4. Context-free Languages

4.1 Context-free Grammars

Let G = (V, P) be a rewriting system.

G is a ***context free grammar****, $G = (N, \Sigma, P, S)$, if*

$N \cup \Sigma = V$, $N \cap \Sigma = \emptyset$, $S \in N$, *and*

$P \subseteq N \times V^*$.

$A, B, C, \ldots, S$	*nonterminal symbols*	(N)
$a, b, c, \ldots, t$	*terminal symbols*	(Σ)
X, Y, Z	*general symbols*	(V)
u, v, w, x, y, z	*terminal strings*	(Σ^*)
$\alpha, \beta, \gamma, \ldots, \omega$	*general strings*	(V^*)
ε	*empty string*	

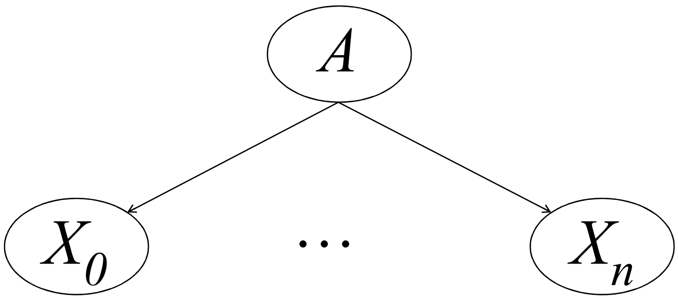
$A \rightarrow \omega_1, \ldots, A \rightarrow \omega_n.$

$A \rightarrow \omega_1 \mid \ldots \mid \omega_n.$

derivation tree (parse tree, syntax tree)

an ***labeled ordered tree****.*

$A \rightarrow X_0 \ldots X_n \in P,\ 0 \le^{\forall} i \le n,\ X_i \in V$



$A < X_i,\ 0 \le i \le n$

$X_i < X_{i+1},\ 0 \le i < n$

sentential forms of *G*

sentential forms of the start symbol S.

(sententail form of $\gamma \in V^*$ *was defined in 1.6)*

derivation of γ ***in*** $G(\gamma \in V^*)$

a derivation of string γ *from S in G.*

sentences of $\gamma(\gamma \in V^*)$

sentential forms of γ *which are terminal strings*

sentences of *G*

sentences of the start symbol S.

language generated by $\gamma(\gamma \in V^*)$ ***in*** *G*

$L_G(\gamma) = \{w \in \Sigma^* \mid \gamma \Rightarrow^* w \text{ in } G\}.$

language generated(described) by *G*

$L(G) = L_G(S) = \{w \in \Sigma^* \mid S \Rightarrow^* w \text{ in } G\}.$

language L ***over*** Σ *is* ***context-free****, if*

$\exists$ *cfg* $G = (N, \Sigma, P, S)$ *such that* $L(G) = L$.

size of *G*

$|G| = \sum |P|$

norm of *G*

$||G|| = |G| \cdot log_2|V|$

storing a grammar G in space O(|G|)

two linear lists, one for lhs, the other for rhs

4.2 Leftmost and Rightmost Derivation

Let $A \rightarrow \omega \in P$. *Then*

$\Rightarrow_{lm} \equiv \{(xA\beta, x\omega\beta) \mid x \in \Sigma^*, \beta \in V^*\}$.

$\Rightarrow_{rm} \equiv \{(\alpha Ay, \alpha\omega y) \mid \alpha \in V^*, y \in \Sigma^*\}$.

γ_1 ***left(right)most derives*** γ_2 *in G using* <u>*rule*</u> *r*

$\gamma_1 \Rightarrow^r_{lm} \gamma_2$, $(\gamma_1 \Rightarrow^r_{rm} \gamma_2)$ $\gamma_1, \gamma_2 \in V^*, r \in P$.

γ_1 ***left(right)most derives*** γ_2 *in G using* <u>*rule string*</u> π

$\gamma_1 \Rightarrow^\pi_{lm} \gamma_2$, $(\gamma_1 \Rightarrow^\pi_{rm} \gamma_2)$ $\gamma_1, \gamma_2 \in V^*, \pi \in P^*$.

$\Rightarrow_{lm} = \cup_{r \in P} \Rightarrow^r_{lm}$

$\Rightarrow_{rm} = \cup_{r \in P} \Rightarrow^r_{rm}$

γ_1 <u>***directly***</u> ***left(right)most derives*** γ_2 *in G, if*

$\gamma_1 \Rightarrow_{lm} \gamma_2$, $(\gamma_1 \Rightarrow_{rm} \gamma_2)$.

γ_1 ***left(right)most derives*** γ_2 *in G, if*

$\gamma_1 \Rightarrow^*_{lm} \gamma_2$, $(\gamma_1 \Rightarrow^*_{rm} \gamma_2)$.

γ_2 *is a* ***left(right) sentential form of*** γ_1.

left(right)sentential form of *G*

left(right) sentential form of the start symbol S.

Let $n \geq 0$, $\gamma_i \in V^*$, $0 \leq i \leq n$. Then ***left(right)most derivation*** *of length n from γ_0 to γ_n.*

A string sequence $(\gamma_0, \ldots, \gamma_n)$

.э. $0 \leq^{\forall} i \leq n$, $\gamma_i \Rightarrow_{lm} \gamma_{i+1}$ $(\gamma_i \Rightarrow_{rm} \gamma_{i+1})$.

