# *Chap. 2 Finite Automata*

## *2.1 An Informal Picture of Finite Automata*

*A man with a wolf, goat, and cabbage is on the left bank of a river*

*A boat carries one man and only one of the other three.*

*The wolf eats the goat without the man.*

*The goat eats the cabbage without the man.*

*Is it possible to cross the river without the goat or cabbage being eaten?*

***states**: occupants of each bank after a crossing*

*16 subsets of the man(m), wolf(W), goat(G), and cabbage(C).*

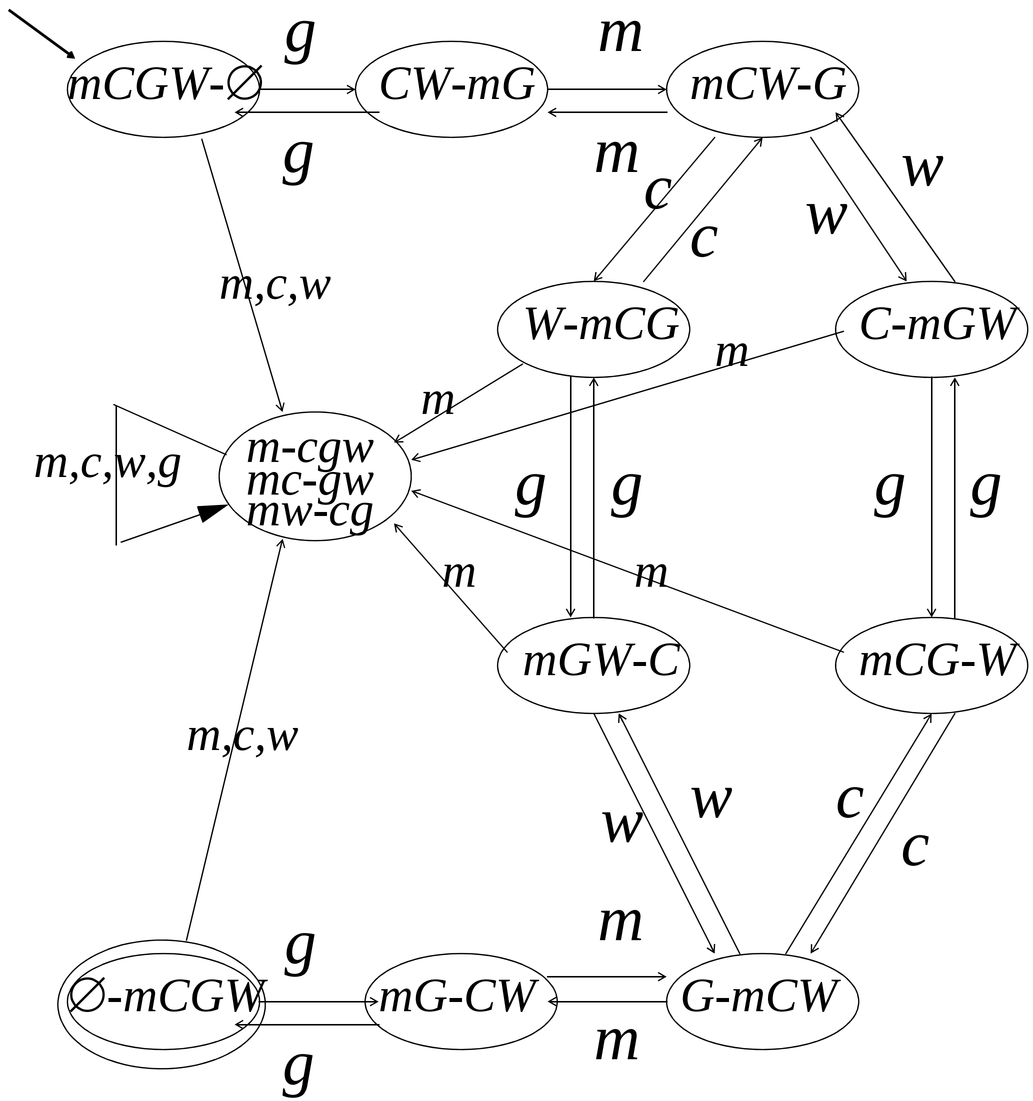
*16 states: mWGC-∅, WC-mG, …, ∅-mWGC*

***input symbol**: four ways of crossing the river*

*man cross alone(m), with the wolf(w), the goat(g), or cabbage(c)*

***input string**: a **sequence** of crossings(input symbols)*

***state transition**: From a **state** to another **state** by a **input symbol***


mCGW-∅
CW-mG
mCW-G
W-mCG
C-mGW
m-cgw
mc-gw
mw-cg
mGW-C
mCG-W
∅-mCGW
mG-CW
G-mCW
g
g
m
m
c
c
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m,c,w
m,c,w,g
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m,c,w
w
w
c
c
g
g
m
m

