5.A Examples of Context-Free Grammars

3.1 and 3.2 of Du & Ko's book(pp. 91 - 108)

Example 3.1 *Consider* $S \rightarrow \varepsilon \mid aSb$. *What is* $L(S)$?

Solution $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \ldots \Rightarrow a^nSb^n \Rightarrow a^nb^n$.

$\therefore L(S) = \{a^nb^n \mid n \geq 0\}$

Example 3.2 *Consider* $S \rightarrow ABA$, $A \rightarrow a \mid bb$, $B \rightarrow bS \mid \varepsilon$. *What is* $L(S)$?

Solution *Derivation of the above grammar are:*

$$S \Rightarrow ABA \Rightarrow A\underline{bS}A \Rightarrow Ab\underline{ABA}A \Rightarrow AbA\underline{bS}AA \Rightarrow AbAb\underline{ABA}AA$$

$$= A(bA)^2BA^3 \Rightarrow \ldots \Rightarrow A(bA)^nBA^{n+1} \Rightarrow \ldots .$$

$\therefore L(S) = \{(a+bb)(ba + bbb)^n(a+bb)^{n+1} \mid n \geq 0\}$

Example 3.3 *Find a cfg for* $L = \{a^n b^{2n} \mid n \geq 0\}$

Solution 1 $S \rightarrow aSbb$ *always generates* **matching** *pair of a and bb.*

$$S \Rightarrow aSbb \Rightarrow aaSbbbb \Rightarrow \ldots \Rightarrow a^n S(bb)^n \Rightarrow a^n b^{2n}.$$

Solution 2 Recursivly *express longer string in L by shorter string in L.*

$a^{n+1}b^{2(n+1)} = a(a^n b^{2n})bb.$ *But fails!*

If $S \Rightarrow^* a^n b^{2n}$, *we* ***need*** *rule* $S \rightarrow aSbb$ *to get* $a(a^n b^{2n})bb$.

The rule $S \rightarrow \varepsilon$ *gets* $S \Rightarrow^* a^n b^{2n}$ *for* $n \geq 0$.

$\therefore S \rightarrow \varepsilon \mid aSbb$

How about $a^{n+1}b^{2(n+1)} = a^n abbb^{2n}$?

Example 3.4 $L = \{x \in (a+b)^* \mid x = x^R\}$ ***palindromes***

If $y \in L$, $y = aya$ *or* $byb \in L$. $S \rightarrow aSa \mid bSb$ *for* $|x| \geq 2$.

$x = a$ *or* $b \in L$ *for* $|x| = 1$, *and* $x = \varepsilon$, *for* $|x| = 0$.

$\therefore S \rightarrow \varepsilon \mid a \mid b \mid aSa \mid bSb.$

Example 3.5 *Find a cfg for*

$L = \{x \in (a+b)^* \mid \text{prefix of } x \text{ has at least as many } a\text{'s as } b\text{'s}\}$

Solution

(1) $\varepsilon \in L$. ***basis***

(2) If $y \in L$, $ay \in L$ but $by \notin L$. ***recursive*** *expression*

(3) If $y \notin L$, then y has a prefix which has ***one more*** b than a.

Let u be the ***shortest such***$(\#_b(u) = \#_a(u) + 1)$ prefix of y

where $\#_a(x)$ denotes the occurence of $a \in \Sigma$ in $x \in \Sigma^*$.

Then $u = wb$ and $w \in L$. $(\#_a(w) = \#_b(w))$

Then $\#_a(awb) = \#_b(awb)$ and if $z \in L$, $awbz \in L$.

$\therefore$ If $w, z \in L$, $awbz \in L$. ***recursive*** *expression*

(4) Nothing else is in L.

$\therefore S \to \varepsilon \mid aS \mid aSbS$.

$S \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSbS \Rightarrow ababS \Rightarrow ababaS \Rightarrow ababa$.

Example 3.6 *$L = \{x \in (a+b)^* \mid x$ has as many a's as b's$\}$*

Let $\#_a(x)$ denotes the number of occurences of symbol a in x.

Solution *1. Let $d(x) = \#_a(x) - \#_b(x)$. $L = \{x \in (a+b)^* \mid d(x) = 0\}$*

*Consider u, the **nonempty shotest** prefix of $x \in L$.э. $d(u) = 0.(x = uz)$*

*and assume u **ends** with b, then u **starts** with a and.*

proof of claim *Let's assume $|u| = k$, $1 \leq {}^{\forall}i \leq k$, u_i be the prefix of u, $|u_i| = i$.*

Consider $(u_1, u_2, \ldots, u_k)$ sequence.