$\gamma \Rightarrow^{\pi}_{lm} \gamma'$ $(\gamma \Rightarrow^{\pi}_{rm} \gamma')$, $\pi = r_1 \ldots r_n$, $0 \leq^{\forall} i \leq n$, $r_i \in P$, *iff*

$0 \leq^{\forall} i \leq n$, $\gamma_i \Rightarrow^{r_i}_{lm} \gamma_{i+1}$ $(\gamma_i \Rightarrow^{r_i}_{rm} \gamma_{i+1})$, $\gamma = \gamma_0$, $\gamma' = \gamma_n$.

left(right)most derivation *of γ in G*

left(right)most derivation of γ from S in G.

left(right) sentence *of G*

left(right) sentential forms of terminal string.

left sentence = right sentence

Lemma 4.1 *Let* $n \geq 0$, $m \geq 1$,

$1 \leq^{\forall} i \leq m$, $\alpha_i \in V^*$, $\beta \in V^*$,

$\alpha_1 \ldots \alpha_m \Rightarrow^n \beta$, $m \geq 1$, $n \geq 0$. *Then*

$^{\exists} n_1, \ldots, n_m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in V^*$ *such that*

$$\alpha_i \Rightarrow^{n_i} \beta_i, \sum^m n_i = n, \text{ and } \beta_1 \ldots \beta_m = \beta.$$

Proof *Induction on n*

i) $n = 0$, $n_1 = \ldots = n_m = 0$ and $\alpha_i = \beta_i$.

ii) $n > 0$, $A \rightarrow \omega \in P$

$\alpha_1 \ldots \alpha_m = \delta A \eta \Rightarrow \delta \omega \eta = \gamma_1 \ldots \gamma_m \Rightarrow^{n-1} \beta$.

$1 \leq^{\exists} k \leq m$, .э. $\alpha_k = \mu A \nu$, $\gamma_k = \mu \omega \nu$; $\mu, \nu \in V^*$, and

$1 \leq^{\forall} i \neq k \leq m$, $\alpha_i = \gamma_i$.

By IH $^{\exists} n'_1, \ldots, n'_m$, $\beta_1, \ldots, \beta_m$ such that

$$\gamma_i \Rightarrow^{n'_i} \beta_i, \sum^m n'_i = n\text{-}1, \text{ and } \beta_1 \ldots \beta_m = \beta.$$

$\alpha_k = \mu A \nu \Rightarrow \mu \omega \nu = \gamma_k$.

$\therefore$ $n_i = n'_i$, if $i \neq k$; and $n_k = n'_k + 1$

$\therefore$ $\alpha_i \Rightarrow^{n_i} \beta_i$, $\sum^m n_i = n$.

Theorem 4.2 *Let* γ *be a general string*($\gamma \in V^*$). *Then*
$\{w \in \Sigma^* \mid \gamma \Rightarrow^n_{lm} w\} = \{w \in \Sigma^* \mid \gamma \Rightarrow^n w\} = \{w \in \Sigma^* \mid \gamma \Rightarrow^n_{rm} w\}$

Proof

$\{w \in \Sigma^* \mid \gamma \Rightarrow^n_{lm} w\} \subseteq \{w \in \Sigma^* \mid \gamma \Rightarrow^n w\}$ *is trivial,*
since $\Rightarrow_{lm} \subseteq \Rightarrow$.

1. prove $X \Rightarrow^n u$ *implies* $X \Rightarrow^n_{lm} u$.

i) if $n=0$, *it is trivial*

ii) if $n>0$

$\exists X \Rightarrow X_1 \ldots X_m \Rightarrow^{n-1} u$

if $m=0$ *it is trivial* ($u=\varepsilon$)

if $m>0$, $\exists m \geq 1$: $u=u_1 \ldots u_m$, $X_i \Rightarrow^{n_i} u_i$, $\sum^m n_i = n-1$. ***L4.1***

Furthermore $X_i \Rightarrow^{n_i}_{lm} u_i$ *by IH*

$\therefore X \Rightarrow_{lm} X_1 \ldots X_m \Rightarrow^{n-1}_{lm} u$.

2. prove $\gamma = Y_1 \ldots Y_k \Rightarrow^n w$ *implies* $\gamma \Rightarrow^n_{lm} w$, $Y_i \in V$.

$Y_i \Rightarrow^{n_i} w_i$, $\sum^k n_i = n$ *by Lemma 4.1.*

$Y_i \Rightarrow^{n_i}_{lm} w_i$, *by the above condition **1**.*

$\therefore \gamma = Y_1 Y_2 \ldots Y_k \Rightarrow^{n_1}_{lm} w_1 Y_2 \ldots Y_k \Rightarrow^{n_2}_{lm} \ldots \Rightarrow^{n_k}_{lm} w_1 w_2 \ldots w_k$

$\therefore \gamma \Rightarrow^n_{lm} w$ $\therefore$ $\{w \in \Sigma^* \mid \gamma \Rightarrow^n_{lm} w\} \supseteq \{w \in \Sigma^* \mid \gamma \Rightarrow^n w\}$

Furthermore $\{w \in \Sigma^* \mid \gamma \Rightarrow^n w\} = \{w \in \Sigma^* \mid \gamma \Rightarrow^n_{rm} w\}$ *is dual*