## *2.2 Deterministic Finite Automata*

*A* ***deterministic finite automaton*** *consists of:*

*1. A* ***finite*** *set of* ***states****, denoted Q,*

*2. A* ***finite*** *set of* ***input symbols****, denoted Σ,*

*3. A* ***transition function****,*

$\delta: Q \times \Sigma \rightarrow Q$

*Let p, q ∈ Q, and a ∈ Σ, if we write δ(q, a) = p, meaning that when* ***current*** *state is in q and* ***input*** *symbol is a, the current state is changed to p, and input symbol is* ***changed*** *to the* ***next****(****right****) one.*

*4. A* ***start****(****initial****) state,* $q_0 \in Q$*, and*

*5. A set of* ***final*** *or* ***accepting*** *states,* $F \subseteq Q$*.*

*A DFA* $A = (Q, \Sigma, \delta, q_0, F)$ *"five-tuple" notation*

## *2.2.2 How a DFA Processes Strings*

*How a DFA decides whether or not "**accept**"*

*a **input string**(**sequence of input symbols**)*

*Suppose* $a_1 a_2 \ldots a_n$ *is a sequence of input symbols*

1. *We start out the DFA with in its **start** state* $q_0$.
2. *We consult the **transition function**,* $\delta$*,*
   *with current state and current input symbol,*
   *asumme in the first trial* $\delta(q_0, a_1) = q_1$*,*
3. ***Repeat** this step with the **next** state* $q_1$ *and the **next** input symbol* $a_2$.
   *Assume* $\delta(q_1, a_2) = q_2$, $\delta(q_2, a_3) = q_3$, $\ldots$, $\delta(q_{n-1}, a_n) = q_n$.
4. ***Finally**, check if* $\delta(q_{n-1}, a_n) = q_n \in F$ *or not.*
   *If* $q_n \in F$*, **accept*** $a_1 a_2 \ldots a_n$*. If* $q_n \notin F$ ***does not accept*** $a_1 a_2 \ldots a_n$.
   $\delta(\delta(\ldots\delta(\delta(q_0, a_1), a_2), \ldots), a_{n-1}), a_n) = q_n$
   *where* $\delta(q_{i-1}, a_i) = q_i$ *for* $1 \leq {}^{\forall} i \leq n$.

***Example*** *2.1 Let* $\Sigma = \{0, 1\}$ *and* $L = \{x01y \in \Sigma^* \mid x, y \in \Sigma^*\}$.

*State* $q_0$*:* ***initial*** *state*

*If it sees 0, go to a* ***new*** *state* $q_1$*. If it sees 1* ***stay*** *in the state* $q_0$*.*

*It has* ***never*** *seen 0.*

*State* $q_1$*: The* ***last symbol*** *it has seen is 0.*

*If it sees 1, go to a* ***new*** *state* $q_2$*. If it sees 0,* ***stay*** *in the state* $q_1$*.*

*State* $q_2$*: It has* ***ever*** *seen is 01.* ***final*** *state*

*Whether it sees 0 or 1, stay in the state* $q_2$*.*

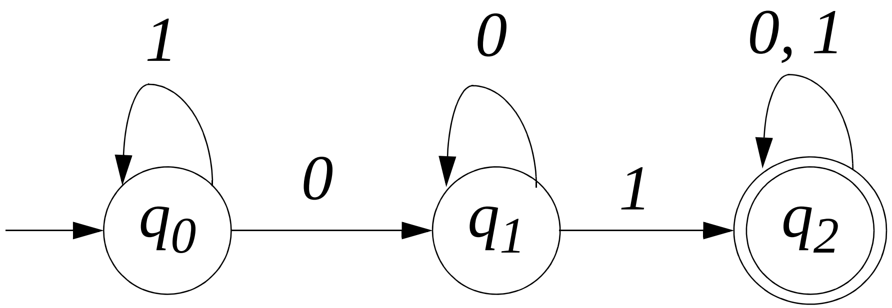
$A = (Q, \Sigma, \delta, q_0, F)$

$Q = \{q_0, q_1, q_2\}$ $\Sigma = \{0, 1\}$

$\delta = \{\delta(q_0, 0)=q_1, \delta(q_0, 1)=q_0, \delta(q_1, 0)=q_1, \delta(q_1, 1)=q_2,$
$\delta(q_2, 0)=q_2, \delta(q_2, 1)=q_2\}$

