

### 6.A.1 Rewriting systems for FA's and PDA's

We have defined a **rewriting system** for a FA

$$F = (Q, T, \delta_F, q_0, F) \text{ with } \delta_F: Q \times T^* \rightarrow 2^Q \text{ as}$$

$$R_F = (Q \times T^*, \rightarrow_F, (q_0, x), \{(f, \varepsilon) \mid x \in T^*, f \in F\} \text{ where}$$

$$\rightarrow_F := \{(q, x) \rightarrow_F (p, \varepsilon) \in (Q \times T^*) \times (Q \times T^*) \mid p \in \delta_F(q, x)\}.$$

Note that  $\rightarrow_F \subseteq (Q \times T^*) \times (Q \times T^*)$  is a binary operation on

**configurations (ID's)  $Q \times T^*$ : configuration rewriting system.**

Compare **configuration rewritings** with **state transition functions**  $\delta_F$

$$(q_0, x_1 x_2 \dots x_n) \Rightarrow_F^{(q_0, x_1) \rightarrow (q_1, \varepsilon)} (q_1, x_2 \dots x_n) \Rightarrow_F \dots \Rightarrow_F$$

$$(q_{n-1}, x_n) \Rightarrow_F^{(q_{n-1}, x_n) \rightarrow (q_n, \varepsilon)} (q_n, \varepsilon), \text{ if and only if,}$$

$$q_n \in \delta(\dots \delta(\delta(q_0, x_1 x_2 \dots x_n)) \dots) = q_1 \in \delta_F(q_0, x_1) \delta(\dots \delta(q_1, x_2 \dots x_n)) \dots) = \dots$$

$$= \delta(q_{n-1}, x_n) = q_n \in \delta_F(q_{n-1}, x_n) q_n.$$

We define a *rewriting system* for a PDA

$P = (Q, T, \Gamma, \delta_P, q_0, Z_0, F)$  with  $\delta_P: Q \times T^* \times \Gamma^* \rightarrow 2^{Q \times T^* \times \Gamma^*}$  as

$R_P = (Q \times T^* \times \Gamma^*, \rightarrow_P, (q_0, x, Z_0), \{(f, \varepsilon, \alpha) \mid f \in F, \alpha \in \Gamma^*\})$

$\rightarrow_P := \{(q, x, \alpha) \rightarrow_F (p, \varepsilon, \beta) \in (Q \times T^* \times \Gamma^*) \times (Q \times T^* \times \Gamma^*)$   
 $\mid (p, \beta) \in \delta_P(q, x, \alpha)\}.$

Note that  $\rightarrow_P \subseteq (Q \times T^* \times \Gamma^*) \times (Q \times T^* \times \Gamma^*)$  is a binary relation on

**configurations (ID's)  $Q \times T^* \times \Gamma^*$ : configuration rewriting system.**

Compare **configuration rewritings** with **state transition functions**  $\delta_P$

$(q_0, x_1 x_2 \dots x_n, \alpha_1 \gamma) \Rightarrow_P^{(q_0, x_1, \alpha_1) \rightarrow (q_1, \varepsilon, \alpha_2)} (q_1, x_2 \dots x_n, \alpha_2 \gamma) \Rightarrow_P \dots \Rightarrow_P$

$(q_{n-1}, x_n, \alpha_{n-1} \gamma')$ , if and only if,

$(q_0, x_1 x_2 \dots x_n, \alpha_1 \gamma) \vdash_P^{(q_1, \alpha_2) \in \delta_P(q_0, x_1, \alpha_1)} (q_1, x_2 \dots x_n, \alpha_2 \gamma) \vdash_P \dots \vdash_P$

$(q_{n-1}, x_n, \alpha_{n-1} \gamma') \vdash_P^{(q_n, \alpha_n) \in \delta_P(q_{n-1}, x_n, \alpha_{n-1})} (q_n, \varepsilon, \alpha_n \gamma').$

We generalize  $\rightarrow_P$  of a rewriting system  $R_P = (Q \times T^* \times \Gamma^*, \rightarrow_P, \iota, \Phi)$ , by