Corollary 4.3 *L(G) coincides with the set of left sentences and right sentences of G.*

4.3 Ambiguity of Grammars

*A grammar is **ambiguous**, iff some sentence has **more than one** leftmost derivation.*

$S \rightarrow a \mid$ **ifcl** $S \mid$ **ifcl** S **else** S

$E \rightarrow a \mid E + E \mid E * E$

$S \rightarrow a \mid S$ *infinite ambiguity*

*A context-free language is **inherently ambiguous**, if every grammar is ambiguous.*

Proposition 4.4(*Parikh (1966)) There exist inherently ambiguous language; one example is the language* $\{a^i b^j c^k \mid i=j \text{ or } j=k\}$.

ambiguity of grammar

leftmost derivation *definition*

rightmost derivation *theorem(proof)*

Fact 4.5 *Let* $\pi \in P^*$.

(1) *if* $\alpha \Rightarrow^{\pi}_{lm} \beta$ *and* $\alpha \Rightarrow^{\pi}_{lm} \beta'$, *then* $\beta = \beta'$.

(2) *if* $\alpha \Rightarrow^{\pi}_{rm} \beta$ *and* $\alpha \Rightarrow^{\pi}_{rm} \beta'$, *then* $\beta = \beta'$.

same rule string $\Rightarrow$ *same sentential form*

Lemma 4.6 *Let* $m \geq 1$, $\pi \in P^*$ *and* $\alpha_1 \ldots \alpha_m \Rightarrow_{lm}^{\pi} w \in \Sigma^*$.
Then $\exists$***unique*** $\pi_1, \ldots, \pi_m$ *and* $w_1, \ldots, w_m$ *such that*

$$\pi = \pi_1 \ldots \pi_m,\ w = w_1 \ldots w_m,\ \alpha_i \Rightarrow_{lm}^{\pi_i} w_i\ 1 \leq i \leq m.$$

Proof

*The existence of such strings are trivial.(**L4.1**)*
Then we should prove the uniqueness.

$\pi_1 \ldots \pi_m = \pi'_1 \ldots \pi'_m = \pi,$
$w_1 \ldots w_m = w'_1 \ldots w'_m = w,$
$\alpha_i \Rightarrow_{lm}^{\pi_i} w_i$, *and* $\alpha_i \Rightarrow_{lm}^{\pi'_i} w'_i$ *for* $1 \leq i \leq m$.
show that $\pi_i = \pi'_i$ *and* $w_i = w'_i$ *for* $1 \leq i \leq m$.

Induction on $m \geq 1$.*(m=1, trivial)*
indunction hypothesis

$\pi_1 \ldots \pi_{j-1} = \pi'_1 \ldots \pi'_{j-1}$ *and* $w_1 \ldots w_{j-1} = w'_1 \ldots w'_{j-1}$.

π_j *is a prefix of* π'_j; *or vice versa*
assume w.o.l.g $\pi'_j = \pi_j \pi''_j$

$\alpha_j \Rightarrow_{lm}^{\pi'_j} w_j'$ *and* $\alpha_j \Rightarrow_{lm}^{\pi_j} w_j \therefore \alpha_j \Rightarrow_{lm}^{\pi_j} w_j \Rightarrow_{lm}^{\pi''_j} w''_j$
$w_j' = w''_j$ *by **Fact 4.5**, and since* $w_j \in \Sigma^*$, $w_j = w''_j$.
so, $\pi''_j = \varepsilon$, $\pi_j = \pi'_j$, *and* $w_j = w_j'$.

Lemma 4.7 *Let* $m \geq 1$ *and* $\alpha_1 \ldots \alpha_m \Rightarrow_{rm}^{\pi} w \in \Sigma^*$. *Then*
$\exists$***unique*** $\pi_1, \ldots, \pi_m$ *and* $w_1, \ldots, w_m$ *such that*

$$\pi = \pi_m \ldots \pi_1,\ w = w_1 \ldots w_m,\ \alpha_i \Rightarrow_{rm}^{\pi_i} w_i\ 1 \leq i \leq m.$$

Lemma 4.8 *Let* $A \in N, \pi \in P^+$. *Then*

$A \Rightarrow^r X_1 \ldots X_m \Rightarrow_{lm}^{\pi'} w$ *where* $\pi = r\pi', r \in P, \pi' \in P^*$.

Moreover there exist a ***unique*** *strings*

$\pi'_1 \ldots \pi'_m = \pi', w_1 \ldots w_m = w$, *such that*

$X_i \Rightarrow_{lm}^{\pi'_i} w_i$ *for* $1 \le i \le m$.

Define *a function f: leftmost deri.* $\rightarrow$ *rightmost deri.*

$f: P^* \rightarrow P^*$.

(1) $f(\varepsilon) = \varepsilon$.

(2) $f(\pi) = f(r \cdot \pi_1 \ldots \pi_m) = r \cdot f(\pi_m) \ldots f(\pi_1)$,

if $r = A \rightarrow X_1 \ldots X_m$, $\pi = r \cdot \pi_1 \ldots \pi_m$, *and*

$\pi_i \in P^*$ *for* $1 \le i \le m (\pi \in P^+)$

Lemma 4.9 *Let* $X \in V \cup \{\varepsilon\}$.

