, Q)$

$Q = \{\langle A, \alpha\rangle \mid A \to \alpha\beta \in P\} \cup \{\langle S', \varepsilon\rangle, \langle S', S\rangle\}$

$R = \{\langle S', \varepsilon\rangle S \to \langle S', S\rangle\}$
$\quad \cup \{\langle S', \varepsilon\rangle \to \langle B, \varepsilon\rangle \mid S \to B\beta \in P\}$
$\quad \cup \{\langle A, \alpha\rangle X \to \langle A, \alpha X\rangle \mid A \to \alpha X\beta \in P\}$
$\quad \cup \{\langle A, \alpha\rangle \to \langle B, \varepsilon\rangle \mid A \to \alpha B\beta \in P\}$

M is a dfa but ε*-moves.*

$M' = (K, N \cup \Sigma, R, \langle S', \varepsilon\rangle, K)$

$K = \{\langle A, \alpha\rangle \mid A \to \alpha\beta \in P, \alpha \neq \varepsilon\} \cup \{\langle S', \varepsilon\rangle, \langle S', S\rangle\}$

$R' = \{\langle S', \varepsilon\rangle S \to \langle S', S\rangle\}$
$\quad \cup \{\langle S', \varepsilon\rangle \to \langle B, X\rangle \mid S \to B\beta \in P, B \to X\gamma \in P\}$
$\quad \cup \{\langle A, \alpha\rangle X \to \langle A, \alpha X\rangle \mid A \to \alpha X\beta \in P, \alpha \neq \varepsilon\}$
$\quad \cup \{\langle A, \alpha\rangle X \to \langle B, X\rangle \mid A \to \alpha B\beta \in P, \alpha \neq \varepsilon,$
$\qquad B \to X\gamma \in P\}$

$|K| = |Q| - |N|$

$R' = \{\langle S', \varepsilon\rangle S \to \langle S', S\rangle\}$
$\quad \cup \{\langle S', \varepsilon\rangle \to \langle B, X\rangle \mid S \to B\beta \in P, B \to X\gamma \in P\}$
$\quad \cup \{\langle A, \alpha\rangle a \to \langle A, \alpha a\rangle \mid A \to \alpha a\beta \in P, \alpha \neq \varepsilon\}$
$\quad \cup \{\langle A, \alpha\rangle B \to \langle A, \alpha B\rangle, \langle A, \alpha\rangle X \to \langle B, X\rangle$
$\qquad \mid A \to \alpha B\beta \in P, \alpha \neq \varepsilon, B \to X\gamma \in P\}$

Let $G = (N, \Sigma, P, S)$ *be a cfg. Then LR(k) automaton,*

$$M_k = (Q, N \cup \Sigma, R, \langle S', \varepsilon, \$^k \rangle, Q)$$

$$Q = \{\langle A, \alpha, x\rangle \mid A \to \alpha\beta \in P, x \in Follow_k(A)\}$$
$$\cup \{\langle S', \varepsilon, \$^k\rangle, \langle S', S, \$^k\rangle\}$$

$$R = \{\langle S', \varepsilon, \$^k\rangle S \to \langle S', S, \$^k\rangle\}$$
$$\cup \{\langle S', \varepsilon, \$^k\rangle \to \langle B, \varepsilon, y\rangle \mid S \to B\beta \in P, y \in First_k(\beta\$^k)\}$$
$$\cup \{\langle A, \alpha, x\rangle X \to \langle A, \alpha X, x\rangle \mid A \to \alpha X\beta \in P\}$$
$$\cup \{\langle A, \alpha, x\rangle \to \langle B, \varepsilon, y\rangle \mid A \to \alpha B\beta \in P, y \in First_k(\beta x)\}$$

M is a dfa but ε*-moves.*

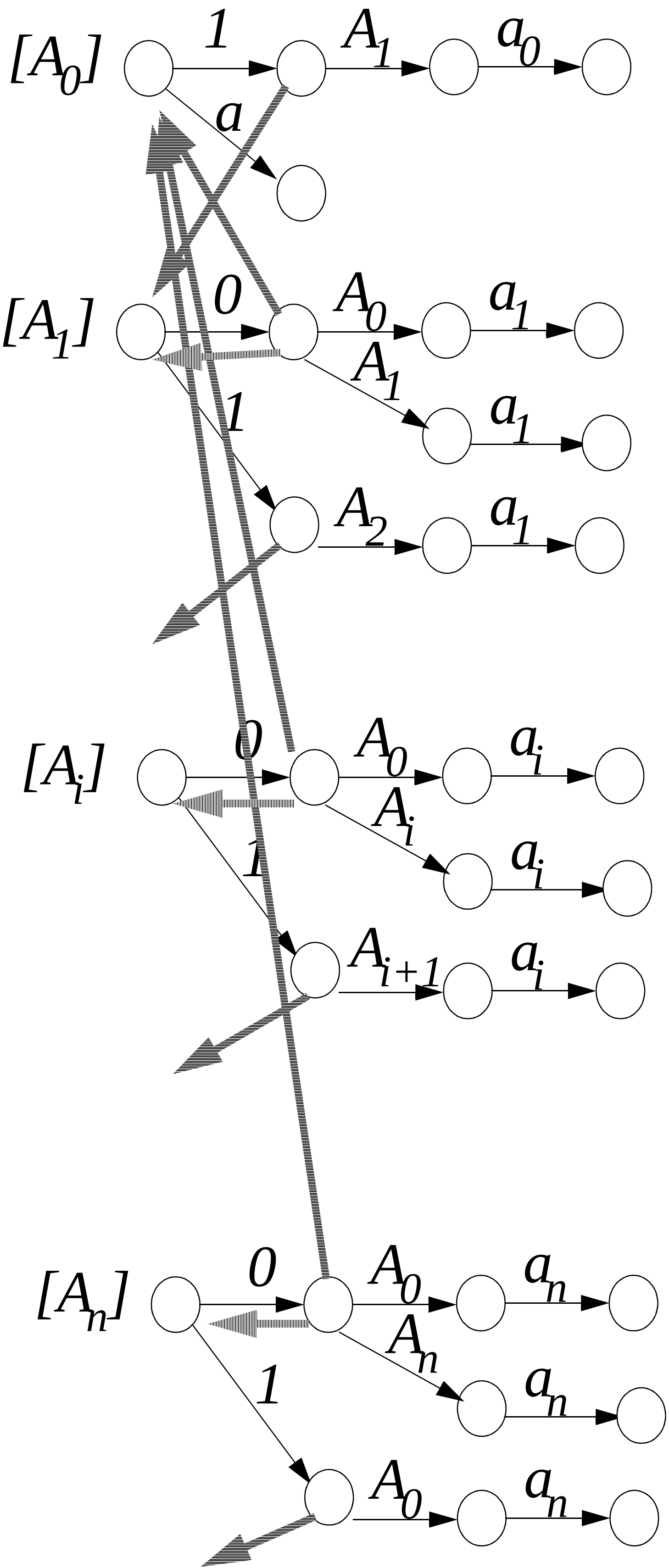
$$M' = (K, N \cup \Sigma, R, \langle S', \varepsilon, \$^k\rangle, K)$$

$$K = \{\langle A, \alpha, x\rangle \mid A \to \alpha\beta \in P, \alpha \neq \varepsilon, x \in Follow_k(A)\}$$
$$\cup \{\langle S', \varepsilon, \$^k\rangle, \langle S', S, \$^k\rangle\}$$

$$R' = \{\langle S', \varepsilon, \$^k\rangle S \to \langle S', S, \$^k\rangle\}$$
$$\cup \{\langle S', \varepsilon, \$^k\rangle \to \langle B, X, y\rangle \mid S \to B\beta \in P, B \to X\gamma \in P, y \in First_k(\beta\$^k)\}$$
$$\cup \{\langle A, \alpha, x\rangle X \to \langle A, \alpha X, x\rangle \mid A \to \alpha X\beta \in P, \alpha \neq \varepsilon\}$$
$$\cup \{\langle A, \alpha, x\rangle X \to \langle B, X, y\rangle \mid A \to \alpha B\beta \in P, \alpha \neq \varepsilon, B \to X\gamma \in P, y \in First_k(\beta x)\}$$

$$|K| = |Q| - |N|$$

$$
\begin{aligned}
R' = \{&\langle S', \varepsilon\rangle S \to \langle S', S\rangle\} \\
&\cup \{\langle S', \varepsilon\rangle \to \langle B, X\rangle \mid S \to B\beta \in P,\ B \to X\gamma \in P\} \\
&\cup \{\langle A, \alpha\rangle a \to \langle A, \alpha a\rangle \mid A \to \alpha a\beta \in P,\ \alpha \neq \varepsilon\} \\
&\cup \{\langle A, \alpha\rangle B \to \langle A, \alpha B\rangle,\ \langle A, \alpha\rangle X \to \langle B, X\rangle \\
&\qquad \mid A \to \alpha B\beta \in P,\ \alpha \neq \varepsilon,\ B \to X\gamma \in P\}
\end{aligned}
$$