$F = \{q_2\}$

# *2.2.3 Simple notations for DFA's*
## *Transition diagram*




## *Transition table*

| | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_1$ | $q_2$ |
| $* q_2$ | $q_2$ | $q_2$ |

## *2.2.4 Extend the domain of transition function from symbols to strings*

$\delta: Q \times \Sigma \to Q$

$\hat{\delta}: Q \times \Sigma^* \to Q$

$\hat{\delta}(q, \varepsilon) =_B q$ **basis**

$\hat{\delta}(q, wa) =_R \delta(\hat{\delta}(q, w), a)$ **recursion**

$q \in Q, w \in \Sigma^*, a \in \Sigma, wa \in \Sigma^+.$

$\hat{\delta}(q_0, a_1 a_2 \ldots a_n) =_R \delta(\hat{\delta}(q_0, a_1 a_2 \ldots a_{n-1}), a_n)$ *1st* **recursion**

$=_R \delta(\delta(\hat{\delta}(q_0, a_1 a_2 \ldots a_{n-2}), a_{n-1}), a_n)$ *2nd* **recursion**

$=_R \cdots$

$=_R \delta(\delta(\ldots\delta(\hat{\delta}(q_0, a_1), a_2), \ldots), a_{n-1}), a_n)$ *(n-1)-th* **recursion**

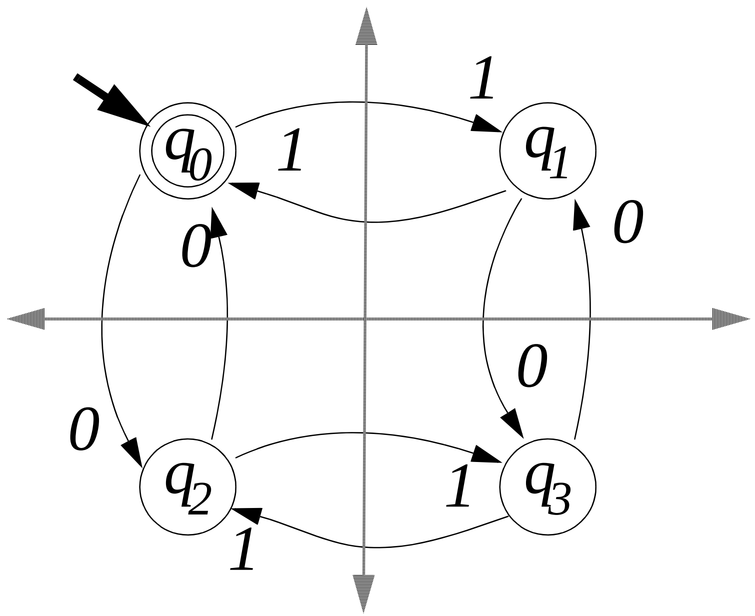
$=_R \delta(\delta(\ldots\delta(\delta(\hat{\delta}(q_0, \varepsilon), a_1), a_2), \ldots), a_{n-1}), a_n)$ *n-th* **recursion**

$=_B \delta(\delta(\ldots\delta(\delta(q_0, a_1), a_2), \ldots), a_{n-1}), a_n)$ **basis**

*If $\hat{\delta}(q, w) = p$, for some $q, p \in Q$ and $w \in \Sigma^*$,*
*$w$ is the* ***path*** *from the state $q$ to the state $p$.*

*Since $\hat{\delta} \supseteq \delta$, we may use $\delta$ instead of $\hat{\delta}$.*

## *Example 2.4*




*even numbers of 0's and even numbers of 1's.*

## *2.2.5 The Language of DFA*

*We **define** the language of a DFA* $A = (Q, \Sigma, \delta, q_0, F)$, $L(A)$,

$$L(A) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\} \text{ or } \{w \in \Sigma^* \mid \delta(q_0, w) \in F\}, \text{ for short.}$$

$$L(A) \subseteq \Sigma^* \text{ or } L(A) \in 2^{\Sigma^*}.$$

***Definition: Regular languages***

*Let L be a language.*

*If L = L(A) for some **DFA** A, then we say L is a **regular language**.*

*a **class of languages***

***type-3 languages** in Chomsky's language hierarchy*

***Definition: Equivalence of Automata***

*Let* $M_1$ *and* $M_2$ *be two automata. We say two automata* $M_1$ *and* $M_2$ *are **equivalent**, if* $L(M_1) = L(M_2)$.