If $X \Rightarrow_{lm}^{\pi} w \in \Sigma^*$ *then* $X \Rightarrow_{rm}^{f(\pi)} w$.

Proof

i) $\pi = \varepsilon$, *trivial*$(f(\varepsilon) = \varepsilon)$.

ii) $|\pi| > 0, \pi = r\pi', X \Rightarrow_{lm}^{r} X_1 \ldots X_m \Rightarrow_{lm}^{\pi'} w$

if $\pi' = \varepsilon$, $X \Rightarrow_{rm}^{f(r)} w$, *since* $f(r) = r$.

if $\pi' \neq \varepsilon$, $\pi'_1 \ldots \pi'_m = \pi'$, $w_1 \ldots w_m = w$, $X_i \Rightarrow_{lm}^{\pi'_i} w_i$ ***(L4.6)***

by IH, $X_i \Rightarrow_{rm}^{f(\pi'_i)} w_i$, *and* $X \Rightarrow_{rm}^{r} X_1 \ldots X_m \Rightarrow_{rm}^{f(\pi'_m) \ldots f(\pi'_1)} w$

$\therefore X \Rightarrow_{rm}^{r f(\pi'_m) \ldots f(\pi'_1)} w \; (= \Rightarrow_{rm}^{f(\pi)} w)$.

Let $w \in \Sigma^*$. *For any* $X \in V \cup \{\varepsilon\}$,

$\Pi_L(X) = \{\pi \mid X \Rightarrow^{\pi}_{lm} w\}$.

Let $\Pi_L = \cup_{X \in V \cup \{\varepsilon\}} \Pi_L(X)$.

$\Pi_R(X) = \{\pi \mid X \Rightarrow^{\pi}_{rm} w\}$.

$\Pi_R = \cup_{X \in V \cup \{\varepsilon\}} \Pi_R(X)$.

f: $\Pi_L(\subseteq P^*) \rightarrow \Pi_R(\subseteq P^*)$.

Lemma 4.10 *Let* $X \in V \cup \{\varepsilon\}$. *If* $X \Rightarrow^{\pi}_{rm} w \in \Sigma^*$ *then*

$X \Rightarrow^{\pi'}_{lm} w$ *where* $\pi = f(\pi')$.

f is onto.

Lemma 4.11 *Let* $X \in V \cup \{\varepsilon\}$. *If* $X \Rightarrow^{\pi}_{rm} w \in \Sigma^*$ *then*

$X \Rightarrow^{\pi'}_{lm} w$, $X \Rightarrow^{\pi''}_{lm} w$, *and* $f(\pi') = f(\pi'') = \pi$,

implies that $\pi' = \pi''$.

f is one-to-one.

$\therefore$ *f is one-to-one onto.*

Since f is a bijection,
ono-to-one correspondence
between left and right parses

Theorem 4.12 *Let G = (N, Σ, P, S) be a grammar. G is* ***ambiguous*** *iff some sentence in G has* ***more than one rightmost derivations****.*

Proof

→) *Assume G is ambiguous, then w∈L(G) has at least two leftmost derivation.*

$$S \Rightarrow_{lm}^{\pi} w \text{ and } S \Rightarrow_{lm}^{\pi'} w$$

$$S \Rightarrow_{rm}^{f(\pi)} w \text{ and } S \Rightarrow_{rm}^{f(\pi')} w$$

by Lemma 4.11, $f(\pi) \neq f(\pi')$ *because* $\pi \neq \pi'$

∴ *w has more than one rightmost derivation*

←) *w∈L(G) has at least two rightmost derivation*

$$S \Rightarrow_{rm}^{\pi} w \text{ and } S \Rightarrow_{rm}^{\pi'} w$$

by Lemma 4.10, $S \Rightarrow_{lm}^{f^{-1}(\pi)} w$ *and* $S \Rightarrow_{lm}^{f^{-1}(\pi')} w$

π *and* π' *are distinct, so* $f^{-1}(\pi)$ *and* $f^{-1}(\pi')$ *are distinct*

∴ *G is ambiguous.*

4.4 Useless and Nullable Symbols

*A symbol $X \in V$ is **useful** if*

$$S \Rightarrow^* \alpha X \beta \Rightarrow^* w,\ \alpha\beta \in V^*,\ w \in \Sigma^*.$$

*Otherwise **useless**.*

*A grammar G is **reduced**, if*

*it contains **no useless** symbols.*

*A nonterminal $A \in N$ is **nullable**, if $A \Rightarrow^+ \varepsilon$ in G.*

*A rule of the form $A \rightarrow \varepsilon$ is called an **ε-rule**.*

*A grammar having no ε-rule is called **ε-free**.*

$W = \{A \mid A \Rightarrow^* \varepsilon\}$

$W_1 = \{A \mid A \rightarrow \varepsilon \in P\}$

$W_k = \{A \mid A \notin \cup_{i=1}^{k-1} W_i,\ A \rightarrow A_1 \ldots A_n \in P,\ {}^\forall A_j \in \cup_{i=1}^{k-1} W_i\}$

$({}^\exists j\ .\!\ni.\ A_j \in W_{k-1})$

Lemma 4.13 $W = \cup_{i=1}^{|W|} W_i$.

