Chap. 6 Pushdown Automata

6.1 Definition of Pushdown Automata

Example 6.2 $L_{ww^R} = \{ww^R \mid w \in (0+1)^*\}$ ***Palindromes*** *over* $\{0, 1\}$.

A cfg $P \rightarrow \varepsilon \mid 0 \mid 1 \mid 0P0 \mid 1P1$.

Consider a FA with a stack(= a Pushdown automaton; PDA).

q_0: *Push input symbol onto stack, and stay in* q_0(***gathering*** *mode) or*

Go to state q_1*(matching mode)* ***nondeterministically***(ε*-moves)*

Guess that it is the ***center*** *of the palrindrome, now.*

q_1: ***If*** *the input symbol is same as the top of the stack,*

then *pop the top of the stack and stay in* q_1*.(matching mode)*

If *the input symbol is* ***not*** *same as the top of the stack,*

then *reject.*

If *no more input symbol and empty stack in state* q_1,

then *go to the final state* q_2 *and* ***accept***, ***else*** *reject.*

6.1.2 The Formal Definition of Pushdown Automata

A pushdown automaton(PDA) $P = (q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *is*

1. q is a **finite** set of **state** alphabet,
2. Σ is a **finite** set of **input** alphabet,
3. Γ is a **finite stack** alphabet,
4. δ is a **transition function**. $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times \Gamma^*}$.

   If $(p, \gamma) \in \delta(q, a, X)$ for $q, p \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\gamma \in \Gamma^*$.

   pop stack **top** X and **push** stack string $\gamma \in \Gamma^*$ onto stack.

   i) $(p, \varepsilon) \in \delta(q, a, X)$ **pop** X

   ii) $(p, X) \in \delta(q, a, X)$ no stack change

   iii) $(p, YX) \in \delta(q, a, X)$ **push** Y

5. $q_0 \in Q$ is an **initial state**,
6. $Z_0 \in \Gamma^*$ is an **initial stack** content,
7. $F \subseteq Q$ is a set of **final states**.

Example 6.2 $P_{ww^R} = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\})$

Fig. 6.2 *in p 230.*

State q_0: If see 0, then push 0 and if see 1, then push 1; stay in q_0.

1. $\delta(q_0, 0, Z_0) = \{(q_0, 0Z_0)\}$ $\delta(q_0, 1, Z_0) = \{(q_0, 1Z_0)\}$
2. $\delta(q_0, 0, 0) = \{(q_0, 00)\}$ $\delta(q_0, 1, 0) = \{(q_0, 10)\}$
   $\delta(q_0, 0, 1) = \{(q_0, 01)\}$ $\delta(q_0, 1, 1) = \{(q_0, 11)\}$ **push** and **gath**.
3. $\delta(q_0, \varepsilon, Z_0) = \{(q_1, Z_0)\}$ $\delta(q_0, \varepsilon, 0) = \{(q_1, 0)\}$ $\delta(q_0, \varepsilon, 1) = \{(q_1, 1)\}$
   Go to the state q_1 on ε input(**nondeterministic**)-guess center of Pal.

State q_1: **match** input symbols against the stack top symbol and **pop** it.

4. $\delta(q_1, 0, 0) = \{(q_1, \varepsilon)\}$. $\delta(q_1, 1, 1) = \{(q_1, \varepsilon)\}$ **match** and **pop**
5. $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$ **Accept!**
   else **error**! $\delta(q_1, 0, 1) = \delta(q_1, 1, 0) = \delta(q_1, 0, Z_0) = \delta(q_1, 1, Z_0) = \emptyset$.

$L(P_{ww^R}) = L_{ww^R}$ but P is **not deterministic**!

Configuration(Instantaneous description) *of PDA*

(current state, remained input string, current stack contents)

$(q, x, \gamma) \in Q \times \Sigma^* \times \Gamma^*$.