(1)  $Q \times T^* \times \Gamma^*$  is a set of configurations

(2) Adding *lookahead* input string ( $y \in T^*$ )

and *lookback* stack string ( $\gamma \in \Gamma^*$ )

$$\rightarrow_P = \{(q, xy, \alpha\gamma) \rightarrow_P (p, y, \beta\gamma) \in (Q \times T^* \times \Gamma^*) \times (Q \times T^* \times \Gamma^*)$$

$$| p, q \in Q, x, y \in T^*, \alpha, \beta, \gamma \in \Gamma^*\} \subseteq (Q \times T^* \times \Gamma^*) \times (Q \times T^* \times \Gamma^*),$$

(3)  $\iota = (q_0, x, \gamma_0) \in Q \times T^* \times \Gamma^*$ ,  $q_0$ : initial states,  $\gamma_0$ : initial stack string

extending initial stack symbol ( $Z_0 \in \Gamma$ ) to stack string ( $\gamma_0 \in \Gamma^*$ ), and

(4)  $\Phi = \{(p, \varepsilon, \phi) | p \in Q, \phi \in \Gamma^*\}$ ,  $p$ : final states,  $\phi$ : final stack strings

Note that the 2nd Conf.  $T^*$  is *read only* input ( $xy \rightarrow y$ ) string and the 3rd Conf.  $\Gamma^*$  is a *read/write stack* ( $\alpha\gamma \rightarrow \beta\gamma$ ) string.

### 6.A.2 Left Parsers for Context-free Grammars

Let  $G = (N, T, P, S)$  be a context-free grammar. Then we define

A PDA  $R_L = (T^* \times (N \cup T)^*, \rightarrow_L, \iota = (x, S), \Phi = \{(\varepsilon, \varepsilon)\})$  as

a **Left (gues-verify) parser** for the context-free grammar  $G$  where

$$\begin{aligned} \rightarrow_L = & \{(\varepsilon, \mathbf{A}) \rightarrow_L (\varepsilon, \alpha) \mid \mathbf{A} \rightarrow \alpha \in P\} && \text{guess } \mathbf{A} \text{ as } \alpha \text{ onto } \textit{stack} \\ & \cup \{(a, a) \rightarrow_L (\varepsilon, \varepsilon) \mid a \in T\}. && \text{verify } a \in \Sigma \text{ with } \textit{input} \text{ and } \textit{stack} \end{aligned}$$

$$L(R_L) = \{x \in \Sigma^* \mid (x, S) \Rightarrow_{L^*} (\varepsilon, \varepsilon) \in \Phi\}.$$

We add  $P$  and output function  $\tau_L: \rightarrow_L^* \rightarrow P^*$  to the left parser as

$$P_L = (T^* \times (N \cup T)^*, \rightarrow_L, (x, S), \{(\varepsilon, \varepsilon)\}, P, \tau_L) \text{ with}$$

$$\tau_L((\varepsilon, \mathbf{A}) \rightarrow_L (\varepsilon, \alpha)) = \mathbf{A} \rightarrow \alpha \in P \text{ and}$$

$$\tau_L((a, a) \rightarrow_L (\varepsilon, \varepsilon)) = \varepsilon. \text{ Then}$$

$(x, S) \Rightarrow_{L^*}^{\theta_L} (\varepsilon, \varepsilon)$ , iff  $S \Rightarrow_{lm}^{\pi_L} x$  where  $\tau_L(\theta_L) = \pi_L \in P^*$  **left parse** of  $x$ ,  
 $|\theta_L| = |\pi_L| + |x|$   $|\pi_L|$ -**guess** actions and  $|x|$ -**verify** actions.

Consider a *leftmost derivations* in  $G$  and *rewritings* in the left parser  $P_L$ .