Proof

$W = \cup_{i=1}^{\infty} W_i$

if $A \in W_k$, $k>1$, then $A \notin \cup_{i=1}^{k-1} W_i$,

${}^\exists A \rightarrow A_1 \ldots A_n$ such that ${}^\forall A_j \in \cup_{i=1}^{k-1} W_i$

some A_j must be in W_{k-1}, thus $W_{k-1} \neq \emptyset$ if $W_k \neq \emptyset$

Theorem 4.14 *The set of nullable nonterminals of a grammar G can be computed in time(O|G|).*

$W_1 := \{A \mid A \rightarrow \varepsilon \in P\};$
$k := 1;$
while $W_k \neq \emptyset$ ***do***
    mark A's in W_k;
    $W_{k+1} := \{A \mid A \rightarrow A_1 \ldots A_m \in P, {}^{\forall}A_i$: marked,
            A: not marked$\}$;
    $k := k + 1$
    od;
$W = \emptyset;$
for $i:=1$ ***to*** $k-1$ ***do*** $W := W \cup W_k$ ***od.***

Lemma 4.15 *Let* $G = (N, \Sigma, P, S)$ *and* $G' = (N, \emptyset, P', S)$ *where*

$$P' = \{A \rightarrow \alpha_1 \ldots \alpha_n \mid A \rightarrow x_0 \alpha_1 x_1 \ldots \alpha_n x_n \in P, 1 \leq {}^{\forall} i \leq n, \alpha_i \in N^+, 0 \leq {}^{\forall} j \leq n, x_j \in \Sigma^*\}.$$

Then $A \Rightarrow^* x \in \Sigma^*$ *in* G, *iff* $A \Rightarrow^* \varepsilon$ *in* G'.

Let ***contains*** *be a relation on* V *defined by*
    A ***contains*** X *if* $A \rightarrow \alpha X \beta \in P$, $A \in N$, $X \in V$.
Lemma 4.16 S ***contains**** X, *iff* $S \Rightarrow^* \alpha X \beta$, $\alpha, \beta \in V^*$.
Proof *If* $X \in$ ***contains***$^n(S)$, $S \Rightarrow^n \alpha X \beta$ *for* $n \geq 0$.
    If $S \Rightarrow^n \alpha X \beta$, $X \in$ ***contains***$^m(S)$, *for* $0 \leq m \leq n$.

Theorem 4.17 *Any grammar G can be transformed into an equivalent* ***reduced*** *grammar in time* $O(|G|)$.

Guarded Commands, Nondeterminancy, and Formal Derivation of Programs

E.W. Dijkstra, CACM 18,8 pp.453-457, (Aug. 75)

A Discipline of Programming, *Prentice-Hall, 1976*

Concurrent assignment

$x, y := y, x$ $\qquad$ $x := y; y := x;$

$\qquad\qquad\qquad\qquad$ $y := x; x := y$

$t := x; x := y; y := t$

$x_1, \ldots, x_n := E_1, \ldots, E_n.$

Nondeterminancy

if $x \geq y \rightarrow m := x$

| $x \leq y \rightarrow m := y$

fi

If $x{=}y$, $m := x$ ***or*** $m := y$

Repeatative construct

$i, S := 1, 0;$ ***do*** $i \leq 100 \rightarrow S := S + i; i := i + 1$ ***od***

$1 \leq {}^{\forall}i \leq 101: S = \Sigma_{j=1}^{i-1} j$

If $i = 101, S = \Sigma_{j=0}^{100} j$

$i, S := 0, 0;$ ***do*** $i \leq 99 \rightarrow i := i + 1; S := S + i$ ***od***

$0 \leq {}^{\forall}i \leq 100: S = \sum_{j=0}^{i} j$

If $i = 100, S = \Sigma_{j=0}^{100} j$

$q_1, q_2, q_3, q_4 := Q_1, Q_2, Q_3, Q_4;$

do $q_1 > q_2 \rightarrow q_1, q_2 := q_2, q_1$

| $q_2 > q_3 \rightarrow q_2, q_3 := q_3, q_2$

| $q_3 > q_4 \rightarrow q_3, q_4 := q_4, q_3$

od.

$P = 1 \leq {}^{\forall}i \leq 4$: q_i's are permutation of Q_i

$\neg BB = \neg((q_1 > q_2) \vee (q_2 > q_3) \vee (q_3 > q_4))$

$= \neg(q_1 > q_2) \wedge \neg(q_2 > q_3) \wedge \neg(q_3 > q_4))$

$= (q_1 \leq q_2) \wedge (q_2 \leq q_3) \wedge (q_3 \leq q_4)$

$= (q_1 \leq q_2 \leq q_3 \leq q_4)$

P Loop invariance condition

BB There exists at least one guard that is true

*If P is true before the loop(**initialization** of P),*
*P remains true in the loop(loop **invariance**), and*
*the loop terminates(loop **terminating**),*
then $P \wedge \neg BB$ is true after the loop.

if $X > 0$ **and** $Y > 0 \rightarrow$

$x, y := X, Y;$

do $x > y \rightarrow x := x - y$

| $x < y \rightarrow y := y - x$ **od fi**; *print(x)*.

Theorem 4.18 *Let $G = (N, \Sigma, P, S)$ be a grammar and $m \geq 1$ is the length of the longest rule in P. Then G can be transformed in time $O(2^m \cdot |G|)$ into a grammar $\hat{G} = (N, \Sigma, \hat{P}, S)$ satisfying*

(1) $\hat{G}$ is ***ε-free****.*

(2) $L_{\hat{G}}(X) = L_G(X) \setminus \{\varepsilon\}$ for all $X \in V$.

