1.A Sets, Relations, Graphs, and Functions

1.A.1 Set *a **collection** of objects(**element**)*

Let A be a set and a be an elements in A, then we write $a \in A$.

How to specify sets

1. *to **enumerate all** of the elements* 원소나열법
2. *to state the **properties** that characterizes the elements.* 조건제시법

$A = \{x \mid p(x)\}$

*p(x) is a **predicate***

*p(x) is either **true** or **false** depending on x*

$A = \{x \in U \mid p(x)\}$

$A \subseteq U$, *U is the **universe** of **discourse***

$x \in U$ *U is the **type** of x in A* *U x in C, Java*

$p(x)$ ***attribute** of x*

3. ***automata**, **grammars**, **programs***

Three cases for two sets A and B

1. ***subset***

$A \subseteq B$ or $B \subseteq A$

$\Leftrightarrow \quad A - B = \emptyset$ or $B - A = \emptyset$

$\Leftrightarrow \quad A \cap \overline{B} = \emptyset$ or $B \cap \overline{A} = \emptyset$

2. ***disjoint***

$A \cap B = \emptyset$

3. *in general(**incomparable**, neither subset nor disjoint)*

not($A \subseteq B$ or $B \subseteq A$) and not ($A \cap B = \emptyset$)

$\Leftrightarrow \quad A \not\subseteq B$ and $B \not\subseteq A$ and $A \cap B \neq \emptyset$.

$\Leftrightarrow \quad A \cap \overline{B} \neq \emptyset$ and $\overline{A} \cap B \neq \emptyset$ and $A \cap B \neq \emptyset$.

$A \cap B \neq \emptyset$

Venn diagram 2^n *regions*

Truth table 2^n *raws*

$\overline{A} \cap \overline{B} \overset{?}{=} \emptyset$

1.A.2 Binary relation

Cartecian product *of two sets, A and B*

$A \times B = \{(a, b) | a \in A, b \in B\}$.

$(a, b) \in A \times B$: an ***ordered pair***.

$|A \times B| = |A| \times |B|$.

Binary relation *R from the set A(**domain**) to the set B(**range**).*

$R \subseteq A \times B$. $a \in A, b \in B, (a, b) \in R$ or ***a R b***.

$|R| \leq |A \times B|$.

Inverse *of a relation R,* $R^{-1} = \{(b, a) \in B \times A | (a, b) \in R\}$

Composition(Product) *of two relations R and S*

where $R \subseteq A \times \underline{B}$ and $S \subseteq \underline{B} \times C$.

$R \cdot S = \{(a, c) | (a, \underline{b}) \in R, (\underline{b}, c) \in S\}$

*Binary relation R **on** A* $R \subseteq A \times A$.

Identity relation *R on A* $id_A = \{(a, a) | a \in A\}$

$\forall R \subseteq A \times A,\ R \cdot id_A = id_A \cdot R = R$.

Repeated composition(product) *of a binary relation R* ***on A***.

Let $R \subseteq A \times A$. *We define*

$R^2 = R \cdot R$, $R^3 = R \cdot R \cdot R$, … $R^n = R \cdot R \cdot \ldots \cdot R$, … *and*

$R = R^1$. *Then we can* ***define***

$R^n R^m = R^{n+m}$, for $(\forall n, m \in \mathbb{N})$, $n, m \geq 1$.

$R^0 = ?$ *If we define* $R^0 = id_A$. *Then we can* ***extend*** *the definition*

$R^n R^m = R^{n+m}$, for $n, m \geq 0$.

*Another (****recursive****) definition for* ***repeated product*** *of binary relations*

$R^0 =_B id_A$. ***basis***

$R^n =_R R \cdot R^{n-1}$, $n \geq 1$. ***recursion***

ex) $R^3 =_R R \cdot R^2 =_R R \cdot R \cdot R^1 =_R R \cdot R \cdot R \cdot R^0 =_B R \cdot R \cdot R \cdot id_A = R \cdot R \cdot R$

1.A.3 *A* ***directed graph*** *$G = (V, E)$ is*

V: a set of ***vertices****,*

$E \subseteq V \times V$: a set of ***edges****,*

E: a ***binary relation*** *on V*

Some properties of the binary relations

1) R is ***reflexive****, if $\forall a \in A$, $a R a$.* $id_A \subseteq R$

R is ***irreflexive****, if $\forall a \in A$, $a \not{R} a$.* $R \cap id_A = \emptyset$