$\vdash_P \subseteq (Q \times \Sigma^* \times \Gamma^*) \times (Q \times \Sigma^* \times \Gamma^*)$

$(q, a\mathbf{x}\, X\beta) \vdash_P (p, \mathbf{x}, \gamma\beta)$, if $(p, \gamma) \in \delta(q, a, X)$

$(q, \mathbf{x}, X\beta) \vdash_P (p, \mathbf{x}, \gamma\beta)$, if $(p, \gamma) \in \delta(q, \varepsilon, X)$

We may use $\vdash$ *instead of* $\vdash_P$ *if* P *is understood.*

$\vdash$ *is a binary relation on* $(Q \times \Sigma^* \times \Gamma^*)$.

$\vdash^*$ *is a **reflexive** and **transitive** closure of* $\vdash$.

Recursive definition of $\vdash^*$.

$\forall I, J, K \in Q \times \Sigma^* \times \Gamma^*$, $I \vdash_B^* I$.

If $I \vdash J$ *and* $J \vdash^* K$, $I \vdash_R^* K$.

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *is a PDA. Then*

Thm. 6.5 *If* $(q, x, \alpha) \vdash^* (p, y, \beta)$ *for* $q,p \in Q$, $x,y \in \Sigma^*$ *and* $\alpha,\beta \in \Gamma^*$.

Then $(q, xw, \alpha\gamma) \vdash^* (p, yw, \beta\gamma)$ *for any* $w \in \Sigma^*$ *and* $\gamma \in \Gamma^*$.

Adding *lookahead input strings and lookback stack strings.*

Thm. 6.6 *If* $(q, xw, \alpha) \vdash^* (p, yw, \beta)$ *for* $q,p\in Q$, $x,y,w\in \Sigma^*$ *and* $\alpha,\beta\in \Gamma^*$.

Then $(q, x, \alpha) \vdash^* (p, y, \beta)$.

Remooving *lookahead input strings.*

6.2 The language of a PDA

6.2.1 Acceptance by Final State

$$L(P) = \{w \in \Sigma^* | (q_0, w, Z_0) \vdash^* (f, \varepsilon, \alpha), f \in F\}$$

6.2.2 Acceptance by Null Stack

$$N(P) = \{w \in \Sigma^* | (q_0, w, Z_0) \vdash^* (f, \varepsilon, \varepsilon)\}$$

6.2.3 From Empty Stack to Final State

***Thm*. 6.9** *If* $L = N(P_N)$ *for some PDA* $P_N = (Q_N, \Sigma, \Gamma_N, \delta_N, q_0^N, Z_0^N, \varnothing)$.

Then there is a PDA P_F *such that* $L = L(P_F)$.

$P_F = (Q_N \cup \{q_0^F, q_F\}, \Sigma, \Gamma_N \cup \{Z_0^F\}, \delta_F, q_0^F, Z_0^F, \{q_F\})$

where $q_0^F, q_F \notin Q_N$, $Z_0^F \notin \Gamma_N$.

δ_F: 1. $\delta_F(q_0^F, \varepsilon, Z_0^F) = \{(q_0^N, Z_0^N Z_0^F)\}$. *push old stack* ***bottom*** Z_0^N.

2. $\delta_F \supseteq \delta_N$, ***simulate*** P_N *with* δ_N.

3. $\forall q \in Q_N$, $\delta_F(q, \varepsilon, Z_0^F) = \{(q_F, \varepsilon(\text{or } Z_0^F \text{ or any } \alpha \in \Gamma_N^*))\}$.

If stack is ***empty***(Z_0^F*: stack top), go to the final* ***final*** *state* q_F.

See ***Fig***. *6.4 in p. 237.*

$p_0 = q_0^F$, $X_0 = Z_0^F$, $Z_0 = Z_0^N$, $q_0/\varepsilon = q_0^N/any$.

6.2.4 From Final State to Empty Stack

Thm. 6.11 *If* $L = L(P_F)$ *for some PDA* $P_F = (Q_F, \Sigma, \Gamma_F, \delta_F, q_0^F, Z_0^F, \boldsymbol{F})$.