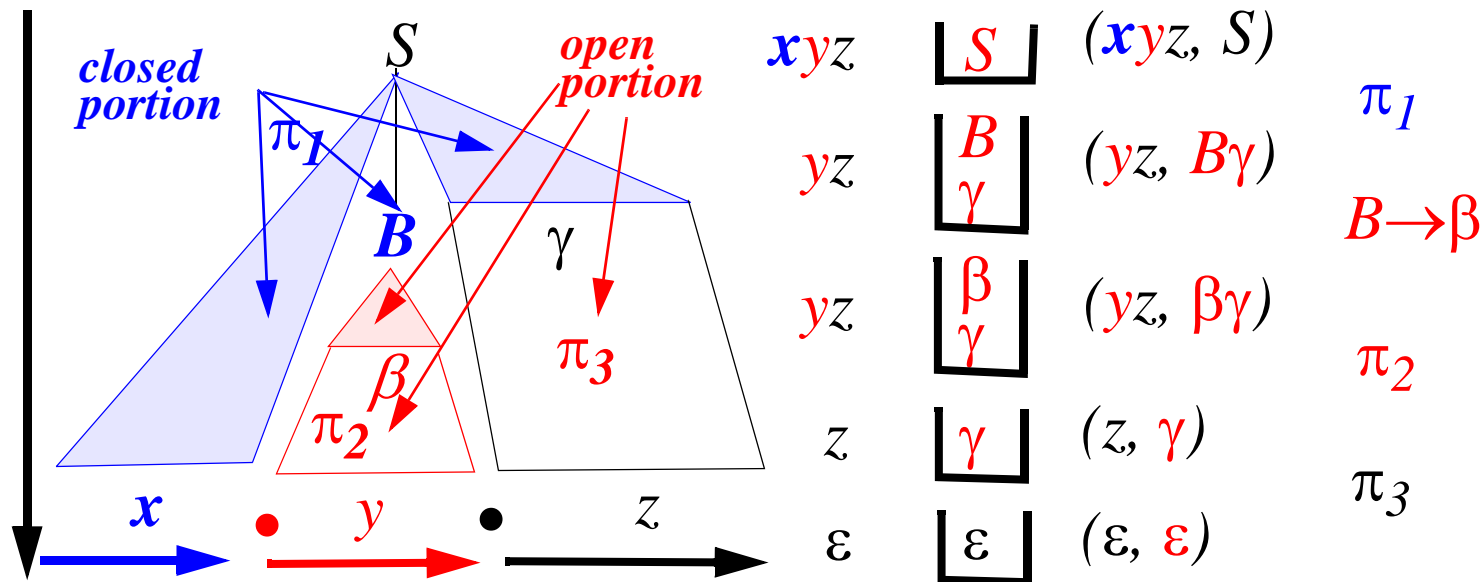
$$S \Rightarrow_{lm}^{\pi_1} \mathbf{x}B\gamma \Rightarrow_{lm}^{B \rightarrow \beta} \mathbf{x}\beta\gamma \Rightarrow_{lm}^{\pi_2} \mathbf{x}y\gamma \Rightarrow_{lm}^{\pi_3} \mathbf{x}y\mathbf{z} \text{ and}$$

$$(\mathbf{x}y\mathbf{z}, S) \Rightarrow_L^{\theta_1} (yz, B\gamma) \Rightarrow_L^{B \rightarrow \beta} (yz, \beta\gamma) \Rightarrow_L^{\theta_2} (z, \gamma) \Rightarrow_L^{\theta_3} (\epsilon, \epsilon).$$

$$|\theta_1| = |\pi_1| + |\mathbf{x}|, |\theta_2| = |\pi_2| + |y|, |\theta_3| = |\pi_3| + |z|. \tau(\theta_1) = \pi_1, \tau(\theta_2) = \pi_2, \tau(\theta_3) = \pi_3.$$

$$\tau(\theta) = \tau(\theta_1 \cdot B \rightarrow \beta \cdot \theta_2 \theta_3) = \pi_1 \cdot B \rightarrow \beta \cdot \pi_2 \pi_3 = \pi.$$

$|\pi_1 \cdot B \rightarrow \beta \cdot \pi_2 \pi_3|$  **guess actions** +  $|\mathbf{x}y\mathbf{z}|$  **verify actions.**



$$\begin{aligned} \text{Example } G_{Uexp}: E &\rightarrow E + T \mid T * F \mid a \mid ( E ) \\ T &\rightarrow T * F \mid a \mid ( E ) \\ F &\rightarrow a \mid ( E ) \end{aligned}$$

$$(a+a*a, E)$$

$$\Rightarrow^{E \rightarrow E+T} (a+a*a, E+T) \text{ or } \Rightarrow^{E \rightarrow T*F} (a+a*a, T*F) \text{ or } \Rightarrow^{E \rightarrow (E)} (a+a*a, (E)) \text{ or } \Rightarrow^{E \rightarrow a} (a+a*a, a)$$

