

1.A Sets, Relations, Graphs, and Functions

1.A.1 Set *a collection of objects(element)*

Let A be a set and a be an elements in A , then we write $a \in A$.

How to specify sets

1. to *enumerate all of the elements* 원소나열법

2. to state the **properties** that characterizes the elements. 조건제시법

$$A = \{x \mid p(x)\}$$

$p(x)$ is a **predicate**

$p(x)$ is either **true** or **false** depending on x

$$A = \{x \in U \mid p(x)\}$$

$A \subseteq U$, U is the **universe of discourse**

$x \in U$ U is the **type** of x in A U x in C , Java

$p(x)$ **attribute** of x

3. **automata, grammars, programs**

Three cases for two sets A and B

1. subset

$$A \subseteq B \text{ or } B \subseteq A$$

$$\Leftrightarrow A - B = \emptyset \text{ or } B - A = \emptyset$$

$$\Leftrightarrow A \cap \bar{B} = \emptyset \text{ or } B \cap \bar{A} = \emptyset$$

2. disjoint

$$A \cap B = \emptyset$$

3. in general (incomparable, neither subset nor disjoint)

$$\text{not}(A \subseteq B \text{ or } B \subseteq A) \text{ and } \text{not}(A \cap B = \emptyset)$$

$$\Leftrightarrow A \not\subseteq B \text{ and } B \not\subseteq A \text{ and } A \cap B \neq \emptyset.$$

$$\Leftrightarrow A \cap \bar{B} \neq \emptyset \text{ and } \bar{A} \cap B \neq \emptyset \text{ and } A \cap B \neq \emptyset.$$

$$A \cap B \neq \emptyset$$

Venn diagram 2^n regions

Truth table 2^n rows

$$\bar{A} \cap \bar{B} \neq \emptyset$$

1.A.2 Binary relation

Cartesian product of two sets, A and B

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

$(a, b) \in A \times B$: an **ordered pair**.

$$|A \times B| = |A| \times |B|.$$

Binary relation R **from** the set A (**domain**) **to** the set B (**range**).

$$R \subseteq A \times B. \quad a \in A, b \in B, (a, b) \in R \text{ or } a R b.$$

$$|R| \leq |A \times B|.$$

Inverse of a relation R , $R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}$

Composition (Product) of two relations R and S

where $R \subseteq A \times B$ and $S \subseteq B \times C$.

$$R \cdot S = \{(a, c) \mid (a, b) \in R, (b, c) \in S\}$$

Binary relation R **on** A $R \subseteq A \times A$.

Identity relation R on A $id_A = \{(a, a) \mid a \in A\}$

$$\forall R \subseteq A \times A, R \cdot id_A = id_A \cdot R = R.$$

Repeated composition(product) of a binary relation R .

Let $R \subseteq A \times A$. We define

$$R^2 = R \cdot R, \quad R^3 = R \cdot R \cdot R, \quad \dots \quad R^n = R \cdot R \cdot \dots \cdot R, \quad \dots \text{ and}$$

$R = R^1$. Then we can **define**

$$R^n R^m = R^{n+m}, \text{ for } (\forall n, m \in \mathbb{N}), n, m \geq 1.$$

$R^0 = ?$ If we define $R^0 = id_A$,. Then we can **extend** the definition

$$R^n R^m = R^{n+m}, \text{ for } n, m \geq 0.$$

Another (**recursive**) definition for **repeated product** of binary relations

$$R^0 =_B id_A. \quad \text{basis}$$

$$R^n =_R R \cdot R^{n-1}, n \geq 1. \quad \text{recursion}$$

$$\text{ex) } R^3 =_R R \cdot R^2 =_R R \cdot R \cdot R^1 =_R R \cdot R \cdot R \cdot R^0 =_B R \cdot R \cdot R \cdot id_A = R \cdot R \cdot R$$