*The **state** of automata*

***summarizes** the information concerning **past inputs***

*that is needed to **determine** the **behavior** of the automata*

*on **subsequent inputs**.*

*Aho and Ullman*

*Let* $M = (Q, \Sigma, \delta, q_0, F)$ *be a DFA and* $q \in Q$.

*We define **accessible** input strings from initial state to the state q.*

$$L(q) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) = q\}$$

$$L(M) = \cup_{q \in F} L(q).$$

*Consider a **binary relation*** $R_M$ *on* $\Sigma^*$ *and* $x\ R_M\ y$ , *if* $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$.

*Then* $R_M$ *is a **quivalent** relation and*

*the **equivalent class*** $[x]_{R_M} = L(q)$, *if* $\hat{\delta}(q_0, x) = q$.

*Since* $R_A$ *is equivalent,* $[x]_{R_M}$ *is a **partion** of* $\Sigma^*$.

## *2.3 Nondeterministic Finite Automaton*

*$A = (Q, \Sigma, \delta, q_0, F)$ is a* ***nondeterministic finite automaton****(NFA), if*

1. *Q is a* ***finite*** *set of* ***states****,*
2. *$\Sigma$ is a* ***finite*** *set of* ***input symbols****, open denoted,*
3. *$\delta$ is a* ***transition function****,*

   $\delta: Q \times \Sigma \rightarrow 2^Q$.

   *If $q \in Q$, $a \in \Sigma$, and $\{p_1, p_2, \ldots, p_n\} \subseteq Q$(or $1 \leq {}^\forall i \leq n$, $p_i \in Q$)*

   *then we write $\delta(q, a) = \{p_1, p_2, \ldots, p_n\}$, meaning that*

   *when current state is in q and input symbol is a,*

   *the state may be changed to* ***any*** *one of $p_1, p_2, \ldots, p_n$,*

   *and input symbol is changed to the next(right) one.*
4. *$q_0 \in Q$ is a* ***start*** *state, and*
5. *$F \subseteq Q$ is a set of* ***final*** *or* ***accepting*** *states.*

   *Fig. 2.9, 2.10, 2.11*

*DFA:* $\delta: Q \times \Sigma \rightarrow Q$

$\forall q \in Q, \forall a \in \Sigma, \exists \delta(q, a) \in Q.$ (**total**) *function*

*Otherwise* ***partial*** *function*

*If* $A = (Q, \Sigma, \delta, q_0, F)$ *be a FA with* $\delta: Q \times \Sigma \rightharpoonup Q$ ***partial*** *function,*

$L(A) = L(B)$ *for DFA* $B = (Q \cup \{d\}, \Sigma, \delta', q_0, F)$ *where* $d \notin Q$,

$\delta' = \delta \cup \{\delta(q, a) = d \mid \delta(q, a) \notin Q\} \cup \{\delta(d, a) = d \mid a \in \Sigma\}$

*d:* ***dead*** *state*

*B is the DFA(with total function)*

| | |
|---|---|
| *DFA:* | $\delta: Q \times \Sigma \rightarrow Q.$ |
| *DFA but partial function:* | $\delta: Q \times \Sigma \rightarrow Q \cup \{\emptyset\}.$ |
| *NFA:* | $\delta: Q \times \Sigma \rightarrow 2^Q.$ |

## 2.3.3 The Extended Transition Function

*DFA*

$\delta: Q \times \Sigma \rightarrow Q$

$\hat{\delta}: Q \times \Sigma^* \rightarrow Q$

*NFA*

$\delta: Q \times \Sigma \rightarrow 2^Q$

$\delta': 2^Q \times \Sigma \rightarrow 2^Q$ *in this lecture*

$\hat{\delta}': 2^Q \times \Sigma^* \rightarrow 2^Q$ *in this lecture* $\hat{\delta}: Q \times \Sigma^* \rightarrow 2^Q$ *in the text*

***Two step extension***

***1. Set extension***: $\delta \Rightarrow \delta'$

*We extend the 1st domain of* $\delta$ *to* ***set of states***.