*$d(u_i)$ and $d(u_{i-1})$ differs by **exactly one**, for $1 < {}^{\forall}i \leq k$.*

$d(u_k)=0$, $d(u_{k-1})=1$, $d(u_i) \geq 1$ for $1 \leq {}^{\forall}i \leq k-2$(shortest). $\therefore u_1=a$.

$u = ayb$, $x = aybz$. Since $d(x) = d(u) = 0$, $d(y) = d(z) = 0$.

$\therefore$ If $y, z \in L$, $aybz \in L$, $byaz \in L$, and $\varepsilon \in L$.

$\therefore S \to \varepsilon \mid aSbS \mid bSaS$.

$$S \Rightarrow aSbS \Rightarrow^n a^n(Sb)^nS \Rightarrow^* a^nb^nS \Rightarrow^m a^nb^nb^m(aS)^m \Rightarrow^* a^nb^{n+m}a^m.$$

Solution 2. *Let* $L(A_k) = \{x \in (a+b)^* \mid d(x) = k\}$. *Then* $L = L(A_0)$

If $x \in L(A_0)$. *Then*

(1) $x = \varepsilon$,

(2) $x = ay$ *where* $y \in L(A_{-1})$, *and*

(3) $x = bz$ *where* $z \in L(A_1)$.

$\therefore A_0 \rightarrow \varepsilon \mid aA_{-1} \mid bA_1$

If $x \in L(A_{-1})$. *Then*

(1) $x = ay$ *where* $y \in L(A_{-2})$,

(2) $x = bz$ *where* $z \in L(A_0)$.

If $z = z_1z_2 \in L(A_{-2})$, *then* $z_1, z_2 \in L(A_{-1})$.

$\therefore \quad A_0 \rightarrow \varepsilon \mid aA_{-1} \mid bA_1, \qquad S \rightarrow \varepsilon \mid aA \mid bB,$

$A_1 \rightarrow aA_0 \mid bA_1A_1, \qquad A \rightarrow aS \mid bAA,$

$A_{-1} \rightarrow bA_0 \mid aA_{-1}A_{-1}. \qquad B \rightarrow bS \mid aBB.$

Example 3.12 $L = \{x \in (a+b)^* \mid x$ *has twice as many b's as a's*$\}$

Solution *Let* $d(x) = 2\#_a(x) - \#_b(x)$.

$$A_0 \to \varepsilon \mid aA_{-1}A_{-1} \mid bA_1,$$
$$A_1 \to aA_{-1} \mid bA_2,$$
$$A_{-1} \to aA_{-1}A_{-1}A_{-1} \mid bA_0,$$
$$A_2 \to aA_0 \mid bA_1 A_2 \mid bA_2A_1.$$

Example 3.13 $L = \{x \in (a+b)^* \mid \#_b(x) = 2\#_a(x) + 3\}$

Solution

Add a new start symbol A_3 *and a rule*

$$A_3 \to A_1A_2 \mid A_2A_1.$$

Example 3.14 $L = \{a^m b^n c^p d^q \mid m + n = p + q\}$

soultion

1. In stage S, match a and d, $S \to aSd$. *Then*

if $m \geq q$, *move to stage A, if* $m \leq q$, *move to stage B,* $S \to A \mid B$,

2. In stage A($m \geq q$*) more a, match a and c, then move to C,* $A \to aAc \mid C$,

3. In stage B($m \leq q$*) more d, match b and d, then move to C,* $B \to bBd \mid C$,

4. In stage C, no more a nor d, match b and c, then finish. $C \to bCc \mid \varepsilon$.

$$S \to aSd \mid A \mid B,$$
$$A \to aAc \mid C,$$
$$B \to bBd \mid C,$$
$$C \to bCc \mid \varepsilon.$$

$$S \Rightarrow^i a^i S d^i \Rightarrow a^i A d^i \Rightarrow^j a^i a^j A c^j d^i \Rightarrow^{k+1} a^{i+j} b^k C c^k c^j d^i \Rightarrow a^{i+j} b^k c^{k+j} d^i.$$
$$\Rightarrow a^i B d^i \Rightarrow^j a^i b^j B d^j d^i \Rightarrow^{k+1} a^i b^j b^k C c^k d^j d^i \Rightarrow a^i b^{j+k} c^k d^{j+i}.$$