2) R is ***symmetric(=)****, if $a R b$* impiles *$b R a$.* $R = R^{-1}$

R is ***asymmetric(<)****, if $a R b$* implies *$b \not{R} a$.* $R \cap R^{-1} = \emptyset$

R is ***antisymmetric(≤)****, if $a R b$* and $a \neq b$ implies *$b \not{R} a$.* $R \cap R^{-1} \subseteq id_A$

R is ***asymmetric*** $\Rightarrow$ *R is* ***irreflexive****.*

R is ***asymmetric*** $\Rightarrow$ *R is* ***antisymmetric****.*

3) R is ***transitive****, if $a R b$* and *$b R c$* implies *$a R c$.* $R \cdot R \subseteq R$

*Let **P** = {reflexive, symmetric, transitive}. Then R' be **P**-closure of R, if*

*i) R' is **P**.*

ii) $R \subseteq R'$.

*iii) R' is the **smallest** set among satisfying i) and ii).*

$\Leftrightarrow$ $^\forall R''$ satisfying i) and ii), $R' \subseteq R''$.

***reflexive closure** of R, $R' = R \cup id_A$.*

***symmetric closure** of R, $R' = R \cup R^{-1}$.*

***transitive closure** of R,*

$R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots = \cup_{i \in N_1} R^i$ where $N_1 = \{1, 2, 3, \ldots\}$.

***reflexive-transitive closure** of R,*

$R^* = R^0 \cup R^1 \cup R^2 \cup R^3 \cup \ldots = \cup_{i \in N_0} R^i$ where $N_0 = \{0, 1, 2, \ldots\}$.

*What is the **reflexive** (, **symmetric**) and **transitive closure** of R in a graph (A, R)?*

Partition *of a set A.*

Let A be a set and $A_1, A_2,\ldots, A_n \subseteq A$*. Then we define* ***partition*** *of a, as*

$Par(A) = \{A_1, A_2, \ldots, A_n\}$ *is called a* ***partition*** *of A, written Par(A),*

if 1) $\cup_{i\in\{1, 2, \ldots, n\}} A_i = A \wedge$ ***exhaustive***

2) $1 \le i\neq j \le n$*:* $A_i \cap A_j = \varnothing$*.* *(pairwise)* ***disjoint***

$|Par(A)|$ *is the* ***size*** *of the partioion.*

Ex. Consider $A = \{a_1, a_2,\ldots, a_n\}$*. What are Par(A)'s?*

Cover *of a set A. Cover(A) =* $\{A_1, A_2, \ldots, A_n\}$ *where*

if $\cup_{i\in\{1, 2, \ldots, n\}} A_i = A$ ***exhaustive***

Power set *of a set A,*

$2^A = P(A) = \{B|\ B \subseteq A\}$ $B \subseteq A \Leftrightarrow B \in 2^A$.

$|2^A| = 2^{|A|}$.

$par(A) \subseteq 2^A$.

A binary relation R on A is ***equivalence****,*

if R is ***reflexive****,* ***symmetric****, and* ***transitive****.*

Par(A) — *partition of A*

*A binary relation R on A is (****(ir)reflexive****)* ***partial order****,*

*if R is (****ir****)****reflexive****,* ***antisymmetric****, and* ***transitive****.*

*A: partially-ordered set(****poset****)*

Let $R \subseteq A \times A$ *be an* ***equivalence****,*

$[a]_R = \{b \in A \mid a\ R\ b\}$ — ***equivalence class****,*

$\{[a]_R \mid a \in A\}$ — *equivalence* ***partition****.*

a set of ***equivalence classes****.*

$\cup_{a \in A} [a]_R = A$, — ***exhaustive***

if $a\ R\ b$, $[a]_R = [b]_R$. — ***same*** *equivalent class*

if $a \not R\ b$. $[a]_R \cap [b]_R = \varnothing$. — *(pairwise)* ***disjoint***