Proof

$\hat{P} = \{A \to \alpha_1 \ldots \alpha_n \mid \exists A_1, \ldots, A_{n-1} .\ni. 1 \leq \forall i < n, A_i \Rightarrow^* \varepsilon,$

$A \to \alpha_1 A_1 \ldots \alpha_{n-1} A_{n-1} \alpha_n \in P, n \geq 1, \alpha_1 \ldots \alpha_n \neq \varepsilon\}$

If $\alpha \Rightarrow_{\hat{G}} \beta$, *then* $\alpha \Rightarrow_G^+ \beta$. $\therefore L_{\hat{G}}(X) \subseteq L_G(X)$.

If $X \Rightarrow_G^n w$, *then* $X \Rightarrow_{\hat{G}}^* w$. $X \in V$ *and* $w \in \Sigma^+$.

assume $X \Rightarrow_G^{n'} w$ *implies* $X \Rightarrow_{\hat{G}}^* w$ *when* $n' < n$.

$X \Rightarrow_G X_1 \ldots X_k \Rightarrow_G^{n-1} w \in \Sigma^+$.

$n\text{-}1 = \sum^k n_i$, $w = w_1 \ldots w_k$, $X_i \Rightarrow_G^{n_i} w_i$ $1 \leq i \leq k$ ***(L4.6)***

Let $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\} (1 \leq \forall m < l, i_m < i_{m+1})$

where $w_{i_1} \ldots w_{i_l} = w$ *and* $(1 \leq \forall j < l, w_{i_j} \neq \varepsilon)$

by IH, $X_{i_j} \Rightarrow_{\hat{G}}^* w_{i_j}$, $X_{i_1} \ldots X_{i_l} \Rightarrow_{\hat{G}}^* w_{i_1} \ldots w_{i_l} = w$

by construction, $\hat{G}$ *has a rule* $X \to X_{i_1} \ldots X_{i_l}$.

$X \Rightarrow_{\hat{G}}^* w$, $\therefore L_G(X) \setminus \{\varepsilon\} \subseteq L_{\hat{G}}(X)$

$\hat{G}$ has at most 2^k rules for $A \to X_1 \ldots X_k$

The transform requires $O(2^m \cdot |G|)$ time.

4.5 Canonical Two-Form Grammar

A Grammar $G = (N, \Sigma, P, S)$ is in ***canonical two-form*** *if its rules are of the forms*

$$A \to BC, \quad A \to B, \quad A \to a, \quad S \to \varepsilon.$$

Furthermore, if $S \to \varepsilon \in P$, S may not occur in the right-hand side of any rule.

Lemma 4.19 *Any Grammar $G = (N, \Sigma, P, S)$ can be transformed in Time $O(|G|)$ into $G' = (N', \Sigma, P', S')$ satisfying*

(1) P' are of the forms

$$A \to BC, \quad A \to B, \quad A \to a, \quad A \to \varepsilon.$$

(2) $N \subseteq N'$

(3) $L_{G'}(X) = L_G(X)$ for all $X \in V$

Proof

Let $A \to X_1 \dots X_m \in P$, $m \geq 2$. Then

$$[A] \to [X_1][X_2 \dots X_m],$$
$$[X_2 \dots X_m] \to [X_2][X_3 \dots X_m],$$
$$\dots,$$
$$[X_{m-1}X_m] \to [X_{m-1}][X_m],$$

and

$$[X_i] \to X_i \qquad 1 \leq i \leq m.$$

$$N' = N \cup \{[X\beta] \mid A \to \alpha X\beta \in P,\ \alpha, \beta \in V^+\}$$
$$\cup \{[X] \mid X \in V,\ A \to \alpha X\beta \in P\}.$$

$P' = \{A \to \alpha \mid A \to \alpha \in P, |\alpha| \le 1\}$
$\cup \{[A] \to [X][\beta] \mid A \to X\beta \in P, \beta \in V^+\}$
$\cup \{[X\beta] \to [X][\beta] \mid A \to \alpha X\beta \in P, \alpha,\beta \in V^+\}$
$\cup \{[X] \to X \mid A \to \alpha X\beta \in P\}$.

i) $L_G(X) \subseteq L_{G'}(X)$ *is trivial, since*
$\alpha \Rightarrow_G \beta$ *implies* $\alpha \Rightarrow^+_{G'} \beta$.
ii) $L_{G'}(X) \subseteq L_G(X)$.
(case1) $X \Rightarrow^n_{G'} w$ *implies* $X \Rightarrow^*_G w$, *and*
(case2) $[\alpha] \Rightarrow^n_{G'} w$ *implies* $\alpha \Rightarrow^*_G w$.