[A0]
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a1
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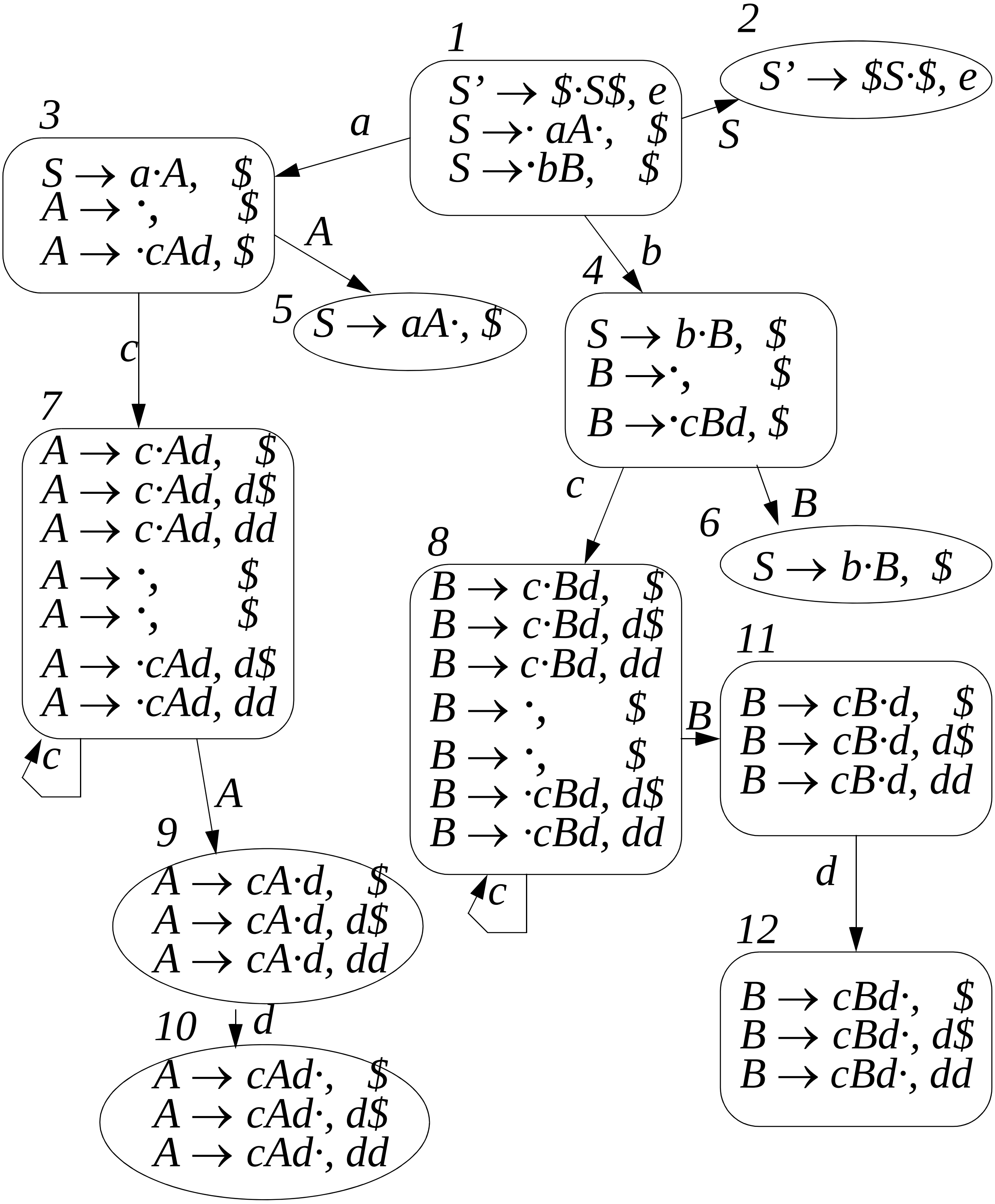
0
A0
ai
Ai
ai
1
Ai+1
ai
[An]
0
A0
an
An
an
1
A0
an

$G_{ab\varepsilon}$: $S \rightarrow aA|bB$, $A \rightarrow \varepsilon|cAd$, $B \rightarrow \varepsilon|cBd$



the partition of the LR(0)-equivalent classes into LR(k) equivalent classes are:

$[]_0 = []_k$

$[S]_0 = [S]_k$

$[a]_0 = [a]_k$

$[a]_0 = [a]_k$

$[aA]_0 = [aA]_k$

$[ac^+]_0 = [ac]_k \cup \ldots \cup [ac^k]_k \cup [ac^k c^+]_k$

$[ac^+A]_0 = [acA]_k \cup \ldots \cup [ac^k A]_k \cup [ac^k c^+ A]_k$

$[ac^+Ad]_0 = [acAd]_k \cup \ldots \cup [ac^k Ad]_k \cup [ac^k c^+ Ad]_k$

$[b]_0 = [b]_k$

$[bB]_0 = [bB]_k$

$[bc^+]_0 = [bc]_k \cup \ldots \cup [bc^k]_k \cup [bdc^k c^+]_k$

$[bc^+B]_0 = [bc]_k \cup \ldots \cup [bc^k]_k \cup [bdc^k c^+]_k$

$[bc^+Bd]_0 = [bcd]_k \cup \ldots \cup [bc^k d]_k \cup [bdc^k c^+ d]_k$

6k + 12 *linear in k*

Theorem 6.47

Let q be the state of LALR(k) machine for G, and q is accessible upon reading string δ.

Then δ *is a viable prefix of G, and*

$^{\exists}\gamma \in V^{*}$.э. $^{\forall}I \in q$, $I \in \langle\gamma\rangle_k$, *where* $\gamma \; \rho_0 \; \delta$.

Conversely,

If $I \in \langle\gamma\rangle_k$,

Then $^{\exists}q$ *s.t.* $I \in q$ *and*

q is accessible upon reading any viable prefix δ

.э. $\delta \; \rho_0 \; \gamma$.

LR(k) states: $\langle[\gamma]_k\rangle_k = \langle\gamma\rangle_k \leftrightarrow [\gamma]_k$

$= \{[A \to \alpha.\beta, x] \in \langle\delta\rangle_k \mid \delta \in [\gamma]_k\}$

LR(0) states: $\langle[\gamma]_0\rangle_0 = \langle\gamma\rangle_k \leftrightarrow [\gamma]_0$

$= \{[A \to \alpha.\beta] \in \langle\delta\rangle_0 \mid \delta \in [\gamma]_0\}$

LALR(k) states: $\langle[\gamma]_0\rangle_k = \langle\gamma\rangle_{k,\,0} \leftrightarrow [\gamma]_0$

$= \{[A \to \alpha.\beta, x] \in \langle\delta\rangle_k \mid \delta \in [\gamma]_0\}$

Since $[\gamma]_k \subseteq [\gamma]_0$,

$\langle\gamma\rangle_k \subseteq \langle\gamma\rangle_{k,\,0}$,

$core(\langle\gamma\rangle_k) = core(\langle\gamma\rangle_{k,\,0}) = core(\langle\gamma\rangle_0)$

Let $G = (N, \Sigma, P, S)$. The ***LALR(k) parser*** *for G is a pushdown transducer $M = ([G]_0, \Sigma, \Gamma, P, \tau, [\varepsilon]_0, \{[\varepsilon]_0[S]_0\}, \$, |)$ where*

$$\Gamma = \{[\delta]_0[\delta X_1]_0\ldots[\delta X_1\ldots X_n]_0 | y \rightarrow [\delta]_0[\delta A]_0 | y \mid [A \rightarrow X_1\ldots X_n\cdot, y] \in \langle[\delta X_1\ldots X_n]_0\rangle_k\} \quad (ra)$$

$$\cup \{[\delta]_0 | ay \rightarrow [\delta]_0[\delta a]_0 | y \mid a \in \Sigma, [A \rightarrow \alpha\cdot a\beta, z] \in \langle[\gamma]_0\rangle_k, y \in First_{max\{k-1, 0\}}(\beta z)\} \quad (sa)$$

Theorem 6.48 *The size of LALR(k) parser for G is $O(2^{|G| + k\log|\Sigma| + \log|G|})$.*

Correctness of LALR(k) parser as a right parser.

L6.49: *right parser* $\Rightarrow$ *LALR(k) parser*

*LR(0) parser(**L6.29**)*

L6.50: *LALR(k) parser* $\Rightarrow$ *right parser*

*LR(k) parser(**L6.32**)*

Theorem 6.51 *For the LALR(k) parser M for G,*

(1) M is a right parser for G

(2) $\forall w \in L(G)$*, M produces all right parses of w in G*

(3) $TIME_G(w) = TIME_M(w) + |w|$.

making LALR(k) parser

from LR(k) parser $\Rightarrow$ *uniting all states with the same set of item cores*

from LR(0) parser $\Rightarrow$ *add suitable k-length lookahead strings to 0-items*

*LALR(k) lookahead set is sufficient and **minimal***

Theorem 6.52

Let $[A \rightarrow \alpha{\cdot}\beta, z] \in q$. Then $\exists x, y$ and $X_1 \ldots X_m$.϶.

$[\varepsilon]|xy \Rightarrow^* [\varepsilon][X_1]\ldots[X_1 \ldots X_m]|y\$$ in M,

where the set of cores in $\langle X_1 \ldots X_m \rangle_0$ is same as in q and $k{:}y\$ = First_k(\beta z)$.

In CLR(k) parser

***every item** $[A \rightarrow \alpha{\cdot}\beta, z]$ in any state q*

***can be "used"** in the parsing of*

***all** terminal strings of the form xy,*

where $k{:}y \in First_k(\beta z)$ and $q = \langle \$\gamma \rangle_0$.

sentence	*LALR(k) is same as LR(k)*
no sentence	*additional **reduce** actions in **LALR(k)***

***Immediate Error Detection Property** in LR(k)*

***reduce stack** for error recovery in LALR(k)*

*$G = (N, \Sigma, P, S)$ is **LALR(k)** if*

its LALR(k) parser is deterministic and

$S \Rightarrow^+ S$ is impossible in G.

A language over alphabet Σ is LALR(k) if

it is generated by an LALR(k) grammar.

Theorem 6.53 *(Characterization of LALR(k) Grammars)Let G' be a augmented grammar.*
The LALR(k) parser of G is deterministic iff
in the LALR(k) machine of G'
no state contains a pair of items
exibiting a reduce-reduce or
a shift-reduce conflict.

Theorem 6.54 *The class of LALR(0) grammars coinsides with the class of LR(0) grammars. For $k \geq 1$ the class of LALR(k) grammars is properly contained in the class of LR(k) grammars.*

Proof

(i) LALR(k) $\subseteq$ LR(k):
uniting of states in CLR(k) machine
can only increase # of reduce-reduce conflicts.

(ii) LALR(k) $\neq$ LR(k)
counter example:

$S \rightarrow aAa|bAb|aBb|bBa,$

$A \rightarrow c,$

$B \rightarrow c.$

This grammar is LR(1)
but not LALR(k) for any k.

Gerneralize the LALR concepts:

LA(k)LR(l) machine

$\rightarrow$ *unite* q_1 *and* q_2

whenever the truncating of the k-length lookahead strings to length $l \leq k$*, yields the same set of l-items.*

- *unite* q_1, q_2 *if* $Trunc_l(q_1) = Trunc_l(q_2)$,

$Trunc_l(q) = \{[A \rightarrow \alpha\cdot\beta, l{:}y] \mid [A \rightarrow \alpha\cdot\beta, y] \in q\}$

Fact 6.55

The LA(k)LR(k) machine is same to LR(k) machine.
The LA(k)LR(0) machine
is same to LALR(k) machine.

Theorem 6.56 *Let q be a state in LA(k)LR(l) machine accessible upon reading string* δ.

Then δ *is a viable prefix of G, and*

$^{\exists}\gamma \in V^*$ *.э.* $^{\forall}I \in q, I \in \langle\gamma\rangle_k$*, where* $\delta\ \rho_l\ \gamma$.