$$\Rightarrow^{E \rightarrow a} (a+a*a, a+T) \quad \Rightarrow^{T \rightarrow T*F} (a+a*a, T*F*F) \Rightarrow_L^{a \neq (} \text{Error!} \quad \Rightarrow^a (+a*a, \epsilon)$$

$$\Rightarrow^a (+a*a, +T) \quad \Rightarrow^{T \rightarrow a} (a+a*a, a*F*F) \text{ or } \Rightarrow^{T \rightarrow (E)} (a+a*a, (E)*F*F) \Rightarrow^{+\neq \epsilon} \text{Error!}$$

$$\Rightarrow^+ (a*a, T) \quad \Rightarrow^a (+a*a, *F*F) \quad \Rightarrow_L^{a \neq (} \text{Error!}$$

$$\Rightarrow^{T \rightarrow T*F} (a*a, T*F) \quad \Rightarrow^{+\neq *} \text{Error!}$$

$$\Rightarrow^{T \rightarrow a} (a*a, a*F)$$

$$\Rightarrow^a (*a, *F)$$

$$\Rightarrow^* (a, F)$$

$$\Rightarrow^{F \rightarrow a} (a, a)$$

$$\Rightarrow^a (\epsilon, \epsilon)$$

**Lemma 6.A.1** If  $(xy, \gamma) \Rightarrow_L^{\theta_L} (y, \delta)$  in  $P_L$ , then  $\gamma \Rightarrow_{lm}^{\tau_L(\theta_L)} x\delta$  in  $G$   
and  $|\theta_L| = |\tau_L(\theta_L)| + |x|$ .

**Proof Induction on  $|\theta_L| \in \mathbb{N}$ .**

i)  $|\theta_L| = 0$ :  $\theta_L = \varepsilon$ .  $(xy, \gamma) = (y, \delta)$ .  $\therefore x = \varepsilon$ ,  $\gamma = \delta$ , and  $\tau_L(\theta_L) = \varepsilon$ .

ii)  $\theta_L = a \cdot \theta_L' \in \rightarrow_L^+$ . where  $a \in \rightarrow_L$  and  $\theta_L' \in \rightarrow_L^*$ .

ii.1)  $a = (\varepsilon, A) \rightarrow_L (\varepsilon, \alpha)$  **guess**  $A \rightarrow \alpha \in P$  action.

$(xy, \gamma) = (xy, A\gamma') \rightarrow_L^{A \rightarrow \alpha} (xy, \alpha\gamma') \Rightarrow_L^{\theta_L'} (y, \delta)$  in  $P_L$ .

$\therefore \alpha\gamma' \Rightarrow_{lm}^{\tau_L(\theta_L')} x\delta$  in  $G$  and  $|\theta_L| = |\tau_L(\theta_L')| + |x|$  by IH.

$\gamma = A\gamma' \Rightarrow_{lm}^{A \rightarrow \alpha} \alpha\gamma' \Rightarrow_{lm}^{\tau_L(\theta_L')} x\delta$  in  $G$  and

$|\theta_L| = 1 + |\tau_L(\theta_L')| + |x| = |\tau_L(\theta_L)| + |x|$ .

ii.2)  $a = (a, a) \rightarrow_L (\varepsilon, \varepsilon)$  **verify**  $a \in T$  action

$(xy, \gamma) = (ax'y, \alpha\gamma') \rightarrow_L^a (x'y, \gamma') \Rightarrow_L^{\theta_L'} (y, \delta)$  in  $P_L$ .

$\therefore \gamma' \Rightarrow_{lm}^{\pi_L'} x'\delta$  in  $G$  and  $|\theta_L| = |\tau_L(\theta_L')| + |x'|$  by IH.

$\gamma = \alpha\gamma' \Rightarrow_{lm}^{\tau_L(\theta_L')} \alpha x'\delta = x\delta$  in  $G$  and

$$|\theta_L| = 1 + |\tau_L(\theta_L')| + |x'| = |\tau_L(\theta_L)| + |x|$$

Apply  $y = \varepsilon$ ,  $\gamma = S$ , and  $\delta = \varepsilon$  in **Lemma A**.

If  $(x, S) \Rightarrow_L^{\theta_L} (\varepsilon, \varepsilon)$  in  $P_L$ , then  $S \Rightarrow_{lm}^{\tau_L(\theta_L)} x$  in  $G$

and  $|\theta_L| = |\tau_L(\theta_L)| + |x|$ .