$\delta': 2^Q \times \Sigma \rightarrow 2^Q$.

*Let* $P \subseteq Q$*, we define*

$$\delta'(P, a) = \{q \in Q \mid p \in P, q \in \delta(p, a)\}$$

*or* $= \cup_{p \in P} \delta(p, a)$

*We may write δ'(p, a)* ***instead*** *of δ'({p}, a) when needed.*

***2. String extension*** : $\delta' \Rightarrow \hat{\delta}'$ (same as $\delta \Rightarrow \hat{\delta}$)

$\hat{\delta}': 2^Q \times \Sigma^* \rightarrow 2^Q$.

$\hat{\delta}'(P, \varepsilon) =_B P$ ***basis***

$\hat{\delta}'(P, wa) =_R \delta'(\hat{\delta}'(P, w), a)$ ***recursion***

$P \subseteq Q$ (or $P \in 2^Q$), $w \in \Sigma^*$, $a \in \Sigma$, $wa \in \Sigma^+$.

*We may use δ instead of δ' or* $\hat{\delta}'$, *since* $\delta \subseteq \delta', \hat{\delta}'$.

*Let* $N = (Q, \Sigma, \delta, q_0, F)$ *be an NFA. Then the* ***language defined*** *by the NFA N, denoted as L(N) is,*

$L(N) = \{w \in \Sigma^* \mid \hat{\delta}'(\{q_0\}, w) \cap F \neq \emptyset\}$.

$L(N) \subseteq \Sigma^*$ or $L(N) \in 2^{\Sigma^*}$.

## 2.3.5 Equivalence of DFA and NFA

*Example 2.10(Fig. 2.12)*

***Set construction***(NFA $\Rightarrow$ DFA)

Given an NFA $N = (Q_N, \Sigma, \delta_N, q_{0N}, F_N)$,

**construct** DFA $D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$.

1. $Q_D = \{P \mid P \subseteq Q_N\} = 2^{Q_N}$. ***A subset*** of NFA ***states*** is ***a state*** of DFA.
3. $\delta_D(P, a) = \{q \in Q_N \mid p \in P \subseteq Q_N, q \in \delta_N(p, a)\}$

   or $= \cup_{p \in P} \delta_N(p, a)$

   or $= \delta_N'(P, a)$ ***set extension*** of $\delta_N$.

   $\delta_D(P, a) = \delta_N'(P, a)$ or $\delta_D = \delta_N'$ in short,

where $\delta_D: Q_D \times \Sigma \to Q_D$, $\delta_N': 2^{Q_N} \times \Sigma \to 2^{Q_N}$, and $Q_D = 2^{Q_N}$.

4. $q_{0D} = \{q_{0N}\}$.
5. $F_D = \{F \in Q_D \mid F \cap F_N \neq \emptyset\}$

***Theorem*** *2.11 L(D) = L(N) in the above set construction*

***Proof*** $\hat{\delta}_D(q_{0D}, w) = \hat{\delta}'_N(\{q_{0N}\}, w)$ *for all* $w \in \Sigma^*$.

$\hat{\delta}_D: Q_D \times \Sigma^* \rightarrow Q_D = 2^{Q_N} \times \Sigma^* \rightarrow 2^{Q_N}$.

$\hat{\delta}'_N: 2^{Q_N} \times \Sigma^* \rightarrow 2^{Q_N}$. *We may use* $q_{0N}$ *instead of* $\{q_{0N}\}$.

***Induction*** *on* $|w|$.

***basis***: *Let* $|w| = 0$, $\hat{\delta}_D(q_{0D}, \varepsilon) = \hat{\delta}_D(\{q_{0N}\}, \varepsilon) = \{q_{0N}\} = \hat{\delta}'_N(\{q_{0N}\}, \varepsilon)$.

***induction***: *Let* $w = xa$ *where* $a \in \Sigma$, $x \in \Sigma^*$, *and* $w \in \Sigma^+$. $(|w| = |x| + 1)$

$\hat{\delta}_D(q_{0D}, xa) = \delta_D(\hat{\delta}_D(\{q_{0N}\}, x), a)$ *by definition of* $\hat{\delta}_D$.