Conversely,

If $I \in \langle\gamma\rangle_k$.

then $^{\exists}q$ *s.t.* $I \in q$ *and*

q is accessible upon reading any viable prefix δ

.э. $\gamma\ \rho_l\ \delta$.

States in LA(k)LR(l) machine

$\langle[\gamma]_l\rangle_k \leftrightarrow [\gamma]_l$

6.6. SLR(k) Parsing

*SLR(k) stands for **S**imple **LR(k)**.*

adding k-lookaheads in a crude, simple way.
seldom minimal lookaheads.

SLR(k) parser *for G is the pushdown transducer M = $([G]_0, \Sigma, \Gamma, P, \tau, [\varepsilon]_0, \{[\varepsilon]_0[S]_0\}, \$, \mid)$ where*

$$\Gamma = \{[\delta]_0[\delta X_1]_0\ldots[\delta X_1\ldots X_n]_0 \mid y \to [\delta]_0[\delta A]_0 \mid y \;\; | \; [A \to X_1\ldots X_n\cdot] \in [\delta X_1\ldots X_n]_0 \text{ and } y \in Follow_k(A)\} \quad (ra)$$

$$\cup \{[\delta]_0 \mid ay \to [\delta]_0[\delta a]_0 \mid y \;\; | \; a \in \Sigma, [A \to \alpha\cdot a\beta] \in [\delta]_0, \; y \in First_{max\{k-1,\, 0\}}(\beta Follow_k(A))\} \quad (sa)$$

Theorem 6.57

The SLR(k) parser M for G is a right parser for G. Moreover, $\forall w \in L(G)$,

M produces all right parses of w in G, and
$TIME_G(w) = TIME_M(w) + |w|$.

Theorem 6.58 *(Characterization of SLR(k) Grammars) The SLR(k) parser of G is deterministic iff for all state q in SLR(k) machine,*

(1) Whenever $[A_1 \to \omega_1\cdot], [A_2 \to \omega_2\cdot] \in q$, *then*
$Follow_k(A_1) \cap Follow_k(A_2) = \emptyset$.

(2) Whenever $[A \to \alpha\cdot a\beta], [B \to \omega\cdot] \in q$, *then*
$First_k(a\beta Follow_k(A)) \cap Follow_k(B) = \emptyset$.

Theorem 6.59

The class of SLR(0) grammars
coinsides with the class of LR(0) grammars.
For $k \geq 1$,
the class of SLR(k) grammars is properly contained in the class of LALR(k) grammars.

(eg) $S \to Ac|bA|bc,$
$A \to \varepsilon$

Time to test SLR(k) property for G: polynomial to $|G|$

Transformation of G into T_k(G), *which is SLR(k)* ***if and only if*** *G is LR(k).*

Idea : replace A by $([\gamma]_k, A)\ldots$

$G = (N, \Sigma, S, P)$

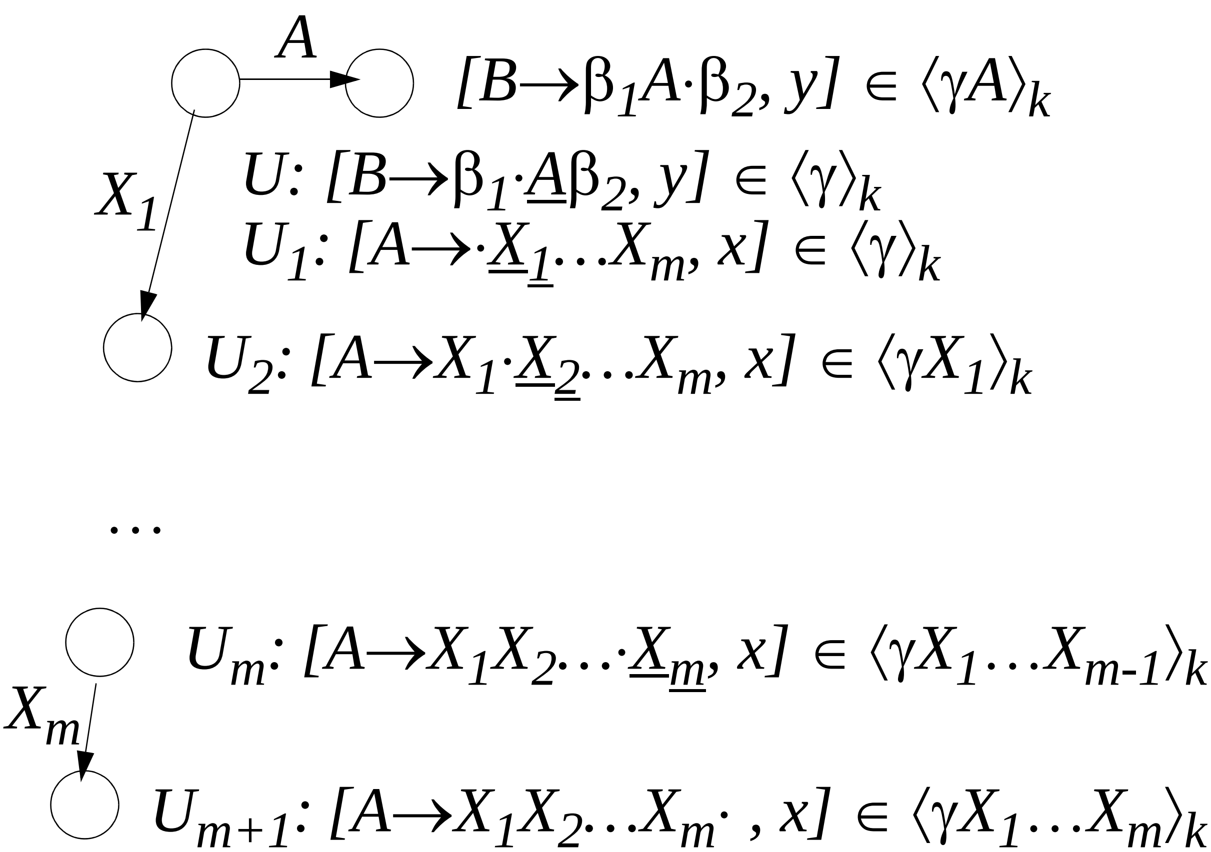
$\mathbf{T_k(G)} = ([G]_k \times N, \Sigma, ([\varepsilon]_k, S), \hat{P})$, *where*

$\hat{P} = \{([\gamma]_k, A) \rightarrow U_1\ldots U_m$

$\quad | \ [B\rightarrow\beta_1\cdot A\beta_2, y]_k \in \langle\gamma\rangle_k,\ A \rightarrow X_1\ldots X_m \in P,$

$\quad 1\leq^{\forall} i\leq m \qquad U_i = ([\gamma X_1\ldots X_{i-1}]_k, X_i)$ if $X_i \in N,$

$\qquad\qquad\qquad\qquad\quad = X_i$ if $X_i \in \Sigma \cup \{\$\}.$



*$([\gamma]_k, A)$ is a **useful** nonterminal in $T_k(G)$, iff*
$[\gamma]_k \in [G]_k$, $A \in N$, and $[B \to \alpha.A\beta] \in [\gamma]_k$.

$S \Rightarrow^*_{rm} \gamma A y$

$([\varepsilon]_k, S) \Rightarrow^*_{rm} \Phi([\gamma]_k, A)y$

$Follow_k(([\gamma]_k, A)) = \{k{:}y \mid S \Rightarrow^*_{rm} \gamma A y\}$

*$T_k(G)$ **right-to-right covers** G*
≡ right parses in $T_k(G)$ are mapped into
right parses in G by a homomorphism h.

Furthermore,
*$T_k(G)$ is **structurally equivalent** to G.*
*≡ parse trees in $T_k(G)$ and G have **same structure***
parse trees are same except for
the labeling of the nonterminal nodes.
*parse trees are **isomorphic***

Cover relations between Grammars

Let x, y ∈ {"left", "right"}. Then

*an **x-to-y cover** of G is a pair* $(\hat{G}, h)$ *where*

$\hat{G} = (\hat{N}, \Sigma, \hat{P}, \hat{S})$ *and* $h: \hat{P}^* \rightarrow P^*$.∋.

i) $^\forall w \in L(\hat{G})$ *and x-parses* $\hat{\pi}$ *of w in* $\hat{G}$,

$h(\hat{\pi})$ *is a y-parse of w in G.*

ii) $^\forall w \in L(G)$ *and y-parses* π *of w in G,*

$^\exists \hat{\pi} \in \hat{P}^*$, $\hat{\pi}$ *is a x-parse of w in* $\hat{G}$ *and* $h(\hat{\pi}) = \pi$.

h maps x-parses *of* $\hat{G}$ ***into y-parses*** *of G*

If $^\exists h$, $(\hat{G}, h)$ *is x-to-y covers of G,*

$\hat{G}$ *x-to-y covers G with respect to h*

If $\hat{G}$ *x-to-y covers G with respect to h,*

$(\hat{G}, h)$ *is x-to-y covers of G,*

Fact 6.60 *If* $\hat{G}$ *x-to-y covers G, then* $L(\hat{G}) = L(G)$.

Fact 6.61 *If (M, τ) is an x parser of* $\hat{G}$ *and if* $\hat{G}$ *x-to-y covers G w.r.t. h, then* $(M, \tau \circ h)$ *is a **y parser** of G.*

$(T_k(G), h_k)$ *right-to-right covers G, if*

$h_k(U \rightarrow U_1 \ldots U_m) = A \rightarrow X_1 \ldots X_m$, *where*

$U = ([\gamma], A), U_i = ([\gamma X_1 \ldots X_{i-1}], X_i)$ if $X_i \in N$,

$= X_i$ if $X_i \in \Sigma$.

(M, τ) is a x-parser of G, if $\tau(\theta) = \pi_x$.
$(\hat{G}, h)$ is a x-to-y cover of G, if $h(\hat{\pi}_x) = \pi_y$.
$(M, \tau \circ h)$ is a x-parser of G.

G

$\hat{G}$ $(\hat{M}, \tau)$ $(M, \tau \circ h)$

Consider a function $M: \Sigma^* \to \{\Gamma^*\}$

$M(w) = \theta$.

M is deterministic if $M: \Sigma^* \to \Gamma^*$.

Consider a function $G_x: \Sigma^* \to \{P^*\}$

$G_x(w) = \pi$.

G is unambiguous, if $G_x: \Sigma^* \to P^*$.

$M \circ \tau$ is a x-parser of G, if $M \circ \tau: \Sigma^* \to \{P^*\}$

$M \circ \tau = G_x$.