$\therefore L(P_L) \subseteq L(G)$ ,  $\tau_L(\theta_L) = \pi_L$ , and  $|\pi_L| = |\theta_L| - |x|$ .

**Lemma 6.A.2** If  $\gamma \Rightarrow_{lm}^{\pi_L} x\delta$  in  $G$ ,  $\delta = \varepsilon$  or  $1:\delta \in N$ , then

$(xy, \gamma) \Rightarrow_L^{\theta_L} (y, \delta)$  in  $P_L$ ,  $\tau_L(\theta_L) = \pi_L$ , and  $|\theta_L| = |\pi_L| + |x|$ .

**Proof Induction on  $|\pi_L| \in \mathbb{N}$ .**

i)  $\pi_L = \varepsilon$ :  $x = \varepsilon$ ,  $\gamma = \delta$ , and  $(y, \gamma) \Rightarrow_L^\varepsilon (y, \gamma)$ .  $\therefore \tau_L(\varepsilon) = \pi_L = \varepsilon$ .

ii)  $\pi_L = \pi_L' \cdot \theta_L'$ ,  $\pi_L' \in P^*$ ,  $\exists \theta_L' \in \rightarrow_L^*$ .  $\exists \tau_L(\theta_L) = \pi_L$ .



$\gamma \Rightarrow_{lm}^{\pi_L'} x'\delta' = x'A\delta'' \Rightarrow_{lm}^{A \rightarrow \alpha} x'\alpha\delta'' = x'z\delta = x\delta$ , where

$$|\theta_L'| = |\pi_L'| + |x'|, \delta' = A\delta'', \alpha\delta'' = z\delta, x'z = x.$$

$\therefore (xy, \gamma) = (x'zy, \gamma) \Rightarrow_L^{\theta_L'} (zy, \delta') = (zy, A\delta'')$

$$\Rightarrow_L^{A \rightarrow \alpha} (zy, \alpha\delta'') = (zy, z\delta) \Rightarrow_L^{|z|} (y, \delta).$$

$\therefore |\theta_L| = |\theta_L'| + 1 + |z| = |\pi_L'| + |x'| + 1 + |z| = |\pi_L| + |x|.$

$$\tau_L(\theta_L) = \pi_L' \cdot A \rightarrow \alpha \cdot \varepsilon = \pi_L.$$

Apply  $y = \varepsilon$ ,  $\gamma = S$ , and  $\delta = \varepsilon$  in **Lemma B**.

If  $S \Rightarrow_{lm}^{\pi_L} x$  in  $G$ , then  $(x, S) \Rightarrow_L^{\theta_L} (\varepsilon, \varepsilon)$  in  $P_L$ ,

$$\text{and } \tau_L(\theta_L) = \pi_L, |\theta_L| = |\pi_L| + |x|.$$

$\therefore L(G) \subseteq L(P_L), \forall \pi_L: \text{left parse of } x \tau_L(\theta_L) = \pi_L, \text{ and } |\theta_L| = |\pi_L| + |x|.$

### Theorem 6.A

$L(G) = L(P_L), \forall x \in L(G) \text{ in } G, \tau_L(\theta_L) = \pi_L \text{ in } P_L, \text{ and } |\theta_L| = |\pi_L| + |x|.$

### 6.A.3 Right Parsers for Context-free Grammars

A PDA  $P_R = (T^* \times (N \cup T)^*, \rightarrow_R, (x, \varepsilon), \{(\varepsilon, S)\})$  is

a **Right (shift-reduce) parser** for the context-free grammar  $G$  where

$\rightarrow_R = \{(a, \varepsilon) \rightarrow_R (\varepsilon, a) \mid a \in T\}$     **shift(push) input  $a \in \Sigma$  onto *stack***

$\cup \{(\varepsilon, \alpha^R) \rightarrow_R (\varepsilon, A) \mid A \rightarrow \alpha \in P\}$ .    **reduce  $\alpha^R$  to  $A$  onto *stack***

$L(P_R) = \{x \in \Sigma^* \mid (x, \varepsilon) \Rightarrow_R^* (\varepsilon, S) \in \Phi\}$ .