$= \delta_N'(\hat{\delta}_D(\{q_{0N}\}, x), a)$ *by* ***Set construction***

$= \delta_N'(\hat{\delta}'_N(\{q_{0N}\}, x), a)$ *by induction hypothesis*

$= \hat{\delta}'_N(\{q_{0N}\}, xa)$ *by definition of* $\hat{\delta}'_N$.

$\hat{\delta}_D(\{q_0\}, x) \in F_D$, *if and only if*, $\hat{\delta}_N(q_0, x) \cap F_N \neq \emptyset$.

*Q.E.D.*

***Theorem 2.12*** *A language L is defined by some DFA, if and only if L is defined by some NFA.(NFA ⇔ DFA)*

***Proof****: (If) Set construction(NFA ⇒ DFA)*

*(Only if) NFA can* ***easily simulate*** *DFA(DFA ⇒ NFA).*

*Let a DFA* $D = (Q, \Sigma, \delta_D, q_0, F)$. *We define an NFA*$(Q, \Sigma, \delta_N, q_0, F)$

$$\delta_N(q, a) = \{p \mid \delta_D(q, a) = p\}.$$

***Proof*** *for* $\forall w \in \Sigma^*$, *if and only if*, $\hat{\delta}_D(q, w) = p$, *then* $\hat{\delta}_D(q, w) = \{p\}$.

*Then* $L(D) = L(N)$.

∴ *L is* ***regular****, iff* $L = L(N)$ *for some* ***NFA*** *N.*

## *2.5 Finite Automata with Epsilon-Transitions*

*A = $(Q, \Sigma, \delta, q_0, F)$ is a* ***fa with epsilon transition****(ε-NFA), if*

*$Q, \Sigma, q_0$, and F are same as NFA, but*

*3. δ is a* ***transition function****,*

$$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q.$$

*If $q \in Q$, and $\{p_1, p_2, \ldots, p_n\} \subseteq Q$,*

*then we write $\delta(q, \varepsilon) = \{p_1, p_2, \ldots, p_n\}$, meaning that*

*when current state is in q and* ***regardless*** *of input symbol*

*the state may be changed to* ***any*** *one of $p_1, p_2, \ldots, p_n$,*

*and next input symbol is* ***not changed****.*

*Ex. 2.16, Fig. 2.18*

## *2.5.3 Epsilon-closure*

***basis:*** *$q \in \varepsilon^*(q)$ (**same as** ECLOSE(q) in the text)*

***recursion**: If $p \in \varepsilon^*(q)$, and $r \in \delta(p, \varepsilon)$, then $r \in \varepsilon^*(q)$.*

*Example 2.18*

*Why $\varepsilon^*(q)$ **instead** of ECLOSE(q)*

*Let $\varepsilon = \{(p, q) \mid q \in \delta(p, \varepsilon)\}$. Then $\varepsilon \subseteq Q \times Q$(a binary relation on Q)*

*and ECLOSE(q) = $\varepsilon^*(q)$ **reflexive-transitive** closure of $\varepsilon$.*

## 2.5.4 The Extended Transition and Language for ε-NFA's

*NFA:* $\delta: Q \times \Sigma \rightarrow 2^Q$.

*ε-NFA:* $\delta: Q \times (\{\varepsilon\} \cup \Sigma) \rightarrow 2^Q$.

*Let's **divide** δ of ε-NFA into $\delta^1$ and $\delta^0$.*

$\delta^1: Q \times \Sigma \rightarrow 2^Q$.

$\delta^0: Q \times \{\varepsilon\} \rightarrow 2^Q$.

$\delta = \delta^1 \cup \delta^0$.

$\delta^1$*: is **same** as δ in NFA, we use δ **instead** of $\delta^1$, since $\delta^1 \subseteq \delta$.*

$\delta^0 \leftrightarrow_f \varepsilon$ *what is the **bijection** f? We use ε **instead** of $\delta^0$.*

$\varepsilon = \{(p, q) \mid q \in \delta(p, \varepsilon)\} \subseteq Q \times Q$ *binary **relation** on Q.*

$\varepsilon^* = \cup_{i \in N_0} \varepsilon^i \subseteq Q \times Q$ *binary **relation** on Q.*

$\varepsilon, \varepsilon^*: Q \rightarrow 2^Q$. ***function** from Q to $2^Q$.*

*Let's **extend** the domain of* $\delta(=\delta^1)$, $\varepsilon$, *and* $\varepsilon^*$ *from **state** to **set of states**.*
*Let* $P \subseteq Q$, *we define*

$\delta': 2^Q \times \Sigma \to 2^Q$.