1.A.3 A directed graph $G = (V, E)$ is

V : a set of vertices,

$E \subseteq V \times V$: a set of edges,

E : a binary relation on V

Some properties of the binary relations

1) R is reflexive, if $\forall a \in A, a R a$.

$$id_A \subseteq R$$

R is irreflexive, if $\forall a \in A, a \not R a$.

$$R \cap id_A = \emptyset$$

2) R is symmetric, if $a R b$ implies $b R a$.

$$R = R^{-1}$$

R is asymmetric, if $a R b$ implies $b \not R a$.

$$R \cap R^{-1} = \emptyset$$

R is antisymmetric, if $a R b$ and $a \neq b$ implies $b \not R a$. $R \cap R^{-1} \subseteq id_A$

R is asymmetric $\Leftrightarrow R$ is irreflexive.

R is asymmetric $\Rightarrow R$ is antisymmetric.

3) R is transitive, if $a R b$ and $b R c$ implies $a R c$. $R \cdot R \subseteq R$

Let $\mathbb{P} = \{\text{reflexive, symmetric, transitive}\}$. Then R' be \mathbb{P} -closure of R , if

i) R' is \mathbb{P} .

ii) $R \subseteq R'$.

iii) R' is the **smallest** set among satisfying i) and ii).

$\Leftrightarrow \forall R''$ satisfying i) and ii), $R' \subseteq R''$.

reflexive closure of R , $R' = R \cup \text{id}_A$.

symmetric closure of R , $R' = R \cup R^{-1}$.

transitive closure of R ,

$$R^+ = R^1 \cup R^2 \cup R^3 \cup \dots = \bigcup_{i \in N_1} R^i \text{ where } N_1 = \{1, 2, 3, \dots\}.$$

reflexive-transitive closure of R ,

$$R^* = R^0 \cup R^1 \cup R^2 \cup R^3 \cup \dots = \bigcup_{i \in N_0} R^i \text{ where } N_0 = \{0, 1, 2, \dots\}.$$

What is the **reflexive** (**, symmetric**) and **transitive closure** of R in a graph (A, R) ?

Partition of a set A .

Let A be a set and $A_1, A_2, \dots, A_n \subseteq A$. Then we define **partition** of A , as

$Par(A) = \{A_1, A_2, \dots, A_n\}$ is called a **partition** of A , written $Par(A)$,

if 1) $\cup_{i \in \{1, 2, \dots, n\}} A_i = A \wedge$ **exhaustive**

2) $1 \leq i \neq j \leq n: A_i \cap A_j = \emptyset.$ **(pairwise) disjoint**

$|Par(A)|$ is the **size** of the partition.

Ex. Consider $A = \{a_1, a_2, \dots, a_n\}$. What are $Par(A)$'s?

Cover of a set A . $Cover(A) = \{A_1, A_2, \dots, A_n\}$ where

if $\cup_{i \in \{1, 2, \dots, n\}} A_i = A$ **exhaustive**

Power set of a set A ,

$$2^A = P(A) = \{B \mid B \subseteq A\}$$

$$B \subseteq A \Leftrightarrow B \in 2^A.$$

$$|2^A| = 2^{|A|}.$$

$$par(A) \subseteq 2^A.$$

A binary relation R on A is **equivalence**,
if R is **reflexive**, **symmetric**, and **transitive**.

$\text{Par}(A)$ partition of A

A binary relation R on A is **(ir)reflexive partial order**,
if R is **(ir)reflexive**, **antisymmetric**, and **transitive**.
 A : **partially-ordered set(po set)**

Let $R \subseteq A \times A$ be an **equivalence**,

$[a]_R = \{b \in A \mid a R b\}$ **equivalence class**,

if $a R b$, $[a]_R = [b]_R$.

$\{[a]_R \mid a \in A\}$ **equivalence partition**.

$\bigcup_{a \in A} [a]_R = A$, **exhaustive**

if $a R b$, $[a]_R \cap [b]_R = \emptyset$. **(pairwise) disjoint**

Let \leq be a partial order(**relation**) on A . $\leq \subseteq A \times A$

Then (A, \leq) is called as partially ordered set or **poset** for short.