Then there is a PDA P_N *such that* $L = N(P_N)$.

$P_N = (Q_F \cup \{q_0^N, q_E\}, \Sigma, \Gamma_N, \delta_N, q_0^E, Z_0^N, \varnothing)$

where $q_0^E, q_E \notin Q_F$, $Z_0^N \notin \Gamma_F$ *and* $\Gamma_N = \Gamma_F \cup \{Z_0^N\}$.

δ_N: 1. $\delta_N(q_0^E, \varepsilon, Z_0^N) = \{(q_0, Z_0^F Z_0^N)\}$ *push old stack* ***bottom*** Z_0^F.

2. $\delta_N \supseteq \delta_F$, ***simulate*** P_F *with* δ_F.

3. $\forall \boldsymbol{f} \in \boldsymbol{F}$, $\forall Z \in \Gamma_N$, $\delta_N(\boldsymbol{f}, \varepsilon, Z) \supseteq \{(q_E, \varepsilon)\}$.

If ***final*** *state, pop a stack symbol and go to the* ***empty*** *state* q_E.

4. $\forall Z \in \Gamma_N$, $\delta_N(q_E, \varepsilon, Z) = \{(q_E, \varepsilon)\}$

Pop *all of the stack symbols in the* ***empty*** *state* q_E.

See ***Fig****. 6.7 in p. 240* $p_0 = q_0^E$, $X_0 = Z_0^N$, $Z_0 = Z_0^F$, $q_0 = q_0^F$, *and*

... *any*/ε = Z/ε *in the state* $p = q_E$.

6.3 Equivalence of PDA's and CFG's

6.3.1 From Grammars to Pushdown Automata

Theorem 6.13 *If $G = (N, T, P, S)$ is a cfg. Then $^{\exists}$PDA P .϶. $L(G) = N(P)$.*

Construct $P = (\{q\}, T, N \cup \Sigma, \delta, q, S, \emptyset)$ ***guess*** *and* ***verify*** *parser*

$^{\forall}A \in N, \delta(q, \varepsilon, A) = \{(q, \alpha) | A \to \alpha \in P\}$ ***guess*** A *as* $\alpha (A \to \alpha \in P)$.

$^{\forall}a \in \Sigma, \delta(q, \mathbf{a}, \mathbf{a}) = \{(q, \mathbf{a}) | \mathbf{a} \in T\}$ ***verify*** $\mathbf{a} \in \Sigma$.

Proof $A \Rightarrow_{lm}^{*} \mathbf{x}\alpha$ *if and only if* $(q, \mathbf{x}, A) \vdash^{*} (q, \varepsilon, \alpha)$, $x \in \Sigma^{*}$, $\alpha \in (N \cup \Sigma)^{*}$.

(If) *If* $(q, \mathbf{x}, A) \vdash^{i} (q, \varepsilon, \alpha)$, *then* $A \Rightarrow_{lm}^{*} x\alpha$ *for* $i \geq 0$.

basis $i = 0$: $\mathbf{x} = \varepsilon$, *and* $A = \alpha$. $\therefore (q, \varepsilon, A) \vdash^{0} (q, \varepsilon, A)$. $\therefore A \Rightarrow_{lm}^{*} A$.

induction *Let* $i \geq 1$, *and consider the next-to-last step.*

i) $(q, \mathbf{x}, A) \vdash^{i-1} (q, \varepsilon, B\gamma) \vdash_{guess}^{B \to \beta} (q, \varepsilon, \beta\gamma) = (q, \varepsilon, \alpha)$

$\therefore A \Rightarrow_{lm}^{*} \mathbf{x}B\gamma$ *by IH and* $B \to \beta \in P$ *by construction of* δ_{guess}.