If G is ambiguous,

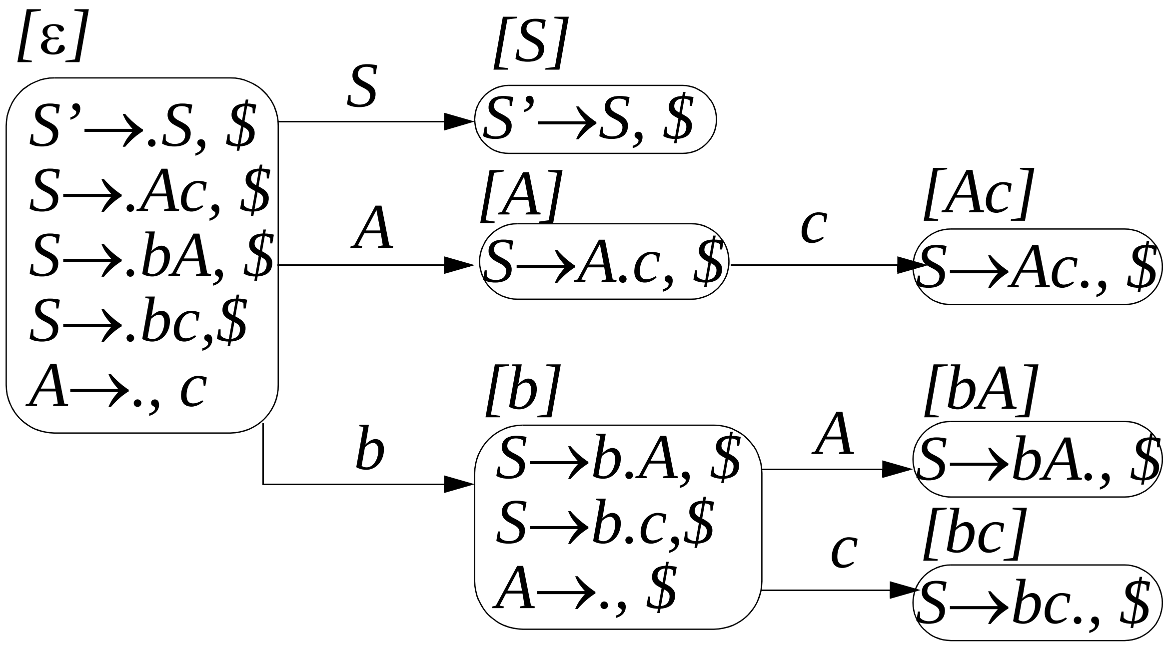
Example

$G: \quad S \rightarrow Ac \mid bA \mid bc$

$\quad\quad A \rightarrow \varepsilon$

Follow(A) = {$, c}

G is not SLR(1) in state [b]



$T_k(G)$: $([\varepsilon], S) \rightarrow ([\varepsilon], A)\ c \mid b\ ([b], A) \mid b\ c$

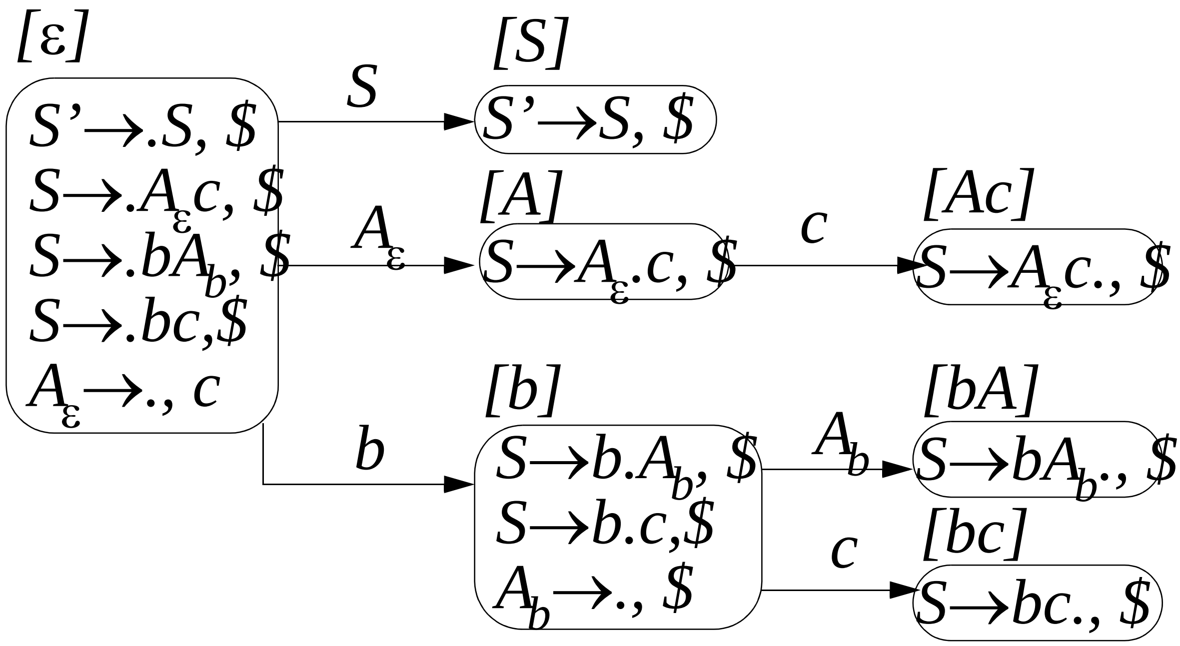
$([\varepsilon], A) \rightarrow \varepsilon$

$([b], A) \rightarrow \varepsilon$

$S_\varepsilon \rightarrow A_\varepsilon\ c \mid b\ A_b \mid b\ c$

$A_\varepsilon \rightarrow \varepsilon$

$A_b \rightarrow \varepsilon$



Follow(A_ε) = {c}

Follow(A_b) = {\$}

$T_k(G)$ *is SLR(1)*

$T_k(G)$ *right-to-right covers G w.r.t.* h_k(**T6.64**)

$\forall w \in L(\hat{G})$ *where* $\hat{\pi}$ *is a right parse of w in* $T_k(G)$,

$h_k(\hat{\pi})$ *is a right parse of w in G.*(**L6.62**)

$\forall w \in L(G)$, *where* π *is a right parse of w in G*

$\exists \hat{\pi}$.э. $h_k(\hat{\pi})=\pi$, $\hat{\pi}$ *is a right parse of w in* $T_k(G)$.

(**L6.63**)

Lemma 6.62 *If* $[\gamma A] \in [G]_k$, $([\gamma], A) \Rightarrow^{\hat{\pi}} \Phi$ *in* $T_k(G)$.
Then let $\Phi = U_1 \ldots U_m y$,

$\gamma X_1 \ldots X_m \in [G]_k$, *and*

$A \Rightarrow^{h_k(\hat{\pi})} X_1 \ldots X_m y$ *in G , where*

$U_i = ([\gamma X_1 \ldots X_{i-1}], X_i)$ *if* $X_i \in N$,

$= X_i$ *if* $X_i \in \Sigma \cup \{\$\}$.

Proof *by induction on* $|\hat{\pi}|$.

IB: $\hat{\pi} = \varepsilon$

IH: $\hat{\pi} = \hat{\pi}_1 \hat{r}$, $\hat{r} = W \rightarrow W_1 \ldots W_p \in \hat{P}$.

$([\gamma], A) \Rightarrow^{\bar{\pi}_1} U_1 \ldots U_n W y$

$\Rightarrow^{\bar{r}} U_1 \ldots U_n W_1 \ldots W_p y = \Phi$.

By definition of $T_k(G)$,

$W = ([\delta], B), \delta = \gamma h_k(U_1 \ldots U_n)$, *and*

$W_i = ([\delta Z_1 \ldots Z_{i-1}], Z_i)$, *if* $Z_i \in N$,

$= Z_i$, *if* $Z_i \in \Sigma \cup \{\$\}$.

By IH, $^{\exists}\gamma X_1 \ldots X_n B \in [G]_k$, *and,*

$A \Rightarrow^{h_k(\hat{\pi}_1)} X_1 \ldots X_n B y$ *in G,*

$[\gamma X_1 \ldots X_n] = [\delta]$, *and by right invariance,*

$[\delta Z_1 \ldots Z_{i-1}] = [\gamma X_1 \ldots X_n Z_1 \ldots Z_{i-1}]$.

If choose $m = n+p$, $1 \le i \le p$,

$U_{n+i} = W_i$, $X_{n+i} = Z_i$, *then the lemma is proved.*

Lemma 6.63 $\gamma A \in [G]_k$, $A \Rightarrow^{\pi} X_1 \ldots X_m y$ *in G, and*
either $X_1 \ldots X_m = \varepsilon$ *or* $X_m \in N$.

Then $\exists \hat{\pi} \in \hat{P}^*$.э. $h_k(\hat{\pi}) = \pi$ *and*

$([\gamma], A) \Rightarrow^{\hat{\pi}} U_1 \ldots U_m y$ *in* $T_k(G)$.

Proof *by induction on* $|\pi|$.

Base: $\pi = \varepsilon$. *Then* $\hat{\pi} = \varepsilon$ *and* $U_1 = ([\gamma], A)$.

Induction Step: $\pi = \pi_1 r$, $r = B \rightarrow Z_1 \ldots Z_p$.

$$A \underset{rm}{\Rightarrow}^{\pi_1} Y_1 \ldots Y_n B y_1 \underset{rm}{\Rightarrow}^{r} Y_1 \ldots Y_n Z_1 \ldots Z_p y_1$$
$$= X_1 \ldots X_m y \text{ in } G. (m=n+p)$$

$\exists \overline{\pi}_1$ *of* $T_k(G)$.э.

$h_k(\overline{\pi}_1) = \pi_1$ *and*

$([\gamma], A) \underset{rm}{\Rightarrow}^{\overline{\pi_1}} U_1 \ldots U_n([\gamma Y_1 \ldots Y_n], B) y_1$

where $U_i = ([\gamma Y_1 \ldots Y_{i-1}], Y_i)$ *if* $Y_i \in N$,

$= Y_i$ *if* $Y_i \in \Sigma$.

Then $\exists \overline{r} = ([\gamma Y_1 \ldots Y_n], B) \rightarrow U_{n+1} \ldots U_{n+p}$ *in* $T_k(G)$

where $U_i = ([\gamma Y_1 \ldots Y_n Z_1 \ldots Z_{i-1}], Z_i)$ *if* $Z_i \in N$,

$= Z_i$ *if* $Z_i \in \Sigma$.