We add  $P$  an output function  $\tau_R: \rightarrow_R^* \rightarrow P^*$  to the right parser as

$P_R = ((T^* \times \Gamma^*), \rightarrow_R, (x, \varepsilon), \{(\varepsilon, S)\}, P, \tau_R)$  with

$\tau_R((a, \varepsilon) \rightarrow_R (\varepsilon, a)) = \varepsilon$  and

$\tau_R((\varepsilon, \alpha^R) \rightarrow_R (A, \varepsilon)) = A \rightarrow \alpha \in P$ . Then

$(x, \varepsilon) \Rightarrow_R^{\theta_R} (\varepsilon, S)$ , iff  $S \Rightarrow_{rm}^{\pi_R^R} x$ ,  $\tau_R(\theta_R) = \pi_R^R \in P^*$  **right parse of  $x$  and**

$|\theta_R| = |\pi_R^R| + |x|$      $|x|$ -**shift** actions and  $|\pi_R^R|$ -**reduce** actions.

$$\begin{aligned} \text{Example } G_{Uexp}: E &\rightarrow E + T \mid T * F \mid a \mid ( E ) \\ T &\rightarrow T * F \mid a \mid ( E ) \\ F &\rightarrow a \mid ( E ) \end{aligned}$$

$$(a+a*a, \varepsilon)$$

$$\Rightarrow^a (+a*a, a)$$

$$\Rightarrow^{E \rightarrow a} (+a*a, E) \text{ or}$$

$$\Rightarrow^+ (a*a, +E)$$

$$\Rightarrow^a (*a, a+E)$$

$$\Rightarrow^{T \rightarrow a} (*a, T+E)$$

$$\Rightarrow^* (a, *T+E) \quad \Rightarrow^* (a, *F+T) \text{ or} \quad \Rightarrow^{T \rightarrow T*F} (*a, T) \quad \Rightarrow^{T \rightarrow T*F} (*a, T)$$

$$\Rightarrow^a (\varepsilon, a*T+E) \quad \Rightarrow^a (\varepsilon, a*F+T) \quad \Rightarrow^* (a, *T)$$

$$\Rightarrow^{F \rightarrow a} (\varepsilon, F*T+E) \text{ or} \Rightarrow^{T \rightarrow a} (\varepsilon, T*T+E) \text{ or} \Rightarrow^{E \rightarrow a} (\varepsilon, E*T+E) \Rightarrow^a (\varepsilon, a*T)$$

$$\Rightarrow^{T \rightarrow T*F} (\varepsilon, T+E) \quad \Rightarrow^? \text{ Error!} \quad \Rightarrow^? \text{ Error!} \quad \Rightarrow^{F \rightarrow a} (\varepsilon, F*T)$$

$$\Rightarrow^{E \rightarrow E+T} (\varepsilon, E)$$

$$\Rightarrow^{T \rightarrow a} (+a*a, T) \text{ or} \Rightarrow^{F \rightarrow a} (+a*a, F)$$

