Chap. 9 Undecidability

9.1 A Language that is Not recursively enumerable

Code for Turing Machine

TM $M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, \{q_2\})$

$Q = \{q_1, q_2, \ldots, q_k\}$

$\Gamma = \{X_1, X_2, \ldots, X_m\}$ $X_1 = 0, X_2 = 1, X_3 = B$.

$L = D_1, R = D_2$.

$\delta(q_i, X_j) = (q_k, X_l, D_m)$ $i, j, k, l, m \in \mathbb{N}$.

$\leftrightarrow 0^i10^j10^k10^l10^m$

$\delta \leftrightarrow \delta_1 11 \delta_2 11 \ldots 11 \delta_n$.

$(M, w) \leftrightarrow code(M)111code(w)$

number of Turing machines is ***countable****.*

We can ***enumerate*** *TM* M_i *for* $i \in \mathbb{N}$.

Diagonalization Language: L_d.

Since both of TM's and strings in Σ^* are **countable**,
we can consider (M_i, w_i) pair for $i \in \mathbb{N}$.

Consider $L_d = \{w_i \in \Sigma^* \mid w_i \notin L(M_i)\}$

Theorem 9.2 L_d is **not** recursively enumerable.

proof Suppose $L_d = L(M)$ for some TM M.

Since M is a TM, $^{\exists}i \in \mathbb{N}$.϶. $M = M_i$.

If $w_i \in L_d$, $w_i \notin L_d$ by definition of L_d. $\therefore$ M does **not** accept w_i.

If $w_i \notin L_d$, $w_i \in L_d$ by definition of L_d. $\therefore$ M **accepts** w_i.

Contradiction! $\therefore$ M does **not** exist.

$\therefore$ L_d is **not recursively enumerable**.

TM's are **countable** whereas languages are **uncountable**!

The halting problem

```
program halt(P: program, I: input)
        if P(I) will stop then print "halts"
                               else print "loops forever" fi
```

Assume the program halt exists and consider a program H

```
program H(P: program)
        if halt(P, P) = "halts" then loops forever
                                     else stop fi
```

Consider H(H)

*if H(H) loops forever ⇒ halt(H, H) prints “**stop**”*
*⇒ But H(H) must stop(**definiton** of H).*
*if H(H) stop ⇒ halt(H, H) prints “**no stop**”*
*⇒ But H(H) must loops forever(**definiton** of H).*

∴ *Contradiction! halt does **not** exist.*

∴ ***halting problem** does **not** exist.*

Languages(sets) that is ***not*** *RE(no TM, no* ***program****)*

$L_d = \{w_i \in \Sigma^*|\ w_i \notin L(M_i)\}$

halting problem

power set of integer is ***uncountable***

Cantor's diagonal arguments

Russel's paradox

$S = \{x|\ x \notin x\}$ $x \in x$, *iff* $x \notin x$. *But* $S \in S$, *iff* $S \notin S$!

Some similar examples in the world

A barber who shave everybody who can not shave himself.

Shall the barber shave himself?

An adjective is heterological, if the adjective does not posess the property it describes.(monosyllabic, polysyllabic)

Is the adjective "heterological" ***heterological****?*

There is a sign that "It is written by me(liar)".Did you(liar) write it?

Self contradiction ***denial of self recursion****!*

9.2 An undecidable problem that is RE

Recursive languages

*If w ∈ L, M **halts** and **accepts**.*
*If w ∉ L, M **halts** and does **not accepts**.*

***subclass** of RE languages*
type 1 in Chomsky's hieratchy

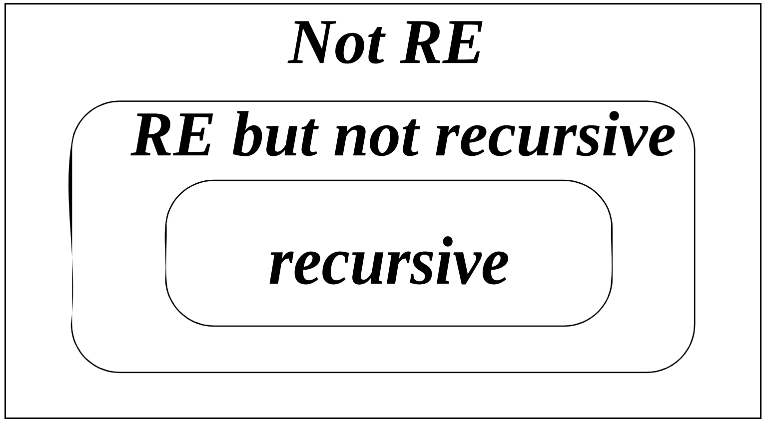
***Recursivly enumerable languages**(RE languages)*