Let be $\leq$ *a partial order(**relation**) on A.* $\leq \subseteq A \times A$

Then $(A, \leq)$ *is called as partially ordered set or **poset** for short.*

Let $(A, \leq)$ *be a **poset**. We define two binary operators on A,*

$ub, lb: A \times A \rightarrow 2^A$

$ub(a, b) = \{c \in A \mid a \leq c, b \leq c\}$ *upper bound(* 배수 *)*

$lb(a, b) = \{c \in A \mid c \leq a, c \leq b\}$ *lower bound(* 약수 *)*

*If a **unique** lub*$(\vee,)$ *and a **unique** glb*$(\wedge)$,

$\vee, \wedge: A \times A \rightarrow A.$ $(A, \leq)$ *is called as a **lattice** and*

$a \vee b = min(ub(a, b))$ *least upper bound(* 최소공배수 *)*

$a \wedge b = max(lb(a, b)).$ *greatest lower bound(* 최대공약수 *)*

$(A, \vee, \wedge)$ *is called an **algebra** induced by the lattice* $(A, \leq)$.

$(\mathbb{N}, lcm, gcd)$ *is an algebra induced by a lattice* $(\mathbb{N}, |)$.

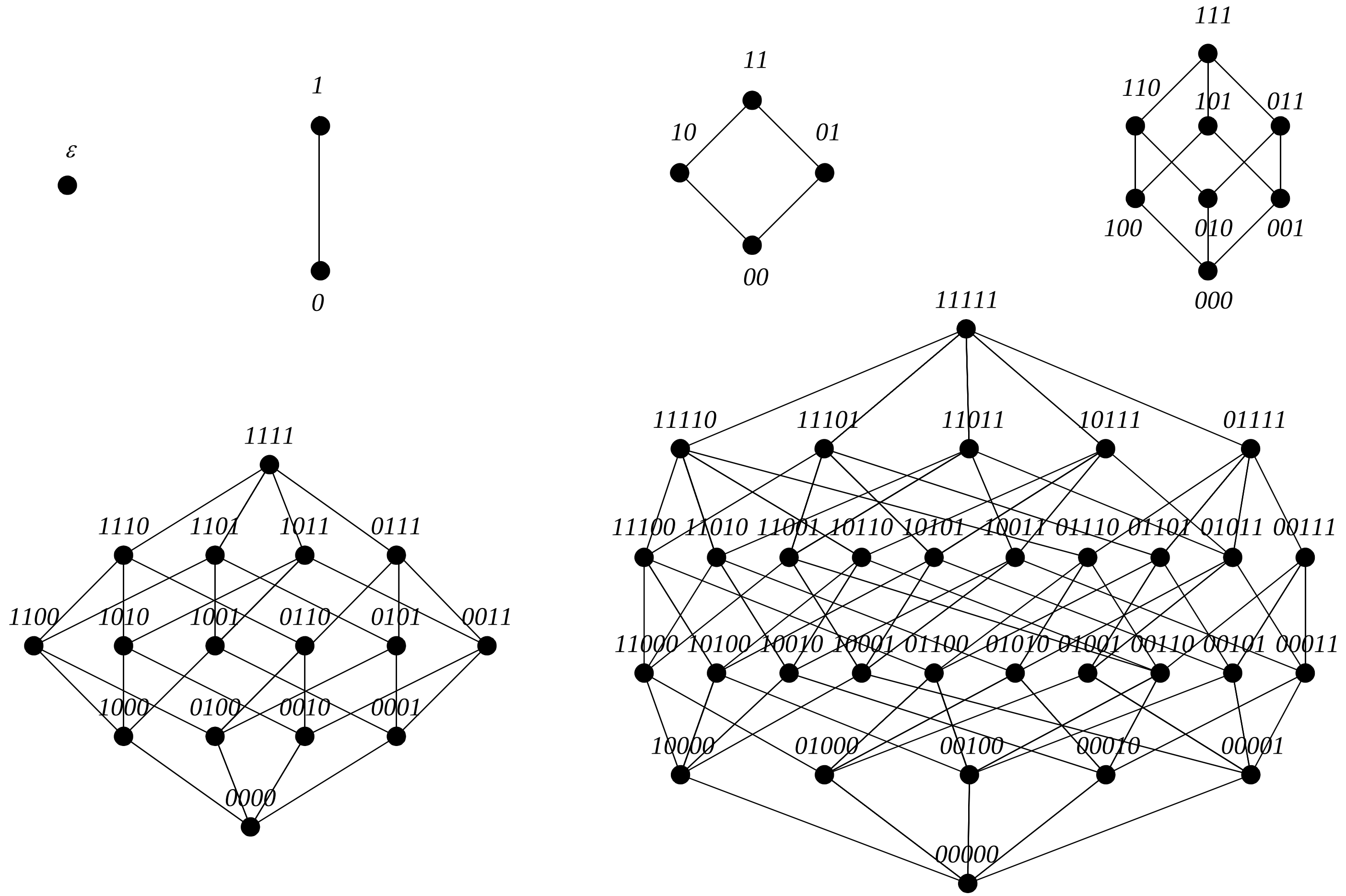
Boolean** algebra,* $(\{f, t\}, \vee, \wedge)$*, is induced by the **lattice $(\{f, t\}, \{f \leq t\})$.