$\delta'(P, a) = \{q \in Q \mid p \in P, \delta(p, a) = q\}$

*or* $= \cup_{p\in P}\ \delta(p, a)$

*We may write* $\delta'(p, a)$ ***instead** of* $\delta'(\{p\}, a)$ *when needed.*

*We may write* $\delta(P, a)$ ***instead** of* $\delta'(P, a)$ *when needed.*

$\varepsilon': 2^Q \to 2^Q$.

$\varepsilon'(P) = \{q \in Q \mid p \in P, \varepsilon(p) = q\}$

*or* $= \cup_{p\in P}\ \varepsilon(p)$

$\varepsilon'^*: 2^Q \to 2^Q$.

$\varepsilon'^*(P) = \{q \in Q \mid p \in P, \varepsilon^*(p) = q\} \quad = \cup_{p\in P}\ \varepsilon^*(p)$

*We write* $\varepsilon(P)$ ***instead** of* $\varepsilon'(P)$ *and* $\varepsilon^*(P)$ ***instead** of* $\varepsilon'^*(P)$.

*Let's* ***extend*** *the of domain of* $\delta$ *from* ***symbol*** *to* ***strings***.

$\hat{\delta}: 2^Q \times \Sigma^* \rightarrow 2^Q$.

$\hat{\delta}(P, \varepsilon) = \varepsilon^*(P)$ ***basis***

$\hat{\delta}(P, wa) = \varepsilon^*(\delta(\hat{\delta}(P, w), a))$ ***recursion***

$P \subseteq Q (\text{or } P \in 2^Q), w \in \Sigma^*, a \in \Sigma, wa \in \Sigma^+$.

$\hat{\delta}(q_0, a_1a_2 \ldots a_n) = \varepsilon^*(\delta(\hat{\delta}(q_0, a_1a_2 \ldots a_{n-1}), a_n))$ *1st* ***recursion***

$= \varepsilon^*(\delta(\varepsilon^*(\delta(\hat{\delta}(q_0, a_1a_2 \ldots a_{n-2}), a_{n-1})), a_n))$ *2nd* ***recursion***

$= \ldots$

$= \varepsilon^*(\delta(\varepsilon^*(\delta(\ldots\varepsilon^*(\delta(\hat{\delta}(q_0, a_1), a_2)), \ldots), a_{n-1})), a_n))$ *(n-1)th* ***recursion***

$= \varepsilon^*(\delta(\varepsilon^*(\delta(\ldots\varepsilon^*(\delta(\varepsilon^*(\delta(\hat{\delta}(q_0, \varepsilon), a_1)), a_2)), \ldots)), a_{n-1})), a_n))$ ***n-th***

$= \varepsilon^*(\delta(\varepsilon^*(\delta(\ldots\varepsilon^*(\delta(\varepsilon^*(\delta(\varepsilon^*(q_0), a_1)), a_2)), \ldots)), a_{n-1})), a_n))$ ***basis***

$\delta^n = \varepsilon^* \circ \delta \circ \varepsilon^* \circ \ldots \circ \delta \circ \varepsilon^* = \varepsilon^* \circ (\delta \circ \varepsilon^*)^n$.

*We may write* $\delta^*$ ***instead*** *of* $\hat{\delta}$, *since* $\hat{\delta} = \cup_{i \in N_0} \delta^i = \delta^*$.

*Let* $E = (Q, \Sigma, \delta, q_0, F)$ *be an* ε*-NFA. Then*

*the* ***language defined by*** *the* ε*-NFA E, denoted as L(E) is,*

$$L(E) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset\}.$$

$$L(E) \subseteq \Sigma^* \text{ or } L(E) \in 2^{\Sigma^*}$$

## 2.5.5 Eliminating ε-Transitions

*Given an ε-NFA* $E = (Q_E, \Sigma, \delta_E, q_0, F_E)$*, construct the* **equivalent** *DFA* $D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$ *such that* $L(D) = L(N)$.

1. $Q_D = \{P \mid P \subseteq Q_E\} = 2^{Q_E}$.