*If w ∈ L, M **halts** and **accepts**.*
*If w ∉ L, M **halts** and does **not accepts** or **loops forever**.*

*Problem P is called **decidable**, if P is **recursive***
*Problem P is called **undecidable**, if P is **not** recursive*
*P may be RE **or** non-RE*

Three ***classes*** *of languages(problems)*

recursive	***decidable***	***countable***
RE *but* ***not recursive***	***undecidable***	***countable***
not RE	***undecidable***	***uncountable***



Not RE

RE but not recursive

recursive

Recursive	***Decidable(algorithm)***
Recursive enumerable	***Turing computable*** ***partial recursive function*** ***programmable(computable)***

Languages and problems

$L: \Sigma^* \rightarrow \{0, 1\}$ $P: \mathbb{N} \rightarrow \{0, 1\}$

Both of languages and problems are ***uncountable****.*

But TM(program) are ***countable****.*

There are problems(languages) that is not recursively enumerable.

halting problem

Russel's paradox

Diagonalization languages(L_d)

complement *of* L_d $\overline{L_d} = \{w_i \in \Sigma^* | w_i \in L(M_i)\}$

RE(but ***not recursive****)*

Complement of recursive and recursively enumerable languages

Theorem 9.3 *If L is* ***recursive****,* $\overline{L}$ *is also* ***recursive****.*

proof *Let L = L(M) for some TM M that always* ***halts****.*

Consider $\overline{M}$

accept *and halt →* ***not accept*** *and halt.*

not accept *and halt →* ***accept*** *and halt.*

$\exists\overline{M}$.϶. $\overline{L} = L(\overline{M})$ *and always* ***halts****.*

More detail

$M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, \{q_2\})$ *no transition from F*

$\overline{M} = (Q \cup \{f\}, \{0, 1\}, \Gamma, \overline{\delta}, q_1, B, \{f\})$

$\overline{\delta} = \delta \cup \{\delta(q, X) = (f, X, S) \mid q \in Q\}$

no transition in M → transition to f

Theorem 9.4 *If both of L and $\overline{L}$ are RE, L is **recursive**(so is $\overline{L}$).*

proof *Let $L = L(M_1)$ and $\overline{L} = L(M_2)$.*

*Consider a **two** tape TM M.*

tape 1 simulates the tape of M_1 and tape 2 simulates the tape of M_2.

*states and state transitions of M simulates M_1 and M_2 in **parallel**.*

*If $x \in L \rightarrow M_1$ **accept** and halt $\rightarrow M$ accept x and **halt**.*

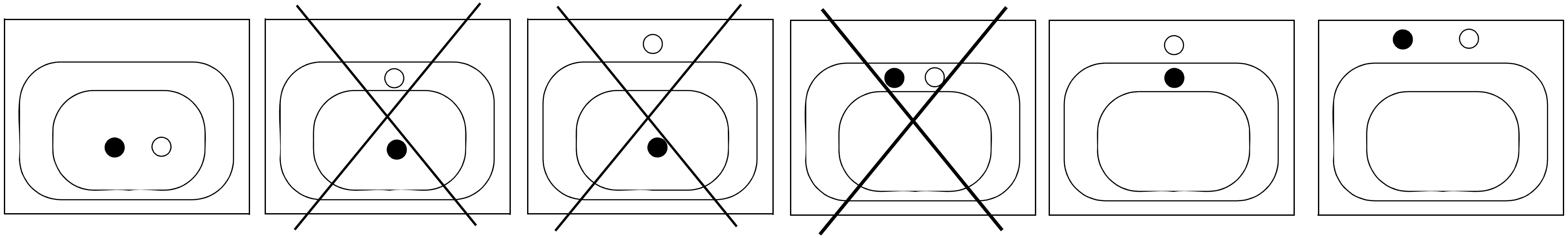
*If $x \notin L \rightarrow M_2$ **accept** and halt $\rightarrow M$ **not** accept x and **halt**.*

$\therefore$ *L is **recursive**.*

*Only **three** case for the **complement** of language (among 6 cases)*

*Thm 9.3: If L is **recursive**, $\overline{L}$ is **recursive** and vice versa.*

*Thm 9.4: L and $\overline{L}$ are **not** both RE and not **recursive**.*



***Universal language**: L_u. complement of L_d.*

$L_u = \{w_i \in \Sigma^* \mid w_i \in L(M_i)\} = \overline{L}_d$.

***Theorem 9.6** L_u is RE but not recursive.*

***proof** Let U, **universal TM**, be a multi tape TM such that $L(U) = L_u$.*

tape 1: (M_i, w_i)

tape 2: simulate the tape of M_i.