***Singleton set** algebra,* $(2^{\{a\}}, \cup, \cap)$*, is **isomorphic** to*

*the **boolean** algebra,* $(\{f, t\}, \vee, \wedge)$ *with respect to **bijection** f.*

CS204 TP of Chap. 13(12) boolean algebra p15 Fig.

*A $|A|$-bit string algebra $(\{0, 1\}^{|A|}, \vee, \wedge)$ induced by $(\{0, 1\}^{|A|}, \leq)$ is **isomorphic** to the set algebra $(2^A, \cup, \cap)$ induced by the **lattice** $(2^A, \subseteq)$.*



1.A.4 *Algebraic system, semi-group and monoid*

Let A be a set and $\oplus$ *be a* ***binary*** *operation on A.*

$\oplus: A \times A \rightarrow A.$

1. *(A,* $\oplus$*) is an* ***algebraic system****.*

   if $^{\forall}a, b \in A, a \oplus b \in A.$ ***closed***

2. *(A,* $\oplus$*) is a* ***semi-group****.*

   if $^{\forall}a, b, c \in A, a \oplus (b \oplus c) = (a \oplus b) \oplus c$ ***associative***

   $a_1 + a_2 + \ldots + a_n = +_{i \in \{1, 2, ..., n\}} a_i.$ *indexed set notation*

   binary operation $\Rightarrow$ ***n-ary*** *operation*

3. *(A,* $\oplus$*, e) is a* ***monoid****.*

   if $^{\exists}e \in A$ *.э.* $^{\forall}a \in A, e \oplus a = a \oplus e = a$ ***identity***

   (A, $\oplus$*, e) is called as a* ***monoid****.*

Let (A, ⊕, e) and (B, ⊗, ε) be two monoids.

If i) h: A → B is a onto function, $|A| \geq |B|$

ii) h(a ⊕ b) = h(a) ⊗ h(b), and *preserve operation*

iii) h(e) = ε. *preserve identity*

Then h is called a ***homomorphism****, and the monoid (B, ◦, ε) is called a* ***homomorphic*** *to the monoid (A, ⊕, e) w.r.t. h.*

(A, ⊕, e) is called ***concretization*** *of (B, ⊗, ε) and (B, ◦, ε) is called* ***abstract interpretation*** *of (A, ⊕, e).*

If h is one-to-one and onto, h is called ***isomorphism****.*

1.A.5 *A binary* ***relation*** *from A to B is a* ***function*** *from A to B, if*

1) $\forall a \in A, \exists (a, b) \in f,$ ***total***

2) $\forall a \in A, \exists_1 (a, b) \in f.$ ***unique***

$f: A \rightarrow B$ $(a, b) \in f$ or $a\ f\ b$ or $f(a) = b$ or $f\ a = b$.

Three ***faces*** *of a binary* ***relation***

i) $R \subseteq A \times B$. $(a, b) \in R$. i) *a* ***set of*** *(ordered)* ***pairs***

ii) $R: A \times B \rightarrow \{$**false, true**$\}$.

$a\ R\ b$, iff $(a, b) \in R$. ii) *a* ***relational operator***$(<, =, \leq)$

iii) $R: A \rightarrow 2^B$.