3. $q_{0D} = \varepsilon^*(\{q_0\})$

4. $\delta_D(P, a) = \varepsilon^*(\delta_E(P, a))$ ***set extension*** *of* $\delta_E$ *and* ***ε-closure***

   *where* $P \subseteq Q_E$ $(P \in Q_D)$, $a \in \Sigma$, *and*

   $\delta_D(P, a) \subseteq Q_E$ (*or* $\delta_D(P, a) \in Q_D$)

5. $F_D = \{F \in Q_D \mid F \cap F_E \neq \emptyset\}$

***Theorem 2.22*** *A language L is accepted by some ε-NFA if and only if L is accepted by some DFA.*

***Proof***: *(If) ε-NFA can* ***easily simulate*** *DFA(DFA ⇒ ε-NFA).*

*(Only if) Eliminating ε-transitions(ε-NFA ⇒ DFA)*

*Let* $E = (Q_E, \Sigma, \delta_E, q_0, F_E)$ *be a* $\varepsilon$*-NFA and*

*DFA* $D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$ *is the one in 2.5.5.*

*We show* $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(q_{0D}, w)$ *by induction on* $|w|$.

***basis***: $\hat{\delta}_E(q_0, \varepsilon) = \varepsilon^*(q_0)$ — *by* basis **definition** *of* $\hat{\delta}_E$.

$= q_{0D} =$ — *by* **construction** *of* $\delta_D$.

$= \hat{\delta}_D(q_{0D}, \varepsilon)$ — *by* basis **definition** *of* $\hat{\delta}_D$.

***Induction***: *Let* $w = xa \in \Sigma^+$, $a \in \Sigma$, *and* $x \in \Sigma^*$.

$\hat{\delta}_E(q_0, xa) = \delta_E(\varepsilon^*(\hat{\delta}_E(q_0, x), a)$ — *by* recursive **definition** *of* $\hat{\delta}_E$.

$= \delta_E(\varepsilon^*(\hat{\delta}_D(q_{0D}, x), a))$ — *by* **induction hypothesis**.

$= \delta_D(\hat{\delta}_D(q_{0D}, x), a)$ — *by* **construction** *of* $\delta_D$.

$= \hat{\delta}_D(q_{0D}, xa)$ — *by* recursive **definition** *of* $\hat{\delta}_D$.

## 2.A Extended Finite Automata(XFA)

*$X = (Q, \Sigma, q_0, \delta, F)$ is an **extended finite state automata**(XFA), if*

$$\delta: Q \times \Sigma^* \rightarrow 2^Q.$$

*Given an XFA $X = (Q_X, \Sigma, q_0, \delta_X, F_X)$, **construct***
*an **equivalent** ε-NFA $E = (Q_E, \Sigma, q_0, \delta_E, F_E)$ such that $L(X) = L(E)$.*

***Construction algorithm***

$Q_E := Q_X;\ F_E := F_X;\ \delta_E := \emptyset;$

***for*** $^\forall(\delta_X(q, x) = P) \in \delta_X$ *where* $x = a_1a_2\ldots a_n$ ***do***

  ***if*** $0 \le n \le 1 \rightarrow \delta_E(q, x) := P$ *where* $x \in \Sigma \cup \{\varepsilon\}$

  $| \ n \ge 2 \quad \rightarrow \delta_E(q, a_1) = \delta_E(q, a_1) \cup \{p_1\};$

    ***for*** $2 \le {}^\forall i \le n\text{-}1$ ***do*** $\delta_E(p_{i-1}, a_i) := \{p_i\}$ ***od***; $\delta_E(p_{n-1}, a_n) = P;$

    $Q_E := Q_E \cup \{p_1, p_2, \ldots, p_{n-1}\}$ *where* $Q_E \cap \{p_1, p_2, \ldots, p_{n-1}\} \neq \emptyset$

  ***fi***

***od***

*The following statements are **equivalent**,*

1. *A language L is **regular**.*
2. *L = L(D) for some DFA D.* $\delta : Q \times \Sigma \to Q$.
3. *L = L(P) for some FA P with partial function δ.*

   $\delta : Q \times \Sigma \to Q \cup \{\emptyset\}$.
4. *L = L(N) for some NFA N.* $\delta : Q \times \Sigma \to 2^Q$.
5. *L = L(E) for some ε-NFA E.* $\delta : Q \times (\{\varepsilon\} \cup \Sigma) \to 2^Q$.
6. *L = L(X) for some XFA X.* $\delta : Q \times \Sigma^* \to 2^Q$.