

Repeated product(composition) of relation(or function), revisited

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The **product(composition)** of R and S , denoted $R \circ S \subseteq A \times C$, is

$$R \circ S = \{(a, c) \in A \times C \mid (a, \underline{b}) \in R, (\underline{b}, c) \in S\}$$

Repeated composition of relations *on* A .

Let $R \subseteq A \times A$. Then we **define**, $R^n \subseteq A \times A$ ($n \geq 0$).

$$R^0 = \text{id}_A, \quad n = 0, \quad R^0 \text{ is an identity relation}$$

$$R^n = R \circ R^{n-1} \quad n \geq 1.$$

We also **define**, R^* , $R^+ \subseteq A \times A$.

$$R^* = \bigcup_{i \in \mathbb{N}_0} R^i,$$

reflexive transitive closure of R

$$R^+ = \bigcup_{i \in \mathbb{N}_1} R^i,$$

transitive closure of R .

Repeated composition of function

Let $f: A \rightarrow B$ and $g: B \rightarrow C$. The **product**(**composition**) of R and S , denoted $f \circ g: A \rightarrow C$, is

$$f \circ g = \{(a, c) \in A \times C \mid (a, b) \in f, (b, c) \in g\}$$

$$\text{or } = \{f \circ g(a) = c \mid f(a) = b, g(b) = c\}$$

$$\text{or } \quad f \circ g(a) = g(f(a)).$$

Repeated composition of functions **on** A .

Let $f: A \rightarrow A$. Then we **define** $f^n: A \rightarrow A$ for $n \geq 0$.

$$f^0 =_B \text{id}_A, \quad n = 0, \quad f^0 \text{ is an identity function}$$

$$f^n =_R f^{n-1} \circ f \quad n \geq 1.$$

We also **define** f^* and f^+ : $A \rightarrow A$.

$$f^* = \cup_{i \in \mathbb{N}_0} f^i \text{ and } f^+ = \cup_{i \in \mathbb{N}_1} f^i.$$

Repeated composition of function for the second domain

Let $f: A \times B \rightarrow A$. We extend $f^2 = f \circ f: A \times B^2 \rightarrow A$.

$$f \circ f = \{((a_0, b_1b_2), a_2) \in (A \times B^2) \times A \mid ((a_0, b_1), a_1) \in f, ((a_1, b_2), a_2) \in f\}$$

$$\text{or} = \{f \circ f(a_0, b_1b_2) = a_2 \mid f(a_0, b_1) = a_1, f(a_1, b_2) = a_2\}$$

$$\text{or} \quad f \circ f(a_0, b_1b_2) = f(f(a_0, b_1), b_2)$$

We define $f^n: A \times B^n \rightarrow A$ for $n \geq 0$. Let $a \in A$, $\varepsilon \in B^0$, $b \in B$, $w \in B^{n-1}$.

$$f^0(a, \varepsilon) =_B a, \quad n = 0.$$

$$f^n(a, bw) =_R f^{n-1} \circ f \quad n \geq 1.$$

$$\text{or} =_R f^{n-1}(f(a, b), w) \quad n \geq 1.$$

We also define, $f^*: A \times B^* \rightarrow A$ and $f^+: A \times B^+ \rightarrow A$.

$$f^* = \cup_{i \in \mathbb{N}_0} f^i \quad f^+ = \cup_{i \in \mathbb{N}_1} f^i.$$

Repeated composition of function whose range and domain are sets.

Let $f: A \times B \rightarrow 2^A$. We extend

$f': 2^A \times B \rightarrow 2^A$. Let $A' \subseteq A$ (or $A' \in 2^A$)

$$f'(A', b) = \cup_{a \in A'} f(a, b).$$

We may write f instead of f' , since $f \subseteq f'$, and we also may write $f(a)$ instead of $f(\{a\})$ for short.

$$f: 2^A \times B \rightarrow 2^A.$$

We define $f^n: 2^A \times B^n \rightarrow 2^A$ for $n \geq 0$. Let $A' \subseteq A$, $\varepsilon \in B^0$, $b \in B$, $w \in B^{n-1}$.

$$f^0(A', \varepsilon) =_B A', \quad n = 0.$$

$$f^n(A', bw) =_R f^{n-1} \circ f \quad n \geq 1.$$

$$\text{or } =_R f^{n-1}(f(A', b), w)$$

We also define $f^*: 2^A \times B^* \rightarrow 2^A$.

$$f^* = \cup_{i \in \mathbb{N}_0} f^i.$$

$f^+: 2^A \times B^+ \rightarrow 2^A$.

$$f^+ = \cup_{i \in \mathbb{N}_1} f^i.$$