$R(a) = \{b_1, b_2, \ldots, b_n\}$, iff $(a, b_1), (a, b_2), \ldots, (a, b_n) \in R$.

$\forall a \in A, \exists_1 \{b_1, b_2, \ldots, b_n\} \subseteq B$ or $\exists_1 \{b_1, b_2, \ldots, b_n\} \in 2^B$.

$\therefore R: A \rightarrow 2^B$. iii) *a* ***set valued function***

Let f: A → B. Is f^{-1}: B → A a function?

No!!!

Function f: A → B is ***onto***(***surjection***; ***correspondence***), *if*

$\forall b \in B, \exists a \in A$.∋. $f(a) = b$. $|A| \geq |B|$

If f is onto, f^{-1} is ***total*** *but* ***not*** *unique function.*

Function f: A → B is ***one-to-one***(***injection***, *1-1), if*

$\forall a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$.

if $\exists b \in B$.∋. $f(a) = b$, $\exists_1 a \in A$. $|A| \leq |B|$

If f is 1-1, f^{-1} is ***unique*** *but* ***not*** *total function.*

Function f: A → B is ***bijective***,

if f is both ***1-1*** *and* ***onto(1-1 correpondence)***.

$\forall b \in B$, *if* $\exists_1 a \in A$.∋. $f(a) = b$.. $|A| = |B|$

If f is 1-1 onto, f^{-1} is both ***total*** *and* ***unique***, *so is a* ***function***.

1.B Set isomorphism and infinite sets

*If there exists a **bijection**(짝짓기 , **1-1 onto**) f from A to B,*
*two sets A and B have same **cardinality**, written |A| = |B|, and*
*two sets A and B are said to be **isomorphic** w.r.t. f, written* $A \cong_f B$.

*A set is said to be **countable**(**enumerable**),*
*if it has the **same cardinality** with a **subset** of ℕ,*
*either **finite** or **infinite***
*and **uncountable**(**uncountably infinite**), otherwise.*
*A set is **countably infinite**, if it has the **same cardinality** with ℕ.*
*the **cardinality** of ℕ is denoted as ℵ, |ℕ| = ℵ.*

*Let A be **countable**. Then we can **enumerate** the set in **numeric** order.*

$A = \{a_0, a_1, \ldots, a_n\}$ — ***finite** for some* $n \geq 0$.

$A = \{a_0, a_1, \ldots\}$ — ***countably infinite**(**enumerable**)*

Consider

$\mathbb{N}_1 = \{1, 2, 3, \ldots\}, \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ $|\mathbb{N}_1| = |\mathbb{N}_0| = \aleph$, but $\mathbb{N}_1 \subset \mathbb{N}_0$.

$E = \{e \in \mathbb{N} \mid e = 2i, i \in \mathbb{N}\}$ $|E| = |\mathbb{N}| = \aleph$, but $E \subset \mathbb{N}$.

$I = \mathbb{N} \cup \{-i \mid i \in \mathbb{N}\}$ $|I| = |\mathbb{N}| = \aleph$, but $\mathbb{N} \subset I$.

$Q = \mathbb{N} \times \mathbb{N}$ $|Q| = |\mathbb{N}| = \aleph$.

enumerate $(i, j) \in \mathbb{N} \times \mathbb{N}$ *in* (***natural***) ***numeric*** *order*

$\mathbb{N} \times \mathbb{N} = \{\underline{(0, 0)}, \underline{(1, 0), (0, 1)}, \underline{(2, 0), (1, 1), (0, 2)}, \underline{(3, 0), (2, 1),} \ldots \}$

$f(i, j) = 1 + 2 + 3 + \ldots + (i+j) + j$ $f(0, 0) = 0$

$= (i+j)(i+j+1)/2 + j$

$f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is 1-1 and onto

Consider $f^{-1}: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ which is also 1-1 and onto.

$\therefore |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph$.

Fact $|\Sigma^*| = \aleph$.

Proof Lexicographical order *for* Σ^*.

in the order of ***size*** *and if same size, in* ***alphabetic order***.