$\therefore A \Rightarrow_{lm}^{*} \mathbf{x}B\gamma \Rightarrow_{lm} \mathbf{x}\beta\gamma = \mathbf{x}\alpha$.

ii) $(q, x, A) = (q, ya, A) \vdash^{i-1} (q, a, a\alpha) \vdash_{verify} (q, \varepsilon, \alpha)$

$(q, y, A) \vdash^{i-1} (q, \varepsilon, a\alpha)$ (Thm 6.5; $(q, \underline{\varepsilon}, a\alpha)$)

$\therefore A \Rightarrow_{lm}^{*} ya\alpha = x\alpha$ by IH

(Only if) If $A \Rightarrow_{lm}^{i} x\alpha$, then $(q, x, A) \vdash^{*} (q, \varepsilon, \alpha)$ for $i \geq 0$.

basis $i = 0$, $A \Rightarrow_{lm}^{0} A$. $\boldsymbol{x} = \varepsilon$ and $A = \alpha$. $(q, \varepsilon, A) \vdash^{0} (q, \varepsilon, A)$.

induction Let $i \geq 1$, and consider the next-to-last step.

$A \Rightarrow_{lm}^{i-1} yB\gamma \Rightarrow_{lm} y\beta\gamma = y\underline{y'\gamma'}\gamma = xA$ where $\beta = y'\gamma'$, $y' \in \Sigma^{*}$, $\gamma' \in (N \cup \Sigma)^{*}$.

$(q, y, A) \vdash^{*} (q, \varepsilon, B\gamma)$ by IH. $\therefore (q, yy', A) \vdash^{*} (q, y', B\gamma)$ (by T.6.5)

$\vdash_{G}^{B\rightarrow\beta} (q, y', \beta\gamma) = (q, y', y'\gamma'\gamma) \vdash_{V}^{|y'|} (q, \varepsilon, \gamma'\gamma) = (q, \varepsilon, \alpha)$

$\therefore A \Rightarrow_{lm}^{*} x\alpha$ ***if and only if*** $(q, x, A) \vdash^{*} (q, \varepsilon, \alpha)$.

If $A = S$, $\alpha = \varepsilon$, then $S \Rightarrow_{lm}^{*} x$ ***if and only if*** $(q, x, S) \vdash^{*} (q, \varepsilon, \varepsilon)$

$\therefore L(G) = N(P)$.

6.3.2 From PDA's to Grammars

Theorem 6.14 *If a PDA* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$.

Then there is a CFG G such that $L(G) = N(P)$. ***null stack***

proof $G = (Q \times \Gamma \times Q \cup \{S\}, \Sigma, P, S)$

$P = \{S \rightarrow [q_0 Z_0 q] \mid {}^{\forall} q \in q\}$ *use* $[q_0 Z_0 q]$ *instead of* (q_0, Z_0, q)

$\cup \{[qAp_m] \rightarrow a\, [pY_1p_1][p_1Y_2p_2] \ldots [p_{m-1}Y_mp_m] \mid$

$(p, Y_1Y_2\ldots Y_m) \in \delta(q, a, A), a \in \Sigma \cup \{\varepsilon\}, {}^{\forall} q \in Q, 1 \le {}^{\forall} i \le m: p_i \in Q, Y_i \in \Gamma\}$

$(\text{if } m = 0, [qAp] \rightarrow a \in P, a \in \Sigma \cup \{\varepsilon\})$

To ***proove*** *that* $[qAp] \Rightarrow_{lm}{}^{*}\, x \in \Sigma^{*}$, *if and only if*

PDA ***Configurations****(ID's)* $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon) \subseteq Q \times \Sigma^{*} \times \Gamma^{*}$.

Nonterminal $[qAp]$ *derives terminal string* x *if and only if*

x *causes PDA P to pop* A *from stack*

starting in the state q *and ending in the state* p.