The case n=0 is trivial.
IH: Assume that n > 0 and it holds for less than n.
case 1) Consider $X \Rightarrow^n_{G'} w$, *since n>0,*
$X \Rightarrow_{G'} \eta \Rightarrow^{n-1}_{G'} w$.
if $\eta=\varepsilon$, $X \to \varepsilon \in P$, *by construction*
if $|\eta|=1$, $\eta=Y$
$Y \Rightarrow^*_G w$ *(IH case1) and* $X \to Y \in P$ *(construction)*
$\therefore X \Rightarrow^*_G w$.
if $|\eta|>1$, $\eta = [Y][\beta]$
$X \to Y\beta \in P$, *by construction*
$Y\beta \Rightarrow^*_G w$, *by Lemma 4.6 and IH case2*
$\therefore X \Rightarrow^*_G w$.

case 2) Consider $[\alpha] \Rightarrow^{n}_{G'} w$. *Then,*

$[\alpha] \Rightarrow_{G'} \eta \Rightarrow^{n-1}_{G'} w$, *for some* $\eta \in V'^*$

if $|\alpha|=1$, $\eta=\alpha$ *and by IH case1*

$\therefore \alpha \Rightarrow^{*}_{G} w$

if $\alpha=X\gamma$, $X \in V$ *and* $\gamma \in V^+$

$[X\gamma] \Rightarrow_{G'} [X][\gamma] \Rightarrow^{n-1}_{G'} w$, *by construction*

$X \Rightarrow^{*}_{G} w_1$ *and* $\gamma \Rightarrow^{*}_{G} w_2$, *where* $w=w_1w_2$

by Lemma 4.6 and IH case2

$\therefore X\gamma \Rightarrow^{*}_{G} w$, *by Lemma 4.6*

$^{\forall}A \rightarrow X_1 \ldots X_m \in P$ *of G of length m+1*

G' has rules of length 3(m-1) + 2m = 5m - 3.

$\therefore$ *Construction of G' is O(|G|).*

Theorem 4.20 *Any G can be transformed into G'*

(1) G' is ***canonical two-form****,*

(2) $N \subseteq N'$,

(3) $L_{G'}(X) = L_G(X) \setminus \{\varepsilon\}\ ^{\forall}X \in V$,

(4) $L_{G'}(S) = L_G(S)$

Proof *$G_1 = (N_1, \Sigma, P_1, S)$ in Lemma 4.19 in $O(|G|)$*

*Find nullable nonterminal of G_1 in $O(|G|)$**(T4.14)***

Transform G_1 into ε-free $G_2 = (N_1, \Sigma, P_2, S)$,

$$L_{G_2}(X) = L_{G_1}(X) \setminus \{\varepsilon\}, \ ^{\forall}X \in V_1.$$

$$O(2^m \cdot |G_1|) = O(2^2 \cdot |G_1|) = O(|G|), \ (\mathbf{T4.18})$$

If S is not nullable condition (4) holds, otherwise,

$$G' = (N_1 \cup \{S'\}, \ \Sigma, P_2 \cup \{S' \to S, S' \to \varepsilon\}, S').$$

A grammar G is ***Chomsky normal form****, if P are*

$A \to BC$, $A \to a$, and $S \to \varepsilon$.

Canonical two-form → Chomsky normal form

Elimination rule of form

$A \to B$ ***unit rules(single rules)***

$^{\forall}A \to BC$, $B \Rightarrow^* B'$, $C \Rightarrow^* C'$.

$A \to B'C'$

eliminating all rules of the form $A \to B$

$O(|G|)$ *for* $B \Rightarrow^* B'$

$O(|N|^2)$ *for* $A \to B'C'$

$O(|G| \cdot |N|^2)$

No better algorithm is known.

4.6 Derivational Complexity

$TIME_G(w) = TIME_G(S, w)$

$SPACE_G(w) = SPACE_G(S, w)$

$O(|w|)$

Consider an ε-free grammar G_n with rules

$A_1 \rightarrow A_2,$

$A_2 \rightarrow A_3,$

...,

$A_n \rightarrow a \mid A_1A_1$

$L(A_1) = a^+,$

$TIME(a^k)=2nk-n$

$SPACE(a^k)=k$

Theorem 4.21 *An ε-free grammar G, |N|=n.*
If $X \in V$, $w \in L(X)$, then TIME(X,w)=2n|w|-n and SPACE(X,w)=|w|. And, these bounds are minimal.

Proof

i) SPACE(X,w) = |w|, because no step decreases the length of the sentential form in ε-free grammar

ii) TIME(X,w) = 2n|w|-n
by induction on |w|
if |w|=1, $X \Rightarrow^{\pi} w$, π contains only $A \rightarrow B$ or $A \rightarrow a$
$|\pi| \le n = 2n|w|-n$, because two rules with same lhs in π make unnecessary loop.

if |w|>1, $X \Rightarrow^{*} w$

$\exists r = A \rightarrow X_1 \ldots X_m$

$X \Rightarrow^{\pi'} A \Rightarrow^{r} X_1 \ldots X_m \Rightarrow^{\pi} w$

$\pi = \pi_1 \ldots \pi_m$, $w = w_1 \ldots w_m$, $X_i \Rightarrow^{\pi_i} w_i$ **(L4.6)**

π' contains only the form $A \rightarrow B$, so $|\pi'| \le n-1$

$|\pi_i| \le 2n|w_i|-n$, by IH

$$|\pi' r \pi| = |\pi'| + 1 + \sum_{i=1}^{m} |\pi_i|$$
$$\le (n-1) + 1 + \sum_{i=1}^{m} (2n|w_i|-n)$$
$$= 2n|w|\ (1 - m)n$$
$$\le 2n|w| - n$$