Let (A, \leq) be a **poset**. We define two binary operators on A ,

$$\vee, \wedge: A \times A \rightarrow 2^A$$

$$a \vee b = \min \{c \in A \mid a \leq c \text{ and } b \leq c\} \quad \text{least upper bound}$$

$$a \wedge b = \max \{c \in A \mid c \leq a \text{ and } c \leq b\}. \quad \text{greatest lower bound}$$

If a **unique lub** and a **unique glb**,

$$\vee, \wedge: A \times A \rightarrow A. \quad (A, \leq) \text{ is called as a } \mathbf{lattice} \text{ and}$$

(A, \vee, \wedge) is called a **algebra** induced by the lattice (A, \leq) .

Boolean algebra, $(\{f, t\}, \vee, \wedge)$, is induced by the **lattice** $(\{f, t\}, \{f \leq t\})$.

Let A be a sets. Then

Set algebra on A , $(2^A, \cup, \cap)$, is induced by the **lattice** $(2^A, \subseteq)$.

Singleton set algebra, $(2^{\{a\}}, \cup, \cap)$, is **isomorphic** to

the **boolean algebra**, $(\{f, t\}, \vee, \wedge)$ with respect to **bijection** g .

What is the **bijective** function g ? *cs204 TP of 12 boolean algebra.*

1.A.4 Algebraic system, semi-group and monoid

Let A be a set and \oplus be a **binary** operation on A .

$$\oplus: A \times A \rightarrow A.$$

1. (A, \oplus) is an **algebraic system**.

$$\text{if } \forall a, b \in A, a \oplus b \in A. \quad \text{closed}$$

2. (A, \oplus) is a **semi-group**.

$$\text{if } \forall a, b, c \in A, a \oplus (b \oplus c) = (a \oplus b) \oplus c \quad \text{associative}$$

$$a_1 + a_2 + \dots + a_n = \sum_{i \in \{1, 2, \dots, n\}} a_i \quad \text{indexed set notation}$$

binary operation \Rightarrow *n-ary operation*

3. (A, \oplus, e) is a **monoid**.

$$\text{if } \exists e \in A \text{ s.t. } \forall a \in A, e \oplus a = a \oplus e = a \quad \text{identity}$$

(A, \oplus, e) is called as a **monoid**.

Let (A, \oplus, e) and $(B, \otimes, \varepsilon)$ be two monoids.

If

- i) $h: A \rightarrow B$ is a onto function, $|A| \geq |B|$
- ii) $h(a \oplus b) = h(a) \otimes h(b)$, and *preserve operation*
- iii) $h(e) = \varepsilon$. *preserve identity*

Then h is called a **homomorphism**, and the monoid $(B, \otimes, \varepsilon)$ is called a **homomorphic** to the monoid (A, \oplus, e) w.r.t. h .

(A, \oplus, e) is called **concretization** of $(B, \otimes, \varepsilon)$ and
 $(B, \otimes, \varepsilon)$ is called **abstract interpretation** of (A, \oplus, e) .

If h is one-to-one and onto, h is called **isomorphism**.

1.A.5 A binary *relation* from A to B is a *function* from A to B , if

$$1) \forall a \in A, \exists (a, b) \in f, \quad \text{total}$$

$$2) \forall a \in A, \exists_1 (a, b) \in f. \quad \text{unique}$$

$$f: A \rightarrow B \quad (a, b) \in f \text{ or } a f b \text{ or } f(a) = b \text{ or } f a = b.$$