$\therefore h_k(\overline{\pi}_1 \overline{r}) = h_k(\overline{\pi}_1) h_k(\overline{r}) = \pi_1 r = \pi$, *and*

$$([\gamma], A) \underset{rm}{\Rightarrow}^{\pi_1 r} U_1 \ldots U_{n+p} y_1 = U_1 \ldots U_m z y_1$$
$$= U_1 \ldots U_m y.$$

Theorem 6.64 *$T_k(G)$ right-to-right covers G w.r.t. h_k.*

Corollary 6.65 *If (M, τ) is a right parser of $T_k(G)$, then (M, τh_k) is a right parser of G.*

Lemma 6.66 *$y \in Follow_k(([\gamma]_k, A))$ in $T_k(G)$ iff $S \Rightarrow^* \delta Az$ in G, $[\delta]_k = [\gamma]_k$, $l{:}z = y$.*

In other words,

y is a follower of $([\gamma]_k, A)$ in $T_k(G)$

iff y is a follower of A in G

in some context LR(k)-equivalent to γ.

Lemma 6.67

$[U \to U_m \ldots U_i \cdot U_{i+1} \ldots U_p, y] \in \langle U_1 \ldots U_i \rangle_l$

in $T_k(G)$ iff

$[A \to X_m \ldots X_i . X_{i+1} \ldots X_p, y] \in \langle X_1 \ldots X_i \rangle_l$,

$U = ([X_1 \ldots X_{m-1}], A)$, and $1 \le j \le p$,

$U_j = ([X_1 \ldots X_{j-1}], X_j)$ if $X_j \in N$,

$= X_j$ if $X_j \in \Sigma \cup \{\$\}$.

Lemma 6.68 *If G is non-LR(k), then so is $T_k(G)$.*

Proof

(i) $S \underset{rm}{\Rightarrow}^{+} S$, then $([\varepsilon]_k, S) \underset{rm}{\Rightarrow}^{+} ([\varepsilon]_k, S)$,

$\therefore T_k(G)$ is non-LR(k).

(ii) $\langle X_1 \ldots X_i \rangle_k$ contains a conflict. Then

$[A \rightarrow X_m \ldots X_i \cdot, y] \in \langle X_1 \ldots X_i \rangle_k$,

$[B \rightarrow X_n \ldots X_i \cdot X_{i+1} \ldots X_p, u] \in \langle X_1 \ldots X_i \rangle_k$,

$y \in First_k(X_{i+1} \ldots X_p u)$.

Then by lemma 6.67,

$[U \rightarrow U_m \ldots U_i \cdot, y] \in \langle U_1 \ldots U_i \rangle_k$, and

$[W \rightarrow U_n \ldots U_i \cdot U_{i+1} \ldots U_p, u] \in \langle U_1 \ldots U_i \rangle_k$,

where $U = ([X_1 \ldots X_{m-1}]_k, A)$,

$W = ([X_1 \ldots X_{n-1}]_k, B)$,

$U_j = ([X_1 \ldots X_{j-1}]_k, X_j)$ if $X_j \in N$,

$= X_j$ if $X_j \in \Sigma \cup \{\$\}$.

$U \neq W$ or $m \neq n$ or $i+1 \leq p$,

$y \in First_k(U_{i+1} \ldots U_p u)$.

$\therefore T_k(G)$ is non-LR(k).

Lemma 6.69 *If $T_k(G)$ is non-SLR(k), then G is non-LR(k).*

Proof *If $T_k(G)$ is non-LR(k), we have*

$[U \to U_m \ldots U_i \cdot] \in \langle U_1 \ldots U_i \rangle_0$,

$[W \to U_n \ldots U_i \cdot U_{i+1} \ldots U_p] \in \langle U_1 \ldots U_i \rangle_0$,

$y \in Follow_k(U) \cap First_k(U_{i+1} \ldots U_p Follow_k(W))$,

$U_{i+1} \in \Sigma$ *whenever* $i+1 \le p$. *Then there exist*

$U = ([X_1 \ldots X_{m-1}]_k, A)$,

$W = ([X_1 \ldots X_{n-1}]_k, B)$, *and*

$U_j = ([X_1 \ldots X_{j-1}]_k, X_j)$ *if* $X_j \in N$,

$= X_j$ *if* $X_j \in \Sigma \cup \{\$\}$, $1 \le \forall j \le p$.

By lemma 6.66,

$S \Rightarrow^* \gamma Az$, $[\gamma]_k = [X_1 \ldots X_{m-1}]_k$, $k{:}z = y$,

$S \Rightarrow^* \delta Bu$, $[\delta]_k = [X_1 \ldots X_{n-1}]_k$, $k{:}xu = y$, *and*

$x \in First_k(U_{i+1} \ldots U_p)$.

(i) If $U_{i+1} \in \Sigma$, $X_{i+1} \in \Sigma$, *too. Then*

$[A \to X_1 \ldots X_i \cdot, k{:}z\$]$, $[B \to X_1 \ldots X_i \cdot X_{i+1} \ldots X_p, k{:}u\$]$

exhibit a conflict.

(ii) $i \le p$, *by the right invariance,* $i = p$, *then reduce-reduce conflict.*

Theorem 6.70 *Any grammar G can be transformed into a structurally equivalent grammar which is SLR(k) iff G is LR(k).*

Theorem 6.71 *For any k ≥ 0,*
the families of LR(k) languages,
LALR(k) languages, and
SLR(k) languages
are all equal.

6.7. Covering LR(k) Grammars by LR(1) Grammars

- *LR(k) language ≡ LR(1) language*
  LR(k) grammar ⇒ LR(1) grammar
  right-to-right cover
- *deterministic language ⇒ SLR(1) parsing*

$T_{k,\,1}(G)$ *right-to-right covers G, and*
$T_{k,\,1}(G)$ *is LR(1), iff G is LR(k+1)*

Idea

- *shift the derivation trees in G k symbols to the right*
- *reduce actions are postponed until 1 symbol lookahead is sufficient to resolve uniquely.*

A is replaced by set of (x, A, y)'s, where
$y \in Follow_k(A)$, $x \in First_k(Ay)$.
$|y| \le k$ *and* $|x| = k$.

$L((x, A, y)) =$
$\{z \mid S \Rightarrow^* uAw \Rightarrow^* uvw,\ y = k{:}w,\ x = k{:}vw,\ xz = vy,\ |x| = k\}$

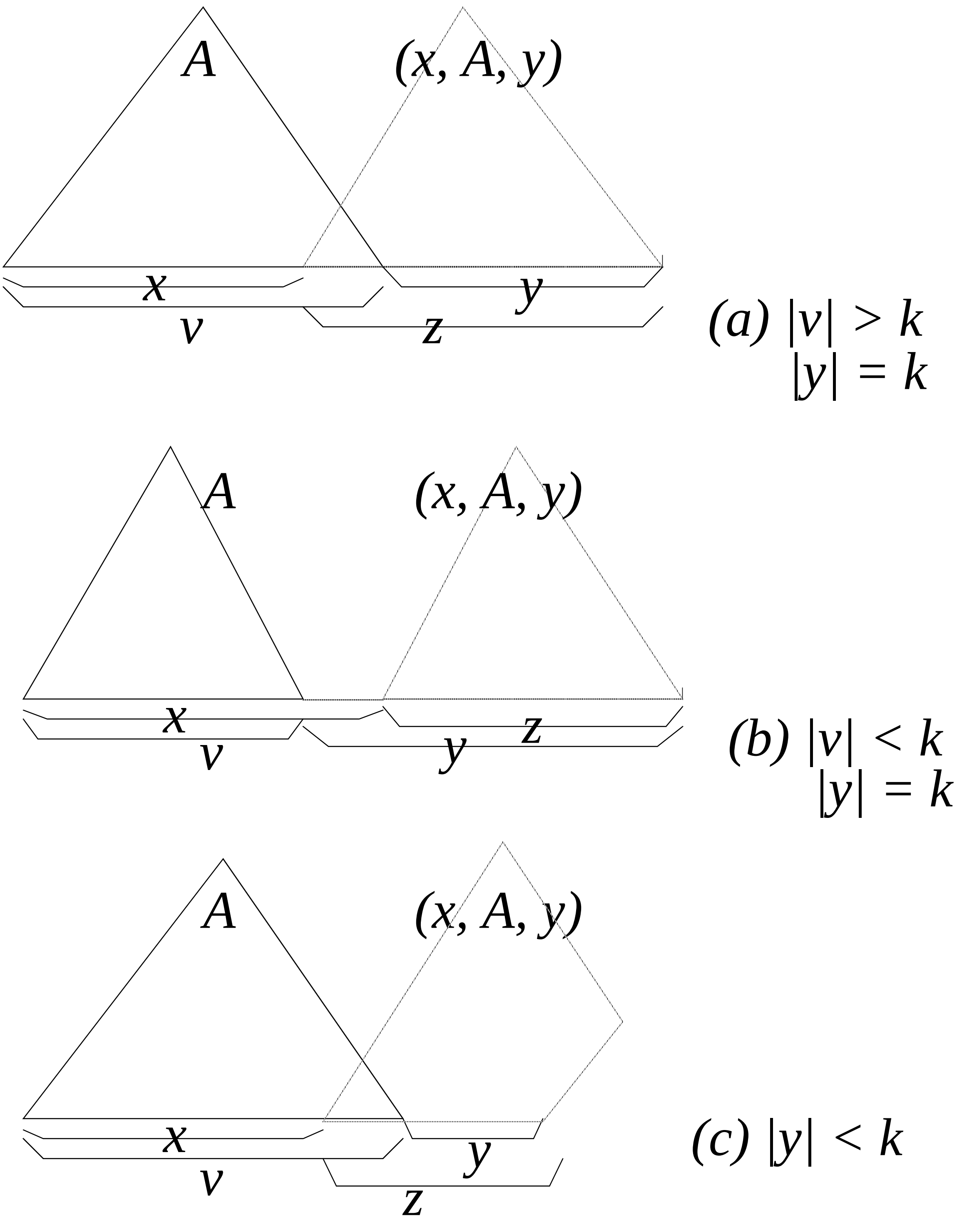


Figure 6.20 (p85)

vy = xz, where A ⇒ v, (x, A, y) ⇒* z, and |x| = k.*

Let $G = (N, \Sigma, P, S)$ *be a grammar.*

$\mathbf{T_{k,1}(G)} = (N', \Sigma, P', S')$*, where*

$N' = \{S'\}$

$\quad \cup \{(x, X, y) \mid y \in Follow_k(X), x \in First_k(Xy)\}$

$P' = \{S' \to x(x, S, \varepsilon) \mid x \in First_k(S)\}$

$\quad \cup \{(y_0, A, y_m) \to$

$\qquad (y_0, X_1, y_1)(y_1, X_2, y_2)\ldots(y_{m-1}, X_m, y_m)$

$\qquad \mid m \geq 0, A \to X_1\ldots X_m \in P, y_m \in Follow_k(A),$

$\qquad\quad 0 \leq {}^{\forall}i < m, y_i \in First_k(X_{i+1}y_{i+1}),$

$\qquad\quad Follow_k, First_k$ *in the context of* $A \Rightarrow^* y_0\ldots\}$