The grammar G_n, $n \ge 1$, show this bound is minimal

Consider a grammar $G_{n,m}$, $n \geq 1$, $m \geq 2$ with rules

$A_1 \rightarrow A_2^{\,m}$,

$A_2 \rightarrow A_3^{\,m}$,

...,

$A_{n-1} \rightarrow A_n^{\,m}$, ,

$A_n \rightarrow \varepsilon$

$$A_i \Rightarrow^{1} A_{i+1}^{\,m} \Rightarrow^{m} A_{i+2}^{\,m^2} \Rightarrow^{m^2} \ldots \Rightarrow^{m^{n-i-1}} A_n^{\,m^{n-i}} \Rightarrow^{m^{n-i}} \varepsilon$$

$$TIME(A_i, \varepsilon) = 1 + m + \ldots + m^{n-i} = (m^{n-i+1} - 1)/(m-1)$$

$$\begin{aligned}
A_i \;&\Rightarrow^*_{lm} A_n^{\,m} A_{n-1}^{\,m-1} \ldots A_{i+1}^{\,m-1} \\
&\Rightarrow^*_{lm} A_n^{\,m} A_{n-1}^{\,m-2} \ldots A_{i+1}^{\,m-1} \\
&\ldots \\
&\Rightarrow^*_{lm} A_n^{\,m} A_{n-1}^{\,m-1} A_{n-2}^{\,m-2} A_{n-3}^{\,m-1} \ldots A_{i+1}^{\,m-1} \\
&\ldots \\
&\Rightarrow^*_{lm} A_n^{\,m} A_{n-1} A_{n-2} \ldots A_{i+1} \\
&\Rightarrow^*_{lm} A_{n-1} A_{n-2} \ldots A_{i+1} \\
&\ldots \\
&\Rightarrow^*_{lm} \varepsilon
\end{aligned}$$

$$SPACE(A_i, \varepsilon) = (n-i)(m-1) + 1$$

$\therefore$ $TIME(A_i, \varepsilon) = (m^{k_i} - 1) / (m-1)$

$SPACE(A_i, \varepsilon) = (k_i - 1)(m-1) + 1$

where $k_i = n - i + 1$.

$W_k = \{A \mid \exists A \Rightarrow^k \varepsilon$, but $\not\exists n$.э. $A \Rightarrow^n \varepsilon, n \leq k\}$

Lemma 4.22 *Let G be a grammar and m≥2 the length of the rhs of the longest rule. Then* $\forall k$ *and* $A \in W_k$ $TIME(A, \varepsilon) = (m^k-1)/(m-1)$ *and*
$SPACE(A, \varepsilon) = (k-1)(m-1)+1$.
And, these bounds are minimal.

Proof *induction on k.*
If $k=1$ and $A \in W_k$, $A \rightarrow \varepsilon \in P$.

$\therefore$ time $1 = (m^1-1)/(m-1)$
space $1 = (1-1)(m-1)+1$.

Assume $k > 1$, $\forall k' < k$, $A' \in W_{k'}$,

time $(m^{k'}-1)/(m-1)$, space $(k'-1)(m-1)+1$.

If $A \in W_k$, $r = A \rightarrow A_1 \ldots A_l \in P$, $\forall A_i \in W_{k_i}$, $k_i < k$,

$A_i \Rightarrow^{\pi_i} \varepsilon$, $|\pi_i| \leq (m^{k_i}-1)/(m-1)$.

$SPACE(A_i, \varepsilon) = (k_i - 1)(m-1)+1$,

$$\therefore TIME(A, \varepsilon) \leq 1 + \sum_{i=1}^{l} (m^{k_i}-1)/(m-1)$$
$$\leq 1 + l(m^{k-1}-1)/(m-1)$$
$$\leq 1 + m(m^{k-1}-1)/(m-1)$$
$$= (m^k-1)/(m-1).$$

4.7 Context-free Language Recognition

*P_{mem}(**G**) : Given a context-free grammar G in the class **G** of all such grammars, and a string w, is w L(G)?*

Theorem 4.27 *given cfg $G=(N, \Sigma, P, S)$ and $w \in \Sigma^*$, $w \in L(G)$?*

$NTIME(O(|G| \cdot |w|))$ and $NSPACE(O(|G|+|w|))$

Proof

assume G is in canonical two-form

guess a derivation $(\gamma_0,\ldots,\gamma_n)$, $\gamma_0=S$, $\gamma_n \in \Sigma^*$, $\gamma_n=w$?

$w=a_1\ldots a_n$, $N(i,j)=\{A|A \text{ derives } a_i\ldots a_j\}$, $1 \le i \le j \le n$

*B **unit-rule** X if $B \to X \in P$*

Fact 4.28 *$N(i,i)=(\textbf{unit-rule}^{-1})^+(a_i)$*

Lemma 4.29 *for all i,j, $1 \le i \le j \le n$,*

$$N(i,j)=\cup_{k=0}^{j-i-1} (\textbf{unit-rule}^{-1})^*(\{A|A \to BC \in P, B \in N(i,i+k), C \in N(i+k+1,j)\})$$

if $S \in N(1,n)$, then $a_1\ldots a_n \in L_G(S)$

Theorem 4.30 *given cfg $G=(N, \Sigma, P, S)$ and $w \in \Sigma^*$, $w \in L(G)$?*

$DTIME(O(|G| \cdot |w|^3))$ and $DSPACE(O(|G| \cdot |w|^2))$