Three faces of a binary relation

$$i) R \subseteq A \times B. \quad (a, b) \in R.$$

i) a set of (ordered) pairs

$$ii) R: A \times B \rightarrow \{\text{false}, \text{true}\}.$$

$$a R b, \text{ iff } (a, b) \in R.$$

ii) a relational operator ($<$, $=$, \leq)

$$iii) R: A \rightarrow 2^B.$$

$$R(a) = \{b_1, b_2, \dots, b_n\}, \text{ iff } (a, b_1), (a, b_2), \dots, (a, b_n) \in R.$$

$$\forall a \in A, \exists_1 \{b_1, b_2, \dots, b_n\} \subseteq B \text{ or } \exists_1 \{b_1, b_2, \dots, b_n\} \in 2^B.$$

$$\therefore R: A \rightarrow 2^B.$$

iii) a set valued function

Let $f: A \rightarrow B$. Is $f^{-1}: B \rightarrow A$ a function?

No!!!

Function $f: A \rightarrow B$ is **onto**(*surjection; correspondence*), if

$$\forall b \in B, \exists a \in A .\exists. f(a) = b. \quad |A| \geq |B|$$

If f is onto, f^{-1} is **total** but **not** unique function.

Function $f: A \rightarrow B$ is **one-to-one**(*injection, 1-1*), if

$$\forall a_1, a_2 \in A, a_1 \neq a_2 \text{ implies } f(a_1) \neq f(a_2).$$

$$\text{if } \exists b \in B .\exists. f(a) = b, \exists_1 a \in A. \quad |A| \leq |B|$$

If f is 1-1, f^{-1} is **unique** but **not** total function.

Function $f: A \rightarrow B$ is **bijective**,

if f is both **1-1** and **onto**(*1-1 correspondence*).

$$\forall b \in B, \text{if } \exists_1 a \in A .\exists. f(a) = b.. \quad |A| = |B|$$

If f is 1-1 onto, f^{-1} is both **total** and **unique**, so is a **function**.

1.B Set isomorphism and infinite sets

If there exists a **bijection** (**짝짓기**, 1-1 onto) f from A to B ,
 two sets A and B have same **cardinality**, written $|A| = |B|$, and
 two sets A and B are said to be **isomorphic** w.r.t. f , written $A \cong_f B$.

A set is said to be **countable**(**enumerable**),
 if it has the same **cardinality** with a **subset** of \mathbb{N} ,
 either **finite** or **infinite**
 and **uncountable**(**uncountably infinite**), otherwise.

A set is **countably infinite**, if it has the same **cardinality** with \mathbb{N} .
 the **cardinality** of \mathbb{N} is denoted as \aleph , $|\mathbb{N}| = \aleph$.

Let A be **countable**. Then we can **enumerate** the set in **numeric order**.

$A = \{a_0, a_1, \dots, a_n\}$ **finite** for some $n \geq 0$.

$A = \{a_0, a_1, \dots\}$ **countably infinite**(**enumerable**)

Consider

$$N_1 = \{1, 2, 3, \dots\}$$

$$|N_1| = |\mathbb{N}| = \aleph, \text{ but } N_1 \subset \mathbb{N}.$$

$$E = \{e \in \mathbb{N} \mid e = 2i, i \in \mathbb{N}\}$$

$$|E| = |\mathbb{N}| = \aleph, \text{ but } E \subset \mathbb{N}.$$

$$I = \mathbb{N} \cup \{-i \mid i \in \mathbb{N}\}$$

$$|I| = |\mathbb{N}| = \aleph, \text{ but } \mathbb{N} \subset I.$$

$$Q = \mathbb{N} \times \mathbb{N}$$

$$|Q| = |\mathbb{N}| = \aleph.$$

enumerate $(i, j) \in \mathbb{N} \times \mathbb{N}$ in *(natural) numeric order*

$$\mathbb{N} \times \mathbb{N} = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0), (2, 1), \dots\}$$

$$f(i, j) = 1 + 2 + 3 + \dots + (i+j) + j \qquad f(0, 0) = 0$$

$$= (i+j)(i+j+1)/2 + j$$

$f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is 1-1 and onto

Consider $f^{-1}: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ which is also 1-1 and onto.

$$\therefore |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph.$$

Fact $|\Sigma^*| = \aleph$.