$\quad \cup \{(ax, a, xb) \to b \mid xb \in Follow_k(a), |xb| = k\}$

$\quad \cup \{(ax, a, x) \to \varepsilon \mid x \in Follow_k(a), |x| < k\}.$

$\qquad\quad y_i \in Follow_k(X_i)$

$h_{k,1}: P' \to P \cup \{\varepsilon\}$

$\quad h_{k,1}(S' \to x(x, S, \varepsilon)) = \varepsilon,$

$\quad h_{k,1}((y_0, A, y_m) \to (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m))$

$\qquad = A \to X_1\ldots X_m,$

$\quad h_{k,1}((ax, a, xb) \to b) = \varepsilon,$ *and*

$\quad h_{k,1}((ax, a, x) \to \varepsilon) = \varepsilon.$

Two problems

i) $(T_{k,1}(G), h_{k,1})$ *is a right-to-right cover of G.*

T6.78(⇒: L6.72-74, ⇐: L6.75-77)

ii) $T_{k,1}(G)$ *is LR(1) if and only if G is LR(k+1).*

T6.85(⇐: L6.79-81, ⇒: L6.82-84)

Example

	$First_1$	$Follow_1$
$S \to Abb \mid Bb$	$\{a\}$	$\{\varepsilon\}$
$A \to aA \mid a$	$\{a\}$	$\{b\}$
$B \to aB \mid a$	$\{a\}$	$\{b\}$

$S_0 \to$	$a\,(a, S, \varepsilon)$	ε
$(a, S, \varepsilon) \to$	$(a, A, b)(b, b, b)(b, b, \varepsilon)$	$S \to Abb$
$\mid$	$(a, B, b)\,(b, b, \varepsilon)$	$S \to Bb$
$(a, A, b) \to$	$(a, a, a)\,(a, A, b)$	$A \to aA$
$\mid$	(a, a, b)	$A \to a$
$(aa, B, b) \to$	$(a, a, a)\,(a, B, b)$	$B \to aB$
$\mid$	(a, a, b)	$B \to a$
$(a, a, a) \to$	a	ε
$(a, a, b) \to$	b	ε
$(b, a, b) \to$	b	ε
$(b, b, \varepsilon) \to$	ε	ε

Example

	$First_2$	$Follow_2$
$S \rightarrow Abb \mid Bb$	$\{aa, ab\}$	$\{\varepsilon\}$
$A \rightarrow aA \mid a$	$\{aa, a\}$	$\{bb\}$
$B \rightarrow aB \mid a$	$\{aa, a\}$	$\{b\}$

$S_0 \rightarrow$	$aa\ (aa, S, \varepsilon)$	ε
$\mid$	$ab\ (ab, S, \varepsilon)$	ε
$(aa, S, \varepsilon) \rightarrow$	$(aa, A, bb)(bb, b, b)(b, b, \varepsilon)$	$S \rightarrow Abb$
$\mid$	$(aa, B, b)\ (b, b, \varepsilon)$	$S \rightarrow Bb$
$(ab, S, \varepsilon) \rightarrow$	$(ab, A, bb)(bb, b, b)(b, b, \varepsilon)$	$S \rightarrow Abb$
$\mid$	$(ab, B, b)\ (b, b, \varepsilon)$	$S \rightarrow Bb$
$(aa, A, bb) \rightarrow$	$(aa, a, aa)\ (aa, A, bb)$	$A \rightarrow aA$
$\mid$	$(aa, a, ab)\ (ab, A, bb)$	$A \rightarrow aA$
$(ab, A, bb) \rightarrow$	(ab, a, bb)	$A \rightarrow a$
$(aa, B, b) \rightarrow$	$(aa, a, aa)\ (aa, B, b)$	$B \rightarrow aB$
$\mid$	$(aa, a, ab)\ (ab, B, b)$	$B \rightarrow aB$
$(ab, B, b) \rightarrow$	(ab, a, b)	$B \rightarrow a$
$(aa, a, aa) \rightarrow$	a	ε
$(aa, a, ab) \rightarrow$	b	ε
$(ab, a, bb) \rightarrow$	b	ε
$(ab, a, b) \rightarrow$	ε	ε
$(bb, b, b) \rightarrow$	ε	ε
$(b, b, \varepsilon) \rightarrow$	ε	ε

Lemma 6.72 *Consider G and $T_{k,1}(G)$ be grammar.*

If $(x, A, y) \Rightarrow^{\pi'} (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m)z$
$= \Phi$ *in* $T_{k,1}(G)$.

Then $y_0 = x$, $y_m z = vy$; *and*

$A \Rightarrow^{\pi} X_1X_2\ldots X_m v$ *in G.*

Moreover, if $|y_m| < k$, *then* $z = \varepsilon$.

Proof *by induction on* $|\pi'|$

IB: $m = 1$, $y_0 = x$, $y_m = y$, $z = v = \varepsilon$, *and* $X_1 = A$.

IH: $\pi' = \pi_1' r'$

$(x, A, y) \Rightarrow^{\pi_1'} (y_0, X_1, y_1)\ldots(y_{n-2}, X_{n-1}, y_{n-1})(y_{n-1}, X, y'_n)z_1 \Rightarrow^{r'} (y_0, X_1, y_1)\ldots(y_{n-2}, X_{n-1}, y_{n-1})\omega z_1 = \Phi$

Moreover, $\exists v_1$ *s.t.*

$\psi_0 = x$, $y_n' z_1 = v_1 y$, *and*

$A \Rightarrow^{\pi} X_1X_2\ldots X_{n-1}Xv_1$ *in G.*

Case 1:

$r' = (y_{n-1}, X, y_n') \rightarrow (y_{n-1}, X_n, y_n)\ldots(y_{m-1}, X_m, y_m)$,
$m \geq n - 1$.

By definition, $y_m = y_n'$ *and* $h(r') = X \rightarrow X_n\ldots X_m \in P$.

Then we have:

$y_0 = x$, $y_m z_1 = y_n' z_1 = v_1 y$, *and*

$A \Rightarrow^{\pi_1'} X_1X_2\ldots X_{n-1}Xv_1 \Rightarrow^{r'} X_1\ldots X_m v_1$ *in* $T_{k,1}(G)$.

$z = z_1$, $v = v_1$.

Case 2: $r' = (au, a, ub) \rightarrow b.\ h(r') = \varepsilon.$

If $m = n - 1$, $z = bz_1$, *and* $v = av_1$, *we then have:*

$$\Phi = (y_0, X_1, y_1)\ldots(y_{n-2}, X_{n-1}, y_{n-1})bz_1$$
$$= (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m)z, \quad (y_n' = ub)$$

$y_0 = x$, $y_m z = y_{n-1}bz_1 = ay_n'z_1 = av_1y = vy$, *and*

$$A \Rightarrow^{h(\pi' r')} X_1\ldots X_{n-1}Xv = X_1\ldots X_m av_1 = X_1\ldots X_m v$$

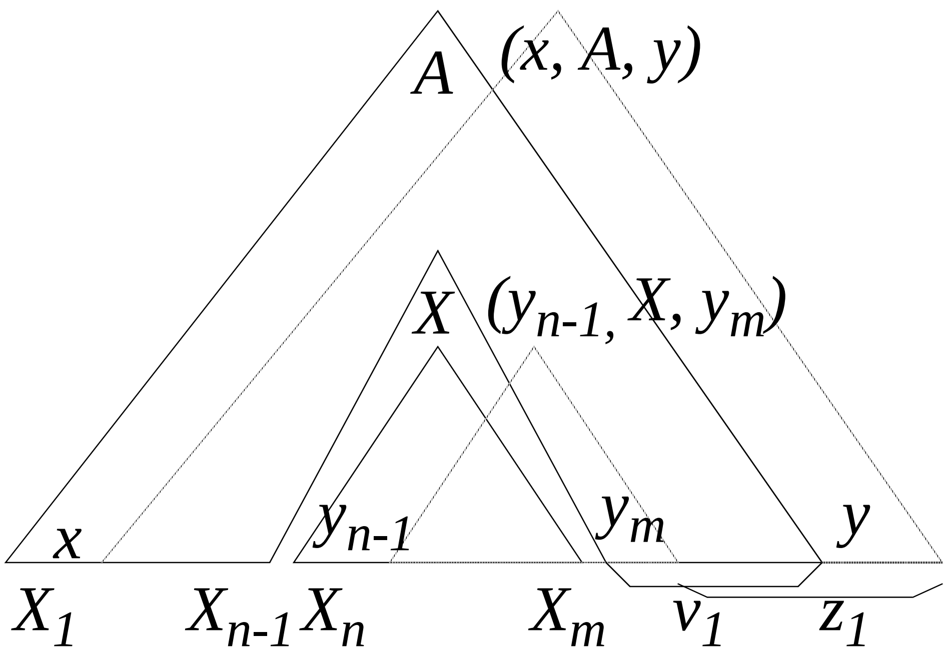
Case 3: $r' = (ay_n', a, y_n') \rightarrow \varepsilon.\ |y_n'| < k,\ h(r') = \varepsilon.$

If $m = n - 1$, $z = z_1$, *and* $v = av_1$, *we then have:*

$$\Phi = (y_0, X_1, y_1)\ldots(y_{n-2}, X_{n-1}, y_{n-1})z_1$$
$$= (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m)z,$$

$y_0 = x$, $y_m z = y_{n-1}z_1 = ay_n'z_1 = av_1y = vy$, *and*

$$A \Rightarrow^{h(\pi' r')} X_1\ldots X_{n-1}Xv = X_1\ldots X_m v_1 = X_1\ldots X_m v$$



Lemma 6.73 *If*

$(x, A, y) \Rightarrow^{\pi'} z$ *in* $T_{k, 1}(G)$, *then*

$A \Rightarrow^{h(\pi')} v$ *in G*, *where* $vy = xz$.