Example 3.15 $L = \{a^m b^n c^p \mid m + 2n \geq p\}$

soultion Consider $L_1 = \{a^m b^n c^p \mid m + 2n = p\}$

$S \to aSc \mid A, \qquad A \to bAcc \mid \varepsilon.$

Now consider $L_2 = \{a^m b^n c^p \mid m + 2n > p\}$

Extra a or b $\quad S \to aS \quad$ or $A \to bAc \mid bA$

$\therefore S \to aSc \mid aS \mid A, \quad A \to bAcc \mid bAc \mid bA \mid \varepsilon.$

Example 3.16 $L = \{a^m b^n \mid 3m \le 5n \le 4m,\ m, n \in \mathbf{N}\}$

soultion *Consider* $L_0 = \{a^m b^n \mid 3m \le 5n \le 4m,\ m \equiv 0\ (mod\ 5)\}$

$m = 5p,\ 3p \le n \le 4p,\ a^{5p} b^n \in L_0.$

$L_0 = \{a^{5p} b^{3p+k} \mid 0 \le k \le p\}$

$A \rightarrow a^5 A b^3 \mid a^5 A b^4 \mid \varepsilon.$

$A \Rightarrow^k a^{5k} A b^{4k} \Rightarrow^{p-k} a^{5k} a^{5(p-k)} A b^{3(p-k)} b^{4k} \Rightarrow a^{5p} b^{3p+k}.$

$L_i = \{a^m b^n \mid 3m \le 5n \le 4m,\ m \equiv i\ (mod\ 5)\}$

$m = 5p+4,\ 3p+3\cdot 4/5 \le n \le 4p+4\cdot 4/5$ *or* $3p + 3 \le n \le 4p + 3$

$a^{5p+4} b^n \in L_4$ *where* $3p + 3 \le n \le 4p + 3.$

if $p = 0,\ m = 4;\ 3 \le n \le 3,\ n = 0,\ a^4 b^3 \in L_4.$

$L_4 = \{a^{5p+4} b^{3p+k+3} \mid 0 \le k \le p\}.$

$S_4 \rightarrow a^4 A b^3.$ *initial condition*

$m = 5p+3,\ 3p + 3\cdot 3/5 \le n \le 4p + 4\cdot 3/5$ or $3p + 2 \le n \le 4p + 2$.

if $p = 0,\ 2 \le n \le 2,\ n = 2,\ a^3b^2 \in L_3$.

$S_3 \to a^3Ab^2$.

$m = 5p+2,\ 3p + 3\cdot 2/5 \le n \le 4p + 4\cdot 2/5$ or $3p + 2 \le n \le 4p + 1$

if $p = 0,\ m = 2;\ 2 \le n \le 1$, no solution.

if $p = 1,\ m = 7; 5 \le n \le 6,\ n = 5$ or $n = 6$. $a^7b^5, a^7b^6 \in L_2$.

$S_2 \to a^7Ab^5$.

$m = 5p+1,\ 3p+3\cdot 1/5 \le n \le 4p+4\cdot 1/5$ or $3p + 1 \le n \le 4p$

if $p = 0,\ m = 1;\ 1 \le n \le 0$, no solution.

if $p = 1,\ m = 6; 4 \le n \le 4,\ n = 4$. $a^6b^4 \in L_1$.

$S_1 \to a^6Ab^4$.

$S \to A \mid a^4Ab^3 \mid a^3Ab^2 \mid a^6Ab^4 \mid a^7Ab^5$,

$A \to a^5Ab^3 \mid a^5Ab^4 \mid \varepsilon$.

Regular grammar

A context-free grammar $G = (N, \Sigma, P, S)$ is a ***regular****, if*

all of the productions are of the form:

$$A \rightarrow aB \in P \text{ or } A \rightarrow a \in P \text{ where } A, B \in V, a \in \Sigma.$$

$$\text{except } S \rightarrow \varepsilon \in P.$$

Extended regular grammar

A context-free grammar $G = (N, \Sigma, P, S)$ is a ***extended regular****, if*

all of the productions are of the form:

$$A \rightarrow xB \in P \text{ or } A \rightarrow x \in P \text{ where } A, B \in N, x \in \Sigma^*.$$

Equivalence of regular and extended regular grammars

Let $G(N, \Sigma, P, S)$ be an extended regular grammar, Then there is a regular grammar $G' = (N', \Sigma, P', S')$ such that $L(G) = L(G')$.

Proof *See Chomsky normal form of context-free grammar.*

Equivalence *of regular grammar and finite automata*

Let $A = (Q, \Sigma, \delta, q_0, F)$ *be a* ***extended*** *finite automata. Then there is a regular grammar* $G = (Q, \Sigma, P, q_0)$ *such that* $L(A) = L(G)$.