$$\Rightarrow^+ (a*a, +T) \quad \Rightarrow^+ (a*a, +F)$$

$$\Rightarrow^a (*a, a+T) \quad \Rightarrow^a (*a, a+F)$$

$$\Rightarrow^{F \rightarrow a} (*a, F+T)$$

$$\Rightarrow^{T \rightarrow T*F} (*a, T)$$

$$\Rightarrow^* (a, *T)$$

$$\Rightarrow^a (\varepsilon, a*T)$$

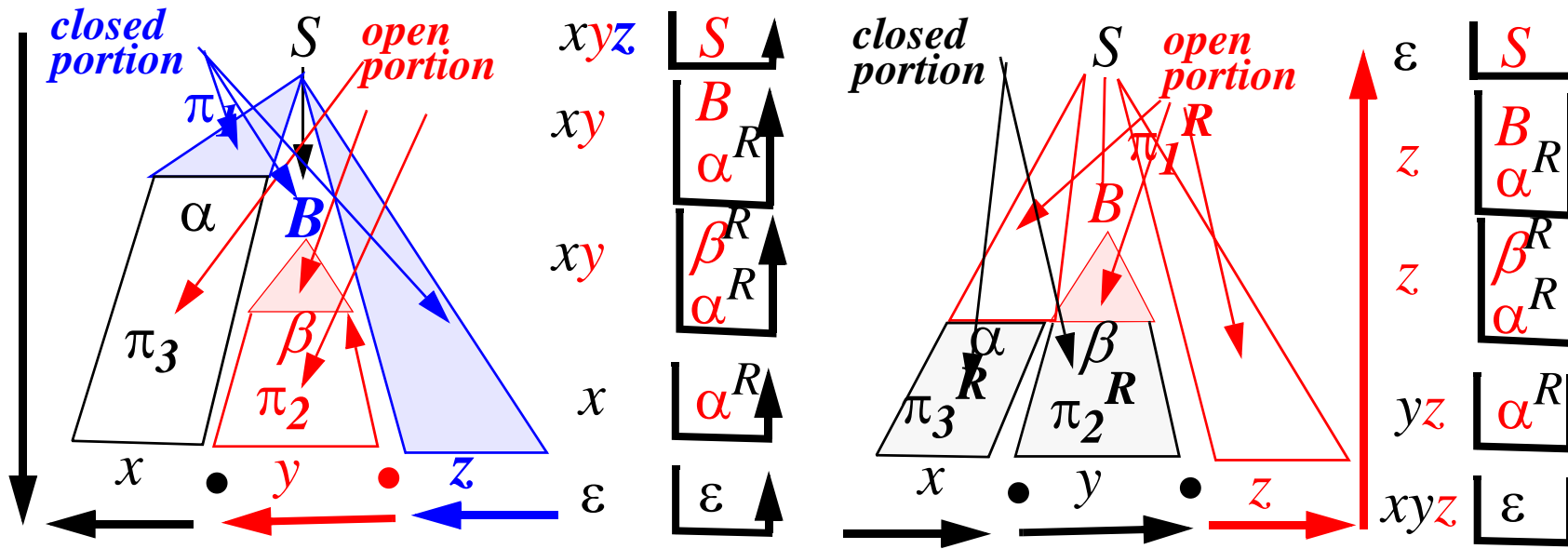
$$\Rightarrow^{F \rightarrow a} (\varepsilon, F*T)$$

Consider a **rightmost derivations** in  $G$  and **rewritings** in **right parser  $P_R$ .**

$$\begin{aligned}
 S &\Rightarrow_{rm}^{\pi_1} \alpha B z \Rightarrow_{rm}^{B \rightarrow \beta} \alpha \beta z \Rightarrow_{rm}^{\pi_2} x y z \Rightarrow_{rm}^{\pi_3} x y z \\
 (x y z, S) &\Rightarrow_R^{\theta_1} (x y, B \alpha^R) \Rightarrow_R^{(\varepsilon, B) \rightarrow (\varepsilon, \beta)} (x y, \beta^R \alpha^R) \Rightarrow_R^{\theta_2} (x, \alpha^R) \Rightarrow_R^{\theta_3} (\varepsilon, S)
 \end{aligned}$$

**Rightmost Derivation in Reversed Order**

$$\begin{aligned}
 x y z &\Rightarrow_{rm}^{\pi_3^R} \alpha y z \Rightarrow_{rm}^{\pi_2^R} \alpha \beta z \Rightarrow_{rm}^{B \leftarrow \beta} \alpha B z \Rightarrow_{rm} S \\
 (x y z, \varepsilon) &\Rightarrow_R^{\theta_3} (y z, \alpha^R) \Rightarrow_R^{\theta_2} (z, \beta^R \alpha^R) \Rightarrow_R^{(\varepsilon, \beta^R) \rightarrow (\varepsilon, B)} (z, B \alpha^R) \Rightarrow_R^{\theta_1} (\varepsilon, S)
 \end{aligned}$$



**Lemma 6.B.1** If  $(xy, \gamma^R) \Rightarrow_R^{\theta_R} (y, \delta^R)$  in  $P_R$ , then  $\delta \Rightarrow_{rm}^{\tau_R(\theta_R)^R} \gamma x$   
 and  $|\theta_R| = |\tau_R(\theta_R)| + |x|$ .