Proof

$\Phi = z$ *in* ***L6.72***.

Lemma 6.74 *If* π' *is a right parse of w in* $T_{k, 1}(G)$, *then* $h(\pi')$ *is a right parse of w in G.*

Lemma 6.75 *Let* $G = (N, \Sigma, P, S)$

$A \Rightarrow^{\pi} X_1X_2\ldots X_m v$ *in* G,

$m = 0$ *or* X_m *is a nonterminal,*

$y \in Follow_k(A)$, $y_m = k{:}vy$, $y_m z = vy$, *and*

$y_i \in First_k(X_{i+1}y_{i+1})$ $0 \le i < m$.

Then there is a rule string π' *of* $T_{k,1}(G)$.э.

$h(\pi') = \pi$, *and*

$(y_0, A, y) \Rightarrow^{\pi'} (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m)z$

in $T_{k,1}(G)$.

Proof *by induction on* $|\pi|$

IB: $m = 1$, $y_0 = x$, $y_m = y$, $z = v = \varepsilon$, *and* $X_1 = A$.

IH: $\pi = \pi_1 r$. *Then*

$A \Rightarrow^{\pi_1} X_1X_2\ldots X_n Bv_1$

$\Rightarrow^{r} X_1X_2\ldots X_nX_{n+1}\ldots X_p v_1$

$= X_1\ldots X_m v$ *in* G

Here, $v = X_{m+1}\ldots X_p v_1$, *because* $r = B \rightarrow X_{n+1}\ldots X_p$.

If $p > m$, *let* $y_p = k{:}v_1y$, $\exists z_1$.э. $y_pz_1 = v_1y$.

And let $y_i = k{:}X_{i+1}y_{i+1}$, $m < i < p$.

Then $y_m = k{:}vy = k{:}X_{m+1}\ldots X_pv_1y = k{:}X_{m+1}y_{m+1}$,

and $y_m \in First_k(By_p)$.

By IH:

$h(\pi_1') = \pi_1$, *and*

$(y_0, A, y) \Rightarrow^{\pi_1'} (y_0, A, y)$

$\Rightarrow^{\pi_1'} (y_0, X_1, y_1)\ldots(y_{n-1}, X, y'_n)z_1$ in $T_{k,1}(G)$.

And $\exists$ r' .э.

$r' = (y_n, X_{n+1}, y_{n+1})\ldots(y_{p-1}, X_p, y_p)$

$h(r') = r$.

Then

$(y_0, A, y) \Rightarrow^{\pi_1' r'} (y_0, X_1, y_1)\ldots(y_{p-1}, X_p, y_p)z_1$

$(y_m, X_{m+1}, y_{m+1})\ldots(y_{p-1}, X_p, y_p) \Rightarrow^{\pi_2'} u \in \Sigma^*$

in $T_{k,1}(G)$, where $\pi_2' = p - m$ rules of the form

$(ax, a, xb) \rightarrow b$ or $(ax, a, x) \rightarrow \varepsilon$.

Then

$(y_0, A, y) \Rightarrow^{\pi_1' r' \pi_2'} (y_0, X_1, y_1)\ldots(y_{m-1}, X_m, y_m)uz_1$

in $T_{k,1}(G)$.

Let $\pi' = \pi_1' r' \pi_2'$, then $h(\pi') = \pi$.

And $y_m uz_1 = vy = y_m z$ implying $uz_1 = z$, as claimed.

Lemma 6.76 *If*

$A \Rightarrow^{\pi} v$ *in G, and*

$y \in Follow_k(A)$, $x = k{:}vy$, *and* $xz = vy$,

then for some π' *of* $T_{k,1}(G)$,

$h(\pi') = \pi$ *and*

$(x, A, y) \Rightarrow^{\pi'} z$ *in* $T_{k,1}(G)$.

Proof

$m=0$ *and* $y_0 = x$ *in* ***L6.75***.

Lemma 6.77 *If* π *is a right parse of w in G, then w has in* $T_{k,1}(G)$ *a right parse* π' .э. $h(\pi') = \pi$.

Theorem 6.78 *For all grammars G and* $k > 0$, $T_{k,1}(G)$ *right-to-right covers G w.r.t. the homomorphism h.*

Lemma 6.79

$(y_0, X_1, y_1)\ldots(y_{n-1}, X_n, y_n) \underset{rm}{\Rightarrow}^* z$ in $T_{k,1}(G)$.

Then $\exists v$.э. $X_1\ldots X_n \underset{rm}{\Rightarrow}^* v$ in G and

$$vy_n = y_0 z.$$

Lemma 6.80

$[U \rightarrow \phi\cdot\psi, d] \in \langle\Phi\rangle_1$.

Then the form of Φ and $[U \rightarrow \phi\cdot\psi, d]$ are

(i) $\Phi = x$, $[S_0 \rightarrow x\cdot y(xy, S, \varepsilon), \$]$.

(ii) $\Phi = x(x, S, \varepsilon)$, $[S_0 \rightarrow x(x, S, \varepsilon)\cdot, \$]$.

(iii) $\Phi = y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, y_r)$,

$[(y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1})\ldots(y_{r-1}, X_r, y_r)\cdot (y_r, X_{r+1}, y_{r+1})\ldots(y_{n-1}, X_n, y_n), d]$,

where $[A \rightarrow X_{m+1}\ldots X_r\cdot X_{r+1}\ldots X_n, y_n d] \in \langle X_1\ldots X_r\rangle_{k+1}$.

(iv) $\Phi = y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, ax)$ and

$[(ax, a, xb) \rightarrow \cdot b, d]$

where $[A \rightarrow \alpha\cdot a\beta, y'] \in \langle X_1\ldots X_r\rangle_{k+1}$

and $xbd \in First_{k+1}(\beta y')$.

(v) $\Phi = y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, ax)b$ and

$[(ax, a, xb) \rightarrow \cdot b, d]$ where... same to (iv).

(vi) $\Phi = y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, ax)b$ and

$[(ax, a, xb) \rightarrow \cdot, d]$

where $[A \rightarrow \alpha\cdot a\beta, y'] \in \langle X_1\ldots X_r\rangle_{k+1}$

and $x\$ \in First_{k+1}(\beta y')$.

Proof *cases on the form of U.*

Case1: $U = S_0$. Then (i) or (ii) is true.

Case2: $U = (x, A, y)$. Then

$$S_0' \underset{rm}{\Rightarrow} S_0\$ \underset{rm}{\Rightarrow} y_0(y_0, S, \varepsilon)\$ \underset{rm}{\Rightarrow}^* y_0\gamma(x, A, y)z\$$$

$$\underset{rm}{\Rightarrow} y_0\gamma\phi\psi z\$ = \Phi\psi z\$ \text{ in } T_{k,1}(G)'$$

and $1{:}z\$ = d$.

$\therefore (y_0, S, \varepsilon) \underset{rm}{\Rightarrow}^* \gamma(x, A, y)z$ in $T_{k,1}(G)$.

$\gamma(x, A, y)z = (y_0, X_1, Y_1)\ldots(y_{m-1}, X_m, x)(x, A, y)z$

and $S \underset{rm}{\Rightarrow}^* X_1\ldots X_m Ayz$ in G.

If $y_m = x$, $y_n = y$, $A \rightarrow X_{m+1}\ldots X_n \in P$,

$U \rightarrow \phi\psi$

$= (y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1})\ldots(y_{n-1}, X_n, y_n)$.

Then

$$S' \underset{rm}{\Rightarrow} X_1\ldots X_m Ayz\$ \underset{rm}{\Rightarrow} X_1\ldots X_m X_{m+1}\ldots X_n yz\$ \text{ in } G'$$

and $[A \rightarrow X_{m+1}\ldots X_r\cdot X_{r+1}\ldots X_n, k+1{:}yz\$]$

$$\langle X_1\ldots X_r\rangle_{k+1}.$$

$\therefore$ (iii) is true.

Case3: $U = (ax, a, xb)$.

By lemma 6.72,

$(y_0, S, \varepsilon)\$ \underset{rm}{\Rightarrow}^+ y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, ax)(ax, a, xb)z$,

and $^\exists A' \rightarrow X_{i+1}\ldots X_n \in P$,

$(y_0, S, \varepsilon)\$ \underset{rm}{\Rightarrow}^+ y_0(y_0, X_1, y_1)\ldots(y_{i-1}, X_i, y_i)(y_i, A', y_n)u$

$\underset{rm}{\Rightarrow} y_0(y_0, X_1, y_1)\ldots(y_{n-1}, X_n, y_n)u$, and

$(y_{r+1}, X_{r+2}, y_{r+2})\ldots(y_{n-1}, X_n, y_n)u \underset{rm}{\Rightarrow}^* z.$

$\therefore S' \underset{rm}{\Rightarrow}^* X_1\ldots X_i A' y_n u\$ \underset{rm}{\Rightarrow} X_1\ldots X_n y_n u\$$

in G by lemma 6.72.

$[A' \rightarrow X_{i+1}\ldots X_r \cdot a X_{r+2}\ldots X_n,\ k+1{:}y_n u\$]$

$\in \langle X_1\ldots X_r\rangle_{k+1},$

and $X_{r+2}\ldots X_n \underset{rm}{\Rightarrow} v$ *in G,* $v y_n u = y_{r+1} z.$

$xbd = k+1{:}y_{r+1}z\$ \in First_{k+1}(X_{r+2}\ldots X_n y_n u\$)$

$= First_{k+1}(X_{r+2}\ldots X_n(k+1{:}y_n u\$)).$

$\therefore$ *one of (iv) and (v) is true.*

<u>Case4</u>: U = (ax, a, x), similar to Case3, (vi) is true.

Lemma 6.81 *If* $T_{k,\,1}(G)$ *is non-LR(1), then G is non-LR(k+1).*

Proof

Let Φ *be a viable prefix .э.* $I, J \in \langle\Phi\rangle_1$

which cause a conflict.