Proof $P = \{q \to xp \mid p \in \delta(q, x), x \in \Sigma^*\} \cup \{f \to \varepsilon \mid f \in F\}$.

Prove $q \Rightarrow^* xp$, *if and only if,* $p \in \delta^*(q, x)$.

$\therefore\ q_0 \Rightarrow^* x$, *if and only if,* $\delta^*(q_0, x) \in F$.

Let $G = (N, T, P, S)$ *be a regular grammar. There is a finite automata* $A = (N \cup \{f\}, T, \delta, S, \{f\})$ *such that* $L(G) = L(A)$.

Proof $\delta = \{B \in \delta(A, x) \mid A \to xB \in P, x \in \Sigma^*\} \cup \{f \in \delta(A, x) \mid A \to x \in P\}$.

Prove $B \in \delta^*(A, x)$, *if and only if,* $A \Rightarrow^* xB$.

$\therefore\ \delta^*(S, x) \in F$, *if and only if,* $S \Rightarrow^* x$.

Consider a ***derivation*** *in regular grammar $G = (N, \Sigma, P, S)$ and* ***state transition*** *in the* ***corresponding*** *finite automaton $M = (Q, \Sigma, \delta, q_S, F)$.*

Let $x, y, z \in \Sigma^*$, $A, B \in N$, $q_A, q_B \in Q$, $q_F \in F$, and $A \leftrightarrow q_A$, $B \leftrightarrow q_B$. Then

$$S \Rightarrow^* xA \Rightarrow xyB \Rightarrow^* xyz. \qquad A \rightarrow yB \in P$$

$$\delta(q_S, xyz) =^* \delta(q_A, yz) = \delta(q_B, z) =^* q_F \in F. \qquad q_B \in \delta(q_A, y)$$

Grammars	***generating*** *terminal strings*
Automata	***consuming*** *terminal strings*

Let $G(N, \Sigma, P, S)$ be an extended regular grammar, Then there is a regular grammar $G' = (N', \Sigma, P', [S])$ such that $L(G) = L(G')$.

proof $N' = \{[Ax] \mid A \to xyB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, y \in \Sigma^*\} \cup \{[\varepsilon]\}$

$P' = \{[A] \to [B] \mid A \to B \in P, B \in N\} \cup$

$\{[Ax] \to a[Axa] \mid A \to xayB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, y \in \Sigma^+, a \in \Sigma\} \cup$

$\{[Ax] \to a[B] \mid A \to xaB \in P, B \in N, x \in \Sigma^*, a \in \Sigma\} \cup$

$\{[Ax] \to a \mid A \to xa \in P, x \in \Sigma^*, a \in \Sigma\}.$

$\{[Ax] \to a[B] \mid A \to xaB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, a \in \Sigma\} \cup$
$\{[\varepsilon] \to \varepsilon\}.$

Assume $A \rightarrow a_1 \ldots a_n B \in P(n \geq 0)$. *Then n-rules,*

$[A] \rightarrow a_1[Aa_1], [Aa_1] \rightarrow a_2[Aa_1a_2], \ldots, [Aa_1 \ldots a_{n-2}] \rightarrow a_{n-1}[Aa_1 \ldots a_{n-1}],$

and $[Aa_1 \ldots a_{n-1}] \rightarrow a_n[B]$.

Assume $A \rightarrow a_1 \ldots a_n \in P(n \geq 0)$. *Then n-rules,*

$[A] \rightarrow a_1[Aa_1], [Aa_1] \rightarrow a_2[Aa_1a_2], \ldots, [Aa_1 \ldots a_{n-2}] \rightarrow a_{n-1}[Aa_1 \ldots a_{n-1}],$

and $[Aa_1 \ldots a_{n-1}] \rightarrow a_n[\varepsilon]$.

$A \Rightarrow^* xB$ *in G, if and only if,* $[A] \Rightarrow^* x[B]$ *in G'.*

basis *If* $A \Rightarrow^0 xB$ *in G,* $x = \varepsilon$, $A = B$. *Then* $[A] \Rightarrow^0 [A] = x[B]$

recursion *If* $A \Rightarrow^n xB$ *in G, Then* $[A] \Rightarrow^* x[B]$ *in G'.*

$A \Rightarrow^{n-1} \Rightarrow yC \Rightarrow xB$ *in G, Then* $C \rightarrow zB \in P$, $x = yz \in \Sigma^*$, $B \in N \cup \{\varepsilon\}$.

$[A] \Rightarrow^* x[B]$