Proof *Lexicographical order for Σ^* .*

in the order of size and if same size, in alphabetic order.

Let $|\Sigma| = k$. Then we can alphabetic order $\Sigma = \{a_1, a_2, \dots, a_k\}$,

and we can order $x \in \Sigma^*$ in *lexicographical order* as follows.

$\Sigma^* = \{(\underline{\varepsilon}), (\underline{a_1}, \underline{a_2}, \dots, \underline{a_k}), (\underline{a_1a_1}, \underline{a_1a_2}, \dots, \underline{a_1a_k}, \underline{a_2a_1}, \dots, \underline{a_ka_k}), (\underline{a_1a_1a_1}, \dots)\}$

If $x = b_1b_2 \dots b_n \in \Sigma^*$, $f: \Sigma^* \rightarrow \mathbb{N}$ is as follows;

$$\begin{aligned} f(x) &= k^0 + k^1 \dots + k^{n-1} + b_1k^{n-1} + b_2k^{n-2} + \dots + b_nk^0 \\ &= (k^n - 1)/(k-1) + b_1k^{n-1} + b_2k^{n-2} + \dots + b_nk^0. \end{aligned}$$

What is f^{-1} ?

We *enumerate* $x = a_1a_2 \dots a_n \in \Sigma^*$ in *numeric order*

$\therefore f: \Sigma^* \rightarrow \mathbb{N}$ is *one-to-one onto*. Q.E.D.

Consider $\{0, 1\}^{\mathbb{N}}$: **infinite binary strings** (See pp.12)

and $2^{\mathbb{N}}$: **power set of natural numbers** (Note that $2^A = \{B \mid B \subseteq A\}$)

Power set of integers infinite binary strings are **isomorphic**.

$$\therefore |2^{\mathbb{N}}| = |\{0, 1\}^{\mathbb{N}}|.$$

Cantor's diagonal argument

Assume $2^{\mathbb{N}}$ is **countable**.

We can **enumerate** $|2^{\mathbb{N}}|$ subsets of \mathbb{N} , in numeric order as follows,

$2^{\mathbb{N}} = \{a_0, a_1, \dots, a_i, \dots\}$ where

$$\forall i \in \mathbb{N}, a_i \leftrightarrow (b_{i0}, b_{i1}, \dots, b_{ij}, \dots) \in 2^{\mathbb{N}}$$

$$\forall j \in \mathbb{N}, \text{ if } b_{ij} = 1 \text{ then } j \in a_i \in 2^{\mathbb{N}},$$

$$\text{ if } b_{ij} = 0 \text{ then } j \notin a_i \in 2^{\mathbb{N}}.$$

\therefore Power set of $\mathbb{N} \leftrightarrow$ infinite binary string

Consider $a \leftrightarrow (\bar{b}_0, \bar{b}_1, \dots, \bar{b}_i, \dots)$ where $\forall i \in \mathbb{N}$, $\bar{b}_i = 1$, if $b_{ii} = 0$,
 $\bar{b}_i = 0$, if $b_{ii} = 1$.

Since $\forall i \in \mathbb{N}$, $\bar{b}_i \neq b_{ii}$, $\therefore \forall i \in \mathbb{N}$, $a \neq a_i$.
 $\therefore a \notin \{a_0, a_1, \dots, a_i, \dots\}$.

But $a \in 2^{\mathbb{N}}$ by the **definition** of power set.

$a \notin \{a_0, a_1, \dots, a_i, \dots\}$ but $a \in 2^{\mathbb{N}}$.

\therefore **Contradiction!!!**

We fail to enumerate $2^{\mathbb{N}} = \{a_0, a_1, \dots, a_i, \dots\}$ in numeric order!

\therefore **We conclude that $|2^{\mathbb{N}}| > |\{a_0, a_1, \dots, a_i, \dots\}| = \aleph$.**

$2^{\mathbb{N}}$ is uncountable.