**Proof Induction on  $|\theta_R|$**

i)  $\theta_R = \varepsilon$ :  $x = \varepsilon$ ,  $\gamma^R = \delta^R$ , and  $\tau_R(\theta_R)^R = \varepsilon$ .

ii.1)  $\theta_R = A \rightarrow \alpha \cdot \theta_R'$       **reduce  $A \rightarrow \alpha \in P$**

$$(xy, \gamma^R) = (xy, \alpha^R \gamma''^R) \Rightarrow_R^{A \rightarrow \alpha} (xy, A \gamma''^R) = (xy, \gamma'^R) \Rightarrow_R^{\theta_R'} (y, \delta^R)$$

$$\text{and } |\theta_R'| = |\tau_R(\theta_R')| + |x|.$$

$$\delta^R \Rightarrow_{rm}^{\tau_R(\theta_R')^R} \gamma'^R x = \gamma''^R A x \Rightarrow_{rm}^{A \rightarrow \alpha} \gamma''^R \alpha^R x = \gamma^R x.$$

$$\therefore |\theta_R| = 1 + |\theta_R'| = 1 + |\tau_R(\theta_R')| + |x| = |\tau_R(\theta_R)| + |x|.$$

ii.2)  $\theta_R = a \cdot \theta_R'$       **shift  $a \in T$**

$$(xy, \gamma^R) = (ax'y, \gamma^R) \Rightarrow_R^a (x'y, a\gamma^R) = (x'y, \gamma) \Rightarrow_R^{\theta_R'} (y, \delta^R)$$

and  $|\theta_R'| = |\tau_R(\theta_R')| + |x|$ .

$\delta^R \Rightarrow_{rm}^{\tau_R(\theta_R')^R} \gamma'^R x' = \gamma^R a x' = \gamma^R x$  in  $G$  and

$$|\theta_R| = 1 + |\theta_R'| = |\tau_R(\theta_R')| + 1 + |x'| = |\tau_R(\theta_R)| + |x|.$$

$\therefore L(G) \subseteq L(P_R), \forall \pi_R: \text{right parse of } x \tau_R(\theta_R) = \pi_R, \text{ and } |\theta_R| = |\pi_R| + |x|.$

**Lemma 6.B.2** If  $\delta \Rightarrow_{rm}^{\pi_R^R} \gamma x$  in  $G$  and  $\gamma = \varepsilon$  or  $\gamma:1 \in N$ , then

$$(xy, \gamma^R) \Rightarrow_R^{\theta_R} (y, \delta^R) \text{ in } P_R \text{ and } |\theta_R| = |\tau_R(\theta_R)| + |x|.$$

**Proof Induction on  $|\pi_R|$**

i)  $\pi_R = \varepsilon: \delta = \gamma x, (xy, \gamma^R) \Rightarrow_R^x (y, \delta^R), |\theta_R| = |x|, \text{ and } \tau_R(\theta_R) = \pi_R = \varepsilon.$

ii)  $\pi_R = A \rightarrow \alpha \cdot \pi_R', \pi_R' \neq \varepsilon: \exists \theta_R'. \exists. \tau_R(\theta_R') = \pi_R'.$

$$\delta \Rightarrow_{rm}^{\pi_R'^R} \gamma' x' = \delta'' A x' \Rightarrow_{rm}^{A \rightarrow \alpha} \delta'' \alpha x' = \gamma x.$$

where  $|\theta_R'| = |\pi_R'| + |x'|$ ,  $\gamma = \delta''A$ ,  $\delta''\alpha = \gamma z$ ,  $zx' = x$ .

$$\begin{aligned} \therefore (xy, \gamma^R) = (zx'y, \gamma^R) &\Rightarrow_R^z (x'y, \gamma^R) = (x'y, \alpha^R \delta''^R) \Rightarrow_R^{A \rightarrow \alpha} (x'y, \\ A\delta''^R) &= (x'y, \gamma'^R) \Rightarrow_R^{x'} (y, \delta^R). \end{aligned}$$

$$\therefore |\theta_R| = |\theta_R'| + I + |z'| = |\pi_R'| + |x'| + I + |z'| = |\pi_R| + |x|.$$

$$\therefore L(P_R) \subseteq L(G), \tau_R(\theta_R) = \pi_R, \text{ and } |\pi_R| = |\theta_R| - |x|.$$

### **Theorem 6.B**

$$L(G) = L(P_R), \forall x \in L(G) \text{ in } G, \tau_R(\theta_R) = \pi_R \text{ in } P_R, \text{ and } |\theta_R| = |\pi_R| + |x|.$$