1) If $(q, x, A) \vdash^{i} (p, \varepsilon, \varepsilon)$, then $[qAp] \Rightarrow_{lm}^{*} x$ for $i \geq 1$.

basis $i = 1$, $(q, x, A) \vdash (p, \varepsilon, \varepsilon)$. $\therefore (p, \varepsilon) \in \delta(q, x, A)$, $x \in \Sigma \cup \{\varepsilon\}$.

$\therefore [qAp] \to x \in P$ where $x \in \Sigma \cup \{\varepsilon\}$. $\therefore [qAp] \Rightarrow_{lm}^{*} x$.

induction $(q, x, A) = (q, ay, A) \vdash (p_1, y, Y_1\ldots Y_m) \vdash^{i-1} (p, \varepsilon, \varepsilon)$.

$\therefore {}^{\exists}p_2, \ldots, p_m, p \in q$ and assume $y = y_1\ldots y_m \in \Sigma^*$.

$(p_1, y_1\ldots y_m, Y_1\ldots Y_m) \vdash^{*} (p_2, y_2\ldots y_m, Y_2\ldots Y_m) \vdash^{*} \ldots (p_m, y_m, Y_m) \vdash (p, \varepsilon, \varepsilon)$.

$1 \leq {}^{\forall}i \leq m$, $(p_i, y_i, Y_i) \vdash^{*} (p_{i+1}, \varepsilon, \varepsilon)$. (Thm 6.5 and y_i depends on Y_i only)

$\therefore [p_iY_ip_{i+1}] \Rightarrow_{lm}^{*} y_i$ by IH.

$\therefore [p_1Y_1p_2][p_2Y_2p_3]\ldots[p_mY_mp] \Rightarrow_{lm}^{*} y_1y_2\ldots y_m = y$

Since $(p_1, Y_1\ldots Y_m) \in \delta(q, a, A)$, $(q, ay, A) \vdash (p_1, y, Y_1\ldots Y_m)$.

$\therefore {}^{\exists}[qAp] \to a\,[p_1Y_1p_2][p_2Y_2p_3]\ldots[p_mY_mp] \in P.({}^{\forall}p_i \in Q)$

$\therefore [qAp] \Rightarrow_{lm} a\,[p_1Y_1p_2][p_2Y_2p_3]\ldots[p_mY_mp] \Rightarrow_{lm}^{*} ay = x$.

2) If $[qAp] \Rightarrow_{lm}^{i} x$, then $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon)$ for $i \geq 1$.

basis $i = 1$, $[qAp] \rightarrow x \in P$, $(p, \varepsilon) \in \delta(q, x, A)$ where $x \in \Sigma \cup \{\varepsilon\}$.

induction $[qAp] \Rightarrow_{lm} a\ [p_1Y_1p_2][p_2Y_2p_3] \ldots [p_mY_mp] \Rightarrow_{lm}^{i-1} x \in \Sigma^{*}$.

$x = ay_1\ldots y_m$ where $1 \leq {}^{\forall}i \leq m$, $[p_iY_ip_{i+1}] \Rightarrow_{lm}^{i} y_i$ where $p_{m+1}=p$.

$\therefore (p_i, y_i, Y_i) \vdash^{*} (p_{i+1}, \varepsilon, \varepsilon)$ by IH.

Since $[qAp] \rightarrow a\ [p_1Y_1p_2][p_2Y_2p_3] \ldots [p_mY_mp] \in P$,

$a\ [p_1Y_1p_2][p_2Y_2p_3] \ldots [p_mY_mp] \in \delta(q, a, A)$ where $a \in \Sigma \cup \{\varepsilon\}$.

$\therefore (q, ay_1\ldots y_m, A) \vdash (p_1, y_1\ldots y_m, Y_1\ldots Y_m) \vdash^{*} \ldots \vdash^{*} (p, \varepsilon, \varepsilon)$.

$[qAp] \Rightarrow_{lm}^{*} x$, ${}^{\forall}q, p \in q$ if and only if $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon)$.