If M_i accept w_i, U accept M_i.

$\therefore$ *L_u is **RE**.(simplified version of 9.2.3)*

*Since L_d is **not** RE.*

$\therefore$ *L_u is **not recursive**.(Three cases for the complement)*

$\therefore$ *L_u is RE but **not** recursive.*

$L_d = \overline{L}_u$ is not RE.(the second case in the diagram of p8)

9.3 Undecidable Problems About Turing Machines

*P is a **decision problem** on the **domain** D, if $\forall d \in D$, $P(d)$ is **yes** or **no**.*

*$P: D \rightarrow \{$**yes**, **no**$\}$*

*Y_P, $N_P \subseteq D$ is called **yes(no) instances** of P, if*

*$Y_P = \{d \in D \mid P(d) =$ **yes**$\}$*

*$N_P = \{d \in D \mid P(d) =$ **no**$\}$, respectively.*

*A **problem** P is **decidable**,*

*if there exists a **decider (program**, **algorithm**, **TM)** that*

***always** tells **yes** or **no** correctly.*

***Undecidable**, otherwise.*

***Assume** D is **countable**.*

*Then $|\{P(D)\}|$ is **uncountable**.*

*But **decider** is **countable**.*

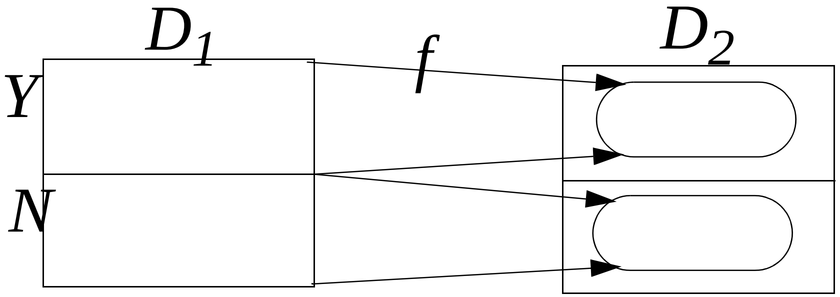
*$\therefore\ \exists P$, .϶. P is **undecidable**.*

Reducing one problem to another

We say a problem P_1 *on* D_1 **reduces to** P_2 *on* D_2*, if*

$$\exists f: D_1 \to D_2,\ .\ni.\ \forall d_1 \in Y_{P_1}(D_1),\ f(d_1) \in Y_{P_2}(D_2)$$

$$\forall d_1 \in N_{P_1}(D_1),\ f(d_1) \in N_{P_2}(D_2)$$



If P_1 *(on* D_1*)* **reduces to** P_2 *(on* D_2*),*

P_2 *is at least as* **hard** *as* P_1*.* $(P_1 \le P_2)$

P_2 *is* **not easier** *than* P_1*.*

Theorem 9.7 *If there is a* ***reduction*** *from* P_1 *to* $P_2(P_1 \leq P_2)$*, then:*

a) If P_1 *is* ***undecidable****, then so is* P_2*. (If* P_2 *is* ***decidable****, so is* P_1*.)*

b) If P_1 *is* ***non-RE****, then so is* P_2*. (If* P_2 *is* ***RE****, so is* P_1*.)*

proof *a) Suppose* P_2 *is* ***decidable****. Then*

$^{\exists}D_2$.϶. $^{\forall}x \in P_1$, $f(x) \in P_2$, D_2 *halts "yes",* D_1 *halts "yes".*

$^{\forall}x \notin P_1$, $f(x) \notin P_2$, D_2 *halts "no",* D_1 *halts "no".*

$\therefore$ P_1 *is* ***decidable****.(Figure 9.7)*

b) Assume P_2 *is* ***RE****. Then*

$^{\exists}M_2$.϶. $^{\forall}x \in P_1$, $f(x) \in P_2$, M_2 *halts "yes",* M_1 *halts "yes".*

$^{\forall}x \notin P_1$, $f(x) \notin P_2$, M_2 *halts "no" or loops forever,*

M_1 *halts "no" or loops forever.*

$\therefore$ P_1 *is* ***RE****.*

If L_u *reduces to P, P is* ***not*** *recursive. (RE or* ***not*** *RE)*

9.3.2 Turing Machine that Accepts the Empty Language

$L_e = \{M \mid L(M) = \emptyset\}$

$L_{ne} = \{M \mid L(M) \neq \emptyset\}$

Theorem 9.8 *L_{ne} is* ***recursively enumerable****.*

proof *Consider a NTM M_{ne}*

1. Guess a TM M and an input string w

2. A TM U test if M accepts w.(U simulates M for w)

3. If M accepts w then M_{ne} accepts M.

$\therefore L(M_{ne}) = L_{ne}$.