Let $|\Sigma| = k$. *Then we can alphabetic order* $\Sigma = \{a_1, a_2 \ldots, a_k\}$,

and we can order $x \in \Sigma^*$ *in* ***lexicographical order*** *as follows.*

$\Sigma^* = \{(\underline{\varepsilon}), (\underline{a_1}, \underline{a_2}, \underline{\ldots}, \underline{a_k}), (\underline{a_1 a_1}, \underline{a_1 a_2}, \underline{\ldots}, \underline{a_1 a_k}, \underline{a_2 a_1}, \underline{\ldots}, \underline{a_k a_k}), (\underline{a_1 a_1 a_1}, \underline{\ldots}\}$

If $x = b_1 b_2 \ldots b_n \in \Sigma^*$, $f: \Sigma^* \to \mathbb{N}$ *is as follows;*

$$f(x) = k^0 + k^1 \ldots + k^{n-1} + b_1 k^{n-1} + b_2 k^{n-2} + \ldots + b_n k^0$$

$$= (k^n - 1)/(k-1) + b_1 k^{n-1} + b_2 k^{n-2} + \ldots + b_n k^0.$$

What is f^{-1}?

We ***enumerate*** $x = a_1 a_2 \ldots a_n \in \Sigma^*$ *in* ***numeric order***

$\therefore f: \Sigma^* \to \mathbb{N}$ *is one-to-one onto. Q.E.D.*

Consider $\{0, 1\}^{\mathbb{N}}$*:* ***infinite*** *binary strings (See pp.12)*

and $2^{\mathbb{N}}$*:* ***power*** *set of natural numbers (Note that* $2^A = \{B \mid B \subseteq A\}$

Power set of integers and infinite binary strings are ***isomorphic****.*

$j \in 2^{\mathbb{N}} \leftrightarrow b_{ij} = 1$ *for* $(b_{i0}, b_{i1}, \ldots b_{ii}, \ldots) \in \{0, 1\}^{\mathbb{N}}$ *where* $b_{ij} \in \{0, 1\}.^{N}$

$\therefore |2^{\mathbb{N}}| \cong_f |\{0, 1\}^{\mathbb{N}}|.$

Cantor's diagonal argument

Assume $\{0, 1\}^{\mathbb{N}}$ *is* ***countable****.*

We can ***enumerate*** $\{0, 1\}^{\mathbb{N}}$*, infinite binary string, in* ***numeric*** *order,*

$b_0 = (b_{00}, b_{01}, \ldots, b_{0n}, \ldots)$

$b_1 = (b_{10}, b_{11}, \ldots, b_{1n}, \ldots)$

…

$b_n = (b_{1n}, b_{1n}, \ldots, b_{nn}, \ldots)$

…

Consider $b = (b_{00}, b_{11}, \ldots, b_{nn}, \ldots)$ *(**diagonal** elements) and*

$\bar{b} = (\bar{b}_{00}, \bar{b}_{11}, \ldots, \bar{b}_{nn}, \ldots)$ *where* $\bar{b}_{ii} = 0$, *if* $b_{ii} = 1$; $\bar{b}_{ii} = 1$, *if* $b_{ii} = 0$.

***complement** of **diagonal** elements*

Then $b = b_i$ *for some* $i \in \mathbb{N}$ *but* $\bar{b} \notin b_i$ *for any* $i \in \mathbb{N}$.

But $b, \bar{b} \in \{0, 1\}^{\mathbb{N}}$*!*

$\therefore$ *The **assumption*** $\{0, 1\}^{\mathbb{N}} = \{b_0, b_1, \ldots, b_n, \ldots\}$ *was **wrong**.*

$\therefore$ *Contradiction!!!*

*We **fail** to **enumerate*** $\{0, 1\}^{\mathbb{N}} = \{b_0, b_1, \ldots, b_n, \ldots\}$ *in **numeric order**.*

$\therefore$ *We **conclude** that* $|\{0, 1\}^{\mathbb{N}}| > |\{b_0, b_1, \ldots, b_n, \ldots\}| = \aleph$.

$\{0, 1\}^{\mathbb{N}}$ *and* $2^{\mathbb{N}}$ *are **uncountable**.*

*Infinite binary strings and power set of integers are **uncountable**.*

$\{0, 1\}^*$ vs $\{0, 1\}^{\mathbb{N}}$.