$\{0, 1\}^*$ vs $\{0, 1\}^{\mathbb{N}}$.

$\{0, 1\}^*$: *all(countably infinite union of finite binary strings*

$$= \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2 \cup \{0, 1\}^3 \cup \dots$$

$$= \{\varepsilon\} \cup \{0, 1\} \cup \{00, 01, 10, 11\} \cup \{000, \dots, 111\} \cup \dots$$

$$= \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots, 111, \dots\}$$

$\{0, 1\}^{\mathbb{N}}$: *all(uncountably infinite union of) infinite binary strings*

$$= \{000\dots000\dots, \quad \leftrightarrow \{\}$$

$$100\dots000\dots, \quad \leftrightarrow \{0\}$$

$$010\dots000\dots, \quad \leftrightarrow \{1\}$$

$$110\dots000\dots, \quad \leftrightarrow \{0, 1\}$$

...

$$111\dots111\dots\} \quad \leftrightarrow \{0, 1, 2, \dots\} = \mathbb{N}$$

$$|\{0, 1\}^{\mathbb{N}}| = |2^{\mathbb{N}}| > |\{0, 1\}^*| = \aleph.$$

Cantor's diagonal argument

Complement of diagonal element

Russel's paradox

$$S = \{x \mid x \notin x\}$$

$x \in S$, iff $x \notin x$.

*But $S \in S$, iff $S \notin S$. **contradictory!***

Halting problem

$H(P)$: if halt(P , P) then loop forever

elses not halt(P , P) then stop fi

What happens if $H(H)$ stops or loops forever?

Denial of self recursion

Σ^ is countable.*

strings are countable

But is 2^{Σ^} uncountable.*

languages are uncountable

class of languages

N. Chomsky

Finite(countable)

Countably infinite

*natural numbers, integers, rational numbers,
finite strings ...*

Uncountable

Cantor's diagonal argument

*power set of natural numbers
infinite strings
real numbers*

*Some informal descriptions on **countable** and **uncountable** infiniteness*

$\aleph \pm k = \aleph$ $\aleph \times k = \aleph$ ***countable***

$\aleph \times \aleph = \aleph$ $\aleph^k = \aleph$ ***countable***

But $k \times k \times \dots = k^{\aleph} > \aleph$ ***uncountable*** ($k \geq 2$)

1.C Symbol, vocabulary, string and language

Let Σ be a set of *symbols* (문자) called as
 a *vocabulary* (어휘 or *alphabet*) of a language.

We define a *string* x over Σ , whose length is n , is defined as

$x = a_1a_2\dots a_n$ where $1 \leq \forall i \leq n: a_i \in \Sigma$. and we write $|x| = n$.

A *string* (문자열) is a *sequence of symbols*.

Example $\Sigma = \{a, b, \dots, z\}$ English alphabet

$x = school$ $y = boy$

$|x| = 6$ $|y| = 3$.

Concatenation(\cdot) of two strings

$x = a_1a_2\dots a_n$ and $y = b_1b_2\dots b_m$.

$x \cdot y = a_1a_2\dots a_nb_1b_2\dots b_m = xy$ justaxaposed

Example $x \cdot y = xy = schoolboy$ $|xy| = 9$.

We define **concatenation** of two strings as

$\therefore \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ a **binary operation on strings**

(1) $\forall x, y \in \Sigma^*, xy \in \Sigma^*$. **closed**

(2) $\exists x, y \in \Sigma^*, xy \neq yx$ **noncommutative**

(3) $\forall x, y, z \in \Sigma^*, x(yz) = (xy)z$ **associative**

(4) We **define an empty string** denoted as ε (or Λ)
as an **identity element** w.r.t. the concatenation operation.

i.e. $\forall x \in \Sigma^*, \varepsilon x = x\varepsilon = x$ where $|\varepsilon| = 0$.