But it is not so easy to find a TM M_e .э. $L(M_e) = L_e$.

Actually there is no TM M_e .э. $L(M_e) = L_e$.

We shall prove that in Thm. 9.9 and 9.10.

Theorem 9.9 L_{ne} is ***not recursive***.

proof *Reduce* L_u *to* L_{ne}.$(L_u \le L_{ne})$

Consider a TM M_{ne}.

1. U simulates M for w.(guess (M, w) pair)

2. If U accepts w $(w \in L(M))$, *then code for* $M \in L_{ne}$.

3. If U does not accepts w $(w \notin L(M))$, *then code for* $M \notin L_{ne}$.

Transform (M, w) pair to M_{ne} .϶. $L(M_{ne}) = \{M \mid w \in L(M)\}$

If $w \in L(M)$, $M \in L_{ne}$.

If $w \notin L(M)$, $M \notin L_{ne}$.

$\therefore$ *We Reduced* L_u *to* L_{ne}.

$\therefore$ L_{ne} is ***not recursive***.

Theorem 9.10 L_e is ***not RE***.

proof *Since* $L_e = \overline{L}_{ne}$ *and* L_{ne} *is RE but not recursive. Case 2 of p7.*

Rice's Theorem and Properties of RE Languages

*Consider a **property** P of a set of languages.*

*property of being **context free** is set of all CFL's.*

*property of being **empty** is $\{\emptyset\}$.*

$P: 2^{\Sigma^*} \rightarrow \{true, false\}$

$P = \{L \subseteq \Sigma^* \mid P(L)\} = \{L \in 2^{\Sigma^*} \mid P(L)\}$.

*A **property** is **trivial**, if it is either **empty** or is **all** of the languages.*

***nontrivial** otherwise.*

$P = \emptyset$ *or* 2^{Σ^*} *are trivial.*

But $P = \{\emptyset\}$ *is nontrivial.*

P *may be represented as set of TM's,* L_P.

$P \leftrightarrow L_P = \{M \in TM \mid L(M) = L\}$

Theorem 9.11 (Rice's Theorem) *Every* ***nontrivial*** *property of the* ***recursively enumerable*** *languages are* ***undecidable****.*

Let P be a ***nontrivial*** *property of RE languages.*

1. Assume $P(\emptyset)$ = false (or $\emptyset \notin P$).

   $\therefore$ $^{\exists}L$.э. $P(L)$ = true or $^{\exists}M_L$.э. $M_L \in L_P$.

   We shall **reduce** L_u to L_P.($L_u \leq L_P$)

   1. U simulate w for M
   2. If U accepts w, L = L(M) $\leftrightarrow$ $^{\exists}M_L \in L_P$.
      - 2.1 M_L simulate for x
      - 2.2 If M_L accepts x, M accepts M_L.
   3. If U does not accepts w, do nothing.

2. Assume $P(\emptyset)$ = true (or $\emptyset \in P$).

   Consider complement property $\overline{P}$, $L_{\overline{P}}$. = $\overline{L_P}$.

   $\therefore$ $\overline{P}$ is undecidable(above). $\therefore$ P is undecidable.(Thm 9.3)

9.3.4 Problems about Turing-Machine Specifications

1. Whether the language accepted by a TM is empty(L_e, L_{ne}).

2. Whether the language accepted by a TM is finite.

3. Whether the language accepted by a TM is regular.

4. Whether the language accepted by a TM is context-free.

Undecidable!

Problem is a language

language	problem	function
Not R.E.	Not computable	
recursively enumerable	computable	partial function
recursive	decidable(alway halt)	(total) function

Three class of languages(problems)

1. Not recursively enumerable
   Not computable
2. Recursively enumerable but not recursive
   Computable but **not** decidable (undecidable)
   partial μ-recursive function
3. recursive
   decidable
   total μ-recursive function

Following systems are equivalent

- *Turing machine*
  - *Turing(1930)*
- μ*-recursive function(partial recursive) function*
  - *Gödel(1934; lecture at Princeton), Herbrand, Kleene(1936)*
- λ*-calculus*
  - *Church(1933-41), Kleene(1935), Rosser(1935)*
- *combinatory logic*
  - *Schönfinkel(1924), Curry(1929)*
- *Post correspondence system*
  - *Post(1936)*
- *type 0 grammar*
  - *Chomsky(1959)*
- *while programs*
  - *Meyer, Ritchie(1967)*

Equivalence of μ-recursive function and λ-calculus

Church(1936) attributes to Kleene

Equivalence of TM and λ-calculus

Turing(1937)

TM for computable

Turing's thesis

Turing-Church's thesis