$\therefore [q_0Z_0p] \Rightarrow_{lm}^{*} x$, ${}^{\forall}q, p \in q$ if and only if $(q_0, x, Z_0) \vdash^{*} (p, \varepsilon, \varepsilon)$.

$S \Rightarrow_{lm} [q_0Z_0p] \Rightarrow_{lm}^{*} x$, ${}^{\forall}q, p \in q$ if and only if $(q_0, x, Z_0) \vdash^{*} (p, \varepsilon, \varepsilon)$.

6.4 Deterministic Pushdown Automata

6.4.1 Definition of a Deterministic PDA

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *with*

$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times \Gamma^*}$ *is* ***deterministic***, *if*

1. $\forall q \in Q, \forall a \in \Sigma \cup \{\varepsilon\}: |\delta(q, a, X)| \leq 1$ *or*

   $\delta: q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightharpoonup Q \times \Gamma^*$.

2. *If* $\exists \delta(q, a, X)$, *then* $\delta(q, \varepsilon, X) = \emptyset$ *or*

   If $\exists (q, \varepsilon, X)$, *then* $\forall a \in \Sigma: \delta(q, a, X) = \emptyset$.

Example 6.16 ***Palindromes*** *over {0, 1} with center marker c.*

$L_{wcw^R} = \{wcw^R \mid w \in (0+1)^*\}$

$P_{wcw^R} = (\{q_0, q_1, q_2\}, \{0, 1, c\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\})$

.э. $L_{wcw^R} = L(P_{wcw^R})$.

*P_{wcw^R} is a **deterministic** PDA!*

1. $\forall X \in \Gamma, \forall \mathbf{a} \in \{0, 1\}: \delta(q_0, \mathbf{a}, X) = \{(q_0, \mathbf{a}X)\}$. *gathering mode(push)*
2. $\delta(q_0, c, X) = \{(q_1, X)$. *goto matching mode*
3. $\forall a \in \{0, 1\}: \delta(q_1, a, a) = \{(q_1, \varepsilon)\}$. *matching mode(pop)*
4. $\delta(q_1, \varepsilon, Z_0) = \{(q_2, \varepsilon)\}$. *end of matching!*

***Stronger** version of **deterministic** PDA*

$$\delta: Q \times \Sigma \times \Gamma \rightharpoonup Q \times \Gamma^*.$$

6.4.2 Regular Languages** and **Deterministic PDA'

***Theorem 6.17** If L is **regular**, the L = L(P) for some PDA.*

***proof** Let $A = (Q, \Sigma, \delta_A, q_0, F)$ is a DFA .э. $L = L(A)$. Then*

$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F)$ with

$\delta_P(q, a, Z_0) = \{(p, Z_0) \mid \delta_A(q, a) = p\}$ *is a **deterministic** PDA.*

DFA is a DPDA and FA is a PDA!

6.4.3 DPDA's and Context-free Languages

*P_{wcw^R} is a DPDA. But L_{wcw^R} is not **regular** but **context-free**.*

$$L_{wcw^R} = L(P_{wcw^R}).$$

*L_{ww^R} is not **regular** but **context-free**. But there is no DPDA P*

$$.\ni.\ L_{ww^R} = L(P)$$

Regular Languages** $\subset$ L(DPDA) $\subset$ **Context-free Languages

6.4.4 DPDA's and Ambiguous Grammars

***Theorem 6.20** If L = L(P) for some DPDA A, Then L has an **unambiguous** context-free grammar.*

***proof** If a PDA P is DPDA, then the CFG in **Thm. 6.16** is **unambiguous**.*

***Theorem 6.21** If L = L(P) for some DPDA A, Then L has an ambiguous grammar.*

***proof** end marker($) vs ε. But forget it! ($\Sigma \cup \{\$\}$ vs $\Sigma \cup \{\varepsilon\}$)*

Context-free Languages = Context-free grammars = Pushdown Automata
⊂ Regular Languages = Regular Grammars = Finite Automata
= Regular Expressions