Then $(\Sigma^*, \cdot, \varepsilon)$ is a noncommutative **monoid**.

We can **extend** the domain and range of the concatenation
from set of **strings** to the set of **languages** (set of set of strings)

$\therefore 2^{\Sigma^*} \times 2^{\Sigma^*} \rightarrow 2^{\Sigma^*}$.

Then $(2^{\Sigma^*}, \cdot, \{\varepsilon\})$ is a (**non induced**) noncommutative **monoid**.

Four terminologies on the formal language theory (FLT).

	<i>element</i>	<i>set</i>
	<i>symbol</i> (문자) $a, b, c \in \Sigma$	<i>vocabulary</i> (어휘) Σ
<u><i>sequence</i></u>	<u><i>string</i></u> (문자열) $x, y, z \in \Sigma^*$	<u><i>language</i></u> (언어) $L, S, T \subseteq \Sigma^*$ $L, S, T \in 2^{\Sigma^*}$

Power of an alphabet revisited

$$\Sigma^0 =_B \{\varepsilon\} \quad \text{basis}$$

$$\Sigma^n =_R \Sigma \Sigma^{n-1} \text{ for } n \geq 1 \quad \text{recursion}$$

$$|\Sigma^n| = |\Sigma|^n.$$

Let $B^A = \{f \mid f: A \rightarrow B\}$. Then $|B^A| = |B|^{|A|}$.

Consider $\Sigma^n = \Sigma^{\{1, 2, \dots, n\}} = \{f \mid f: \{1, 2, \dots, n\} \rightarrow \Sigma\}$

Example $x: \{1, 2, \dots, 6\} \rightarrow \{a, b, \dots, z\}$ and $y: \{1, 2, 3\} \rightarrow \{a, b, \dots, z\}$

$x = \text{school} \in \Sigma^6$ where $x(1)=s, x(2)=c, \dots, x(6)=l$; or $x = (s, c, h, o, o, l)$

$y = \text{boy} = (b, o, y) \in \Sigma^3$ where $y(1) = b, y(2) = o, y(3) = y$.

strings and functions are **isomorphic!**

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \dots = \cup_{i \in N_1} \Sigma^i$ where $N_1 = \{1, 2, 3, \dots\}$.

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \dots = \cup_{i \in N_0} \Sigma^i$ where $N_0 = \{0, 1, 2, \dots\}$.

Σ^* is a **universe** of strings including **identity**(ε)

$(\Sigma^*, \cdot, \varepsilon)$ is a (**free**) **monoid** over Σ . (**unique representation**)

2^{Σ^*} is a **universe** of languages

$(2^{\Sigma^*}, \cdot, \{\varepsilon\})$ is a (**free**) **induced monoid** over Σ .

Reversal, prefix and suffix of strings

Let $x = a_1a_2 \dots a_n$ be a string of length $n \geq 0$ and $k \geq 0$.

$$x^R = a_n a_{n-1} \dots a_2 a_1. \quad \text{reversal of } x.$$

recursive definition of reversal of $x \in \Sigma^$.*

$$\varepsilon^R =_B \varepsilon.$$

Let $a \in \Sigma$ and $x \in \Sigma^*$. Then $(ax)^R =_R x^R a$.

Ex. $abc^R =_R bc^R a =_R c^R ba =_R \varepsilon^R cba =_B \varepsilon cba = cba$.

Prefix and suffix of a string. For $k \in \mathbb{N}$,

$k:x = a_1 a_2 \dots a_k$, if $k \leq n$; **prefix** of x with length k .
 $= x$, otherwise.

$x:k = (k:x^R)^R$. **suffix** of x with length k .

Ex. $3:school = sch$, $boy:2 = oy$, $boy:6 = boy$.

For $k \geq 0$: $k:, :k: \Sigma^* \rightarrow \Sigma^{\leq k} = \{\varepsilon\} \cup \Sigma \cup \Sigma^2 \cup \dots \cup \Sigma^k$.