$\{0, 1\}^*$: **countably** infinite union of **finite** binary strings

$= \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2 \cup \{0, 1\}^3 \cup \ldots$

$= \{\varepsilon\} \cup \{0, 1\} \cup \{00, 01, 10, 11\} \cup \{000, \ldots, 111\} \cup \ldots$

$= \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots, 111, \ldots\}$

$\{0, 1\}^{\mathbb{N}}$: **uncountaby** infinite union of) **infinite** binary strings

$= \{000\ldots000\ldots,$ $\leftrightarrow \{\}$

$100\ldots000\ldots,$ $\leftrightarrow \{0\}$

$010\ldots000\ldots,$ $\leftrightarrow \{1\}$

$\ldots,$

$101\ldots010\ldots,$ $\leftrightarrow \{0, 2, \ldots, n, \ldots\}$

$\ldots,$

$111\ldots111\ldots\}$ $\leftrightarrow \{0, 1, 2, \ldots\} = \mathbb{N}$

$\ldots$

$|\{0, 1\}^{\mathbb{N}}| = |2^{\mathbb{N}}| > |\{0, 1\}^*| = \aleph.$

Cantor's diagonal argument

Complement of diagonal element

Russel's paradox

$S = \{x \mid x \notin x\}$

$x \in S$, *iff* $x \notin x$. *But* $S \in S$, *iff* $S \notin S$. ***contradictory!***

Halting problem

H(P): ***if*** *halt(P, P) then loop forever*

elses not *halt(P, P) then stop* ***fi***

What happens if H(H) stops or loops forever?

Denial of self recursion

Σ^* *is* ***countable***. ***strings*** *are* ***countable***

But is 2^{Σ^*} ***uncountable***. ***languages*** *are* ***uncountable***

class of languages

N. Chomsky

Finite(countable)

Countably infinite

natural numbers, integers, rational numbers,

***finite** strings …*

Uncountable

Cantor's diagonal argument

***power set** of natural numbers*

***infinite** strings*

real numbers

*Some informal descriptions on **countable** and **uncountable** infiniteness*

$\aleph \pm k = \aleph$ $\quad \aleph \times k = \aleph$ ***countable***

$\aleph \times \aleph = \aleph$ $\quad \aleph^k = \aleph$ ***countable***

But $k \times k \times \ldots = k^{\aleph} > \aleph$ ***uncountable**(k ≥ 2)*

1.C Symbol, vocabulary, string and language

*Let Σ be a set of **symbols** (문자) called as*

*a **vocabulary**(어휘 or **alphabet**) of a language.*

*We define a **string** x over Σ, whose length is n, is defined as*

$x = a_1a_2\ldots a_n$ *where* $1 \leq \forall i \leq n$: $a_i \in \Sigma$. *and we write* $|x| = n$.

*A **string**(문자열) is a **sequence** of **symbols.***

Example $\Sigma = \{a, b, ..., z\}$ *English alphabet*

$x = school$ $y = boy$

$|x| = 6$ $|y| = 3$.

***Concatenation**(·) of two strings*

$x = a_1a_2\ldots a_n$ *and* $y = b_1b_2\ldots b_m$.

$x \cdot y = a_1a_2\ldots a_nb_1b_2\ldots b_m = xy$ *justaxaposed*

Example $x \cdot y = xy = schoolboy$ $|xy| = 9$.

We define ***concatenation*** *of two strings as*

$\cdot: \Sigma^* \times \Sigma^* \to \Sigma^*$ *a* ***binary operation*** *on* ***strings***

(1) $^\forall x, y \in \Sigma^*, xy \in \Sigma^*.$ ***closed***

(2) $^\exists x, y \in \Sigma^*, xy \neq yx$ ***noncommutative***

(3) $^\forall x, y, z \in \Sigma^*, x(yz) = (xy)z$ ***associative***

(4) We ***define*** *an* ***empty string*** *denoted as* ε *(or* Λ*)*

as an ***identity*** *element w.r.t. the concatene operation.*

i.e. $^\forall x \in \Sigma^*, \varepsilon x = x\varepsilon = x$ *where* $|\varepsilon| = 0$*.*

Then $(\Sigma^*, \cdot, \varepsilon)$ *is a noncommutative* ***monoid****.*

We can ***extend*** *the domain and range of the concatenation*

from set of ***strings*** *to the set of* ***languages****(set of* ***set of strings****)*