<u>Case1</u>: $\Phi = x(x, S, \varepsilon)$, $I = [S_0 \rightarrow x(x, S, \varepsilon)\cdot, \$]$,

$J = [(x, A, \varepsilon)\rightarrow (x, S, \varepsilon)\cdot, \$].$

Then $[A\rightarrow S\cdot, \$] \in \langle S\rangle_{k+1}$,

$S' \underset{rm}{\Rightarrow} S\$ \underset{rm}{\Rightarrow}^* A\$ \underset{rm}{\Rightarrow} S\$$ *in G', and* $S \underset{rm}{\Rightarrow}^+ S$ *in G.*

<u>Case2</u>: $\Phi = x(x, S, \varepsilon)$, $I = [S_0 \rightarrow x(x, S, \varepsilon)\cdot, \$]$,

$J = [(x, A, \varepsilon)\rightarrow \cdot, \$].$

Then $[A \rightarrow\cdot, \$] \in \langle S\rangle_{k+1}$, *and* $S' \underset{rm}{\Rightarrow} S\$ \underset{rm}{\Rightarrow}^* A\$ \underset{rm}{\Rightarrow} S\$$ *in G.*

<u>Case3</u>: $\Phi = y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, y_r)$,

$I=[(y_m,A,y_r)\rightarrow(y_m,X_{m+1},y_{m+1})\ldots(y_{r-1}, X_r, y_r)\bullet, d]$

$J=[(y_p,B,y_r)\rightarrow(y_p,X_{p+1},y_{p+1})\ldots(y_{r-1}, X_r, y_r)\bullet, d]$.

Then $[A \rightarrow X_{m+1}\ldots X_r\bullet, d]$ and $[B\rightarrow X_{p+1}\ldots X_r\bullet, d]$ cause a conflict in $\langle X_1\ldots X_r\rangle_{k+1}$.

<u>Case4</u>: $\Phi = y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, ax)$,

$I=[(y_m,A,y_r)\rightarrow(y_m,X_{m+1},y_{m+1})\ldots(y_{r-1}, X_r,ax)\bullet, b]$

$J = [(ax, a, xb) \rightarrow\bullet b, d]$.

Then $[A \rightarrow X_{m+1}\ldots X_r\bullet, axb]$ and $[B \rightarrow\alpha\bullet a\beta, y']$ cause a s-r conflict in $\langle X_1\ldots X_r\rangle_{k+1}$ because $xbd \in First_{k+1}(\beta y')$.

<u>Case5</u>: $\Phi = y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, ax)$,

$I=[(y_m,A,y_r)\rightarrow(y_m,X_{m+1},y_{m+1})\ldots(y_{r-1}, X_r,ax)\bullet, \$]$

$J = [(ax, a, xb) \rightarrow\bullet, \$]$.

Then $[A \rightarrow X_{m+1}\ldots X_r\bullet, ax\$]$ and $[B \rightarrow\alpha\bullet a\beta, y']$ cause a s-r conflict in $\langle X_1\ldots X_r\rangle_{k+1}$ bacause $x\$ \in First_{k+1}(\beta y')$.

Lemma 6.82 *Let*

$X_i \underset{rm}{\Rightarrow}^* v_i$, $y_i \in Follow_k(X_i)$, and $y_{i-1} = k{:}v_i y_i$.

Then

$(y_0, X_1, Y_1)\ldots(y_{n-1}, X_n, y_n) \underset{rm}{\Rightarrow}^* z$,

where $v_1\ldots v_n y_n = y_0 z$.

Lemma 6.83 *Let*

$[A \rightarrow X_{m+1}\ldots X_r{\cdot}X_{r+1}\ldots X_n, y_n d]$
$\in \langle X_1\ldots X_r\rangle_{k+1}$,

$X_i \underset{rm}{\Rightarrow}^* v_i$ and $y_{i-1} = k{:}v_i y_i$, $1 \le {}^\forall i \le n$, $0 \le m \le r \le n$.

Then

(b) $[(y_m, A, y_n) \rightarrow (y_m, X_{m+1}, y_{m+1})\ldots(y_{r-1}, X_r, y_r)\cdot$
$(y_r, X_{r+1}, y_{r+1})\ldots(y_{n-1}, X_n, y_n), d]$
$\in \langle y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$.

Moreover, if $r < n$ *and*

$(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb)$, then

(c) $[(ax, a, xb) \rightarrow \cdot b, 1{:}ud]$
$\in \langle y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$,
where $xbu = v_{i+1}\ldots v_n y_n$.

Similarly, if $r < n$ *and*

$(y_r, X_{r+1}, y_{r+1}) = (ax, a, x)$, then

(d) $[(ax, a, xb) \rightarrow \cdot, \$]$
$\in \langle y_0(y_0, X_1, Y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$.

Proof

(i) By definition, $^{\exists}v \in \Sigma^*$.э.

$S' \Rightarrow S\$ \Rightarrow^* X_1 \ldots X_m Av\$ \Rightarrow X_1 \ldots X_m X_{m+1} \ldots X_n v\$$

and $k+1{:}v\$ = y_n d$.

Then

$(y_0,S,\varepsilon) \Rightarrow^* (y_0,X_1,y_1) \ldots (y_{m-1},X_m,y_m)(y_m,A,y_n)z$,

$(y_m,A,y_n) \rightarrow (y_m,X_{m+1},y_{m+1}) \ldots (y_{n-1},X_n,y_n)$

in $T_{k,1}(G)$.

Then

$S_0' \Rightarrow S_0\$ \Rightarrow y_0(y_0,S,\varepsilon)\$$

$\Rightarrow^* y_0(y_0,X_1,y_1) \ldots (y_{m-1},X_m,y_m)(y_m,A,y_n)z\$$

$\Rightarrow y_0(y_0,X_1,y_1)(y_{n-1},X_n,y_n)z\$$.

$\therefore$ (b) is true.

(ii) Assume that $r < n$, $(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb)$.

Then $(ax, a, xb) \rightarrow b$ in $T_{k,1}(G)$ and

$X_i \Rightarrow^* v_i$, $y_i = k{:}v_{i+1} \ldots v_n y_n \in Follow_k(X_i)$,

$(y_{r+1}, X_{r+2}, y_{r+2}) \ldots (y_{n-1},X_n,y_n) \Rightarrow^* u$ by lem6.82

where $v_{r+2} \ldots v_n y_n = y_{r+1} u = xbu$.

Then by (b) and lemma 6.17, (c) is true.

(iii) in a same manner (d) is true.

Lemma 6.84 *Let* $G = (N, \Sigma, P, S)$ *and* $k \geq 0$.
If G is non-LR(k+1), then $T_{k,1}(G)$ *is non-LR(1).*
Proof
(i) if $S \Rightarrow^+ S$ in G, then $T_{k,1}(G)$ is ambiguous.

$$S \Rightarrow^+ A_1 \ldots A_m S,\ A_i \Rightarrow^* \varepsilon \text{ for all } i.$$

then $\exists x$ in $T_{k,1}(G)$,

$$(x,S,\varepsilon) \Rightarrow^+ (x,A_1,x) \ldots (x,A_m,x)(x,S,\varepsilon).$$

By lemma6.75, $A_i \Rightarrow^* \varepsilon$ implies $(x,A_i,x) \Rightarrow^* \varepsilon$ for all i.
$\therefore$ (x, S, ε) derives itself and $T_{k,1}(G)$ is ambiguous.
(ii)

$$[A \rightarrow X_{m+1} \ldots X_r \cdot, w'],\ [B \rightarrow X_{p+1} \ldots X_r \cdot, w'] \in \langle X_1 \ldots X_r \rangle_{k+1}.$$

then for $y_r d = w'$, $y_i = k{:}v_{i+1}y_{i+1}$,

$$[(y_m, A, y_r) \rightarrow (y_m, X_{m+1}, y_{m+1}) \ldots (y_{r-1}, X_r, y_r)\cdot, d],$$
$$[(y_p, A, y_r) \rightarrow (y_p, X_{p+1}, y_{p+1}) \ldots (y_{r-1}, X_r, y_r)\cdot, d],$$
$$\in \langle y_0 (y_0, X_1, y_1) \ldots (y_{r-1}, X_r, y_r) \rangle_{k+1}.$$

(iii)

$$[A \rightarrow X_{m+1} \ldots X_r \cdot, w_1], [B \rightarrow X_{p+1} \ldots X_r \cdot X_{r+1} \ldots X_n, w_2]$$
$$\in \langle X_1 \ldots X_r \rangle_{k+1},\ w_1 \in First_{k+1}(X_{r+1} \ldots X_n w_2).$$

then $\exists v_i$ .. $X_i \Rightarrow^* v_i$ and $k+1{:}v_{r+1} \ldots v_n w_2 = w_1$.
let $y_i = k{:}v_{i+1}y_{i+1}$, then

$$y_r = k{:}v_{r+1}y_{i+1} = k{:}v_{r+1}v_{r+2}y_{r+2}$$
$$= \ldots = k{:}v_{r+1} \ldots v_n y_n,$$

and $w_1 = k+1{:}v_{r+1}\ldots v_n w_2$
$= k+1{:}v_{r+1}\ldots v_n y_n d_2$.

Then by lem6.83,

$[(y_m, A, y_r) \to (y_m, X_{m+1}, y_{m+1})\ldots(y_{r-1}, X_r, y_r)\cdot, d_1]$
$\in \langle y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$.

If $(y_r, X_{r+1}, y_{r+1}) = (ax, a, xb)$,

$[(ax, a, xb) \to \bullet b, 1{:}ud_2]$
$\in \langle y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$,
$xbu \in v_{r+1}\ldots v_n y_n$.

Then $w_1 = k+1{:}v_{r+1}\ldots v_n y_n d_2 = k+1{:}av_{r+2}\ldots v_n y_n d_2$
$= k+1{:}axbud_2 = k+1{:}y_r bud_2$.

$\therefore\ ^{\exists}$a shift-reduce conflict.

If $(y_r, X_{r+1}, y_{r+1}) = (ax, a, x)$, by lem6.83, $d_2 = \$$,m

$[(ax, a, x) \to \bullet, \$]$
$\in \langle y_0(y_0, X_1, y_1)\ldots(y_{r-1}, X_r, y_r)\rangle_1$.

$y_{r+1} = k{:}v_{r+2}\ldots v_n y_n = v_{r+2}\ldots v_n y_n$,

$y_r = ay_{r+1} = v_{r+1}\ldots v_n y_n$.

$\therefore\ d_1 = \$$ and $^{\exists}$a reduce-reduce conflict.

Theorem 6.85

For any reduced grammar $G = (N, \Sigma, P, S)$ and $k \geq 0$,
$T_{k,1}(G)$ is LR(1) iff G is LR(k+1).

Theorem 6.86 *For $k \geq 1$, any reduced grammar G,*
G can be transformed into G' .϶.
G' is an equivalent grammar,
G' right-to-right covers G, and
G' is LR(1) iff G is LR(k).

Theorem 6.87 *For any alphabet Σ,*
the family of deterministic languages over Σ
coincides with
the family of SLR(1) languages over Σ.