$\cdot: 2^{\Sigma^*} \times 2^{\Sigma^*} \to 2^{\Sigma^*}.$

Then $(2^{\Sigma^*}, \cdot, \{\varepsilon\})$ *is a (n* ***induced****) noncommutative* ***monoid****.*

Four terminologies *on the* ***formal language theory****(FLT).*

	element	***set***
	symbol(문자)	***vocabulary***(어휘)
	$a, b, c \in \Sigma$	Σ
sequence(열)	*string*(문자열)	***language***(언어)
	$x, y, z \in \Sigma^*$	$L, S, T \subseteq \Sigma^*$
		$L, S, T \in 2^{\Sigma^*}$

Power of an alphabet revisited

$\Sigma^0 =_B \{\varepsilon\}$ ***basis***

$\Sigma^n =_R \Sigma\Sigma^{n-1}$ *for* $n \geq 1$ ***recursion***

$|\Sigma^n| = |\Sigma|^n.$

$\Sigma^* = \cup_{i\in\mathbb{N}0} \Sigma^i = \{\varepsilon\} \cup \Sigma \cup \Sigma^2 \ldots$ $\Sigma^+ = \cup_{i\in\mathbb{N}1} \Sigma^i = \Sigma \cup \Sigma^2 \ldots$

Let $B^A = \{f \mid f: A \rightarrow B\}$. *Then* $|B^A| = |B|^{|A|}$.

Consider $\Sigma^n = \Sigma^{\{1, 2, \ldots, n\}} = \{f \mid f: \{1, 2, ..., n\} \rightarrow \Sigma\}$

Example $x: \{1, 2, ..., 6\} \rightarrow \{a, b, ..., z\}$ *and* $y: \{1, 2, 3\} \rightarrow \{a, b, ..., z\}$

$x = school \in \Sigma^6$ *where* $x(1)=s, x(2)=c, ..., x(6)=l$; *or* $x = (s, c, h, o, o, l)$

$y = boy = (b, o, y) \in \Sigma^3$ *where* $y(1) = b, y(2) = o, y(3) = y$.

strings and functions are ***isomorphic****!*

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots = \cup_{i \in N_1} \Sigma^i$ *where* $N_1 = \{1, 2, 3, \ldots\}$.

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots = \cup_{i \in N_0} \Sigma^i$ *where* $N_0 = \{0, 1, 2, \ldots\}$.

Σ^* *is a* ***universe*** *of strings including* ***identity***(ε)

$(\Sigma^*, \cdot, \varepsilon)$ *is a (****free****)* ***monoid*** *over* Σ. *(unique representation)*

2^{Σ^*} *is a* ***universe*** *of languages*

$(2^{\Sigma^*}, \cdot, \{\varepsilon\})$*is a (****free****) induced* ***monoid*** *over* Σ.

Reversal, ***prefix*** *and* ***suffix*** *of* ***strings***

Let $x = a_1a_2 \ldots a_n$ *be a string of length* $n \geq 0$ *and* $k \geq 0$.

$x^R = a_na_{n-1} \ldots a_2a_1$. ***reversal*** *of x.*

recursive definition *of* ***reversal*** *of* $x \in \Sigma^*$.

$\varepsilon^R =_B \varepsilon$.

Let $a \in \Sigma$ *and* $x \in \Sigma^*$. *Then* $(ax)^R =_R x^Ra$.

Ex. $abc^R =_R bc^Ra =_R c^Rba =_R \varepsilon^Rcba =_B \varepsilon cba = cba$.

Prefix *and* ***suffix*** *of a string. For* $k \in \mathbb{N}$,

$k{:}x = a_1a_2 \ldots a_k$, *if* $k \leq n$; ***prefix*** *of x with length k.*

$= x$, *otherwise.*

$x{:}k = (k{:}x^R)^R$. ***suffix*** *of x with length k.*

Ex. $3{:}school = sch$, $boy{:}2 = oy$, $boy{:}5 = boy$.

For $k \geq 0$: $k{:}$, $:k$: $\Sigma^* \rightarrow \Sigma^{\leq k} = \{\varepsilon\} \cup \Sigma \cup \Sigma^2 \cup \ldots \cup \Sigma^k$.