Chap. 6 Pushdown Automata

6.1 Definition of Pushdown Automata

Example 6.1 $L = \{wcw^R \mid w \in (0+1)^*\}$

$P \rightarrow c \mid 0P0 \mid 1P1$

1. Start at state q_0, push input symbol onto stack, and stay in q_0.
2. If input symbol is c in state q_0, go to state q_1.
3. If input symbols is same as top of the stack, pop it and stay in q_1.
4. If no more input symbol and empty stack in state q_1, accept.

A pushdown automaton(PDA) $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is

1. *Q is a **finite** set of **states**,*
2. *Σ is a **finite** set of **input** symbols,*
3. *Γ is a **finite stack** alphabet,*
4. *δ is a **transition function**.*

   $$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times \Gamma^*}.$$

5. *$q_0 \in Q$ is an **initial state**,*
6. *$Z_0 \in \Gamma^*$ is an **initial stack** content,*
7. *$F \subseteq Q$ is a set of **final states**.*

Example $P = (\{q_0, q_1, q_2\}, \{0, 1, c, \$\}, \{Z, O, Z_0\}, \delta, q_0, Z_0, \{q_2\})$

$\delta(q_0, 0, Z_0) = \{(q_0, ZZ_0)\}$

$\delta(q_0, 0, Z) = \{(q_0, ZZ)\}$ $\delta(q_0, 0, O) = \{(q_0, ZO)\}$

If see 0, then push Z in q_0.

$\delta(q_0, 1, Z_0) = \{(q_0, OZ_0)\}$

$\delta(q_0, 1, Z) = \{(q_0, OZ)\}$ $\delta(q_0, 1, O) = \{(q_0, OO)\}$

If see 1, then push O in q_0.

$\delta(q_0, c, Z_0) = \{(q_1, Z_0)\}$

$\delta(q_0, c, Z) = \{(q_1, Z)\}$ $\delta(q_0, c, O) = \{(q_1, O)\}$

If see c, go to q_1.

$\delta(q_1, 0, Z) = \{(q_1, \varepsilon)\}$ *If see 0, then pop Z in q_1.*

$\delta(q_1, 1, O) = \{(q_1, \varepsilon)\}$ *If see 1, then pop O in q_1.*

$\delta(q_1, \$, Z_0) = \{(q_2, \varepsilon)\}$ *If see \$, go to the final state q_2.*

\$: end of string marker

Instantaneous description *of PDA*

(current state, remained input string, stack contents)

$(q, x, \gamma) \in Q \times \Sigma^* \times \Gamma^*$.

$\vdash_P \subseteq (Q \times \Sigma^* \times \Gamma^*) \times (Q \times \Sigma^* \times \Gamma^*)$

$(q, ax, X\beta) \vdash_P (p, x, \gamma\beta)$, *if* $(p, \gamma) \in \delta(q, a, X)$

$(q, x, X\beta) \vdash_P (p, x, \gamma\beta)$, *if* $(p, \gamma) \in \delta(q, \varepsilon, X)$

We may use $\vdash$ *instead of* $\vdash_P$ *if P is understood.*

$\vdash$ *is a binary relation on* $(Q \times \Sigma^* \times \Gamma^*)$.

$\vdash^*$, *reflexive transitive closure of* $\vdash$.

Recursive definition of $\vdash^*$.

$\forall I \in Q \times \Sigma^* \times \Gamma^*, I \vdash^* I$.

If $I \vdash J$ *and* $J \vdash^* K$, $I \vdash^* K$.

Theorem 6.5 *If* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *is a PDA and*

$(q, xy, \alpha) \vdash^* (p, y, \beta)$ *for* $q,p \in Q$, $x,y \in \Sigma^*$ *and* $\alpha,\beta \in \Gamma^*$. *Then* $(q, xyw, \alpha\gamma) \vdash^* (p, yw, \beta\gamma)$ *for any* $w \in \Sigma^*$ *and* $\gamma \in \Gamma^*$.

Theorem 6.6 *If* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *is a PDA and*

$(q, xyw, \alpha) \vdash^* (p, yw, \beta)$ *for* $q,p \in Q$, $x,y,w \in \Sigma^*$ *and* $\alpha,\beta \in \Gamma^*$. *Then* $(q, xy, \alpha) \vdash^* (p, y, \beta)$.

6.2 The language of a PDA

$L(P) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (f, \varepsilon, \alpha), f \in F\}$

language accepted by final state

$N(P) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (f, \varepsilon, \varepsilon)\}$

language accepted by null stack

6.2.3 From Empty Stack to Final State

Theorem 6.9 *If $L = N(P_N)$ for some PDA P_N.*

Then there is a PDA P_F such that $L = L(P_F)$.

Let $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_N, \emptyset)$.

$P_F = (Q \cup \{q_0', q_F\}, \Sigma, \Gamma \cup \{Z_F\}, \delta_F, q_0', Z_F, \{q_F\})$

where $q_0', q_F \notin Q$, $Z_F \notin \Gamma$.

δ_F: 1. $\delta_F(q_0', \varepsilon, Z_F) = \{(q_0, Z_N Z_F)\}$. *push old stack* ***bottom*** Z_N.

2. $\delta_F \supseteq \delta_N$, ***simulate*** P_N *with* δ_N.

3. $\forall q \in Q$, $\delta_F(q, \varepsilon, Z_F) = \{(q_F, Z_F)\}$.

If stack is ***empty***(Z_F), *go tothe final* ***final*** *state* q_F.

6.2.4 From Final State to Empty Stack

Theorem 6.11 *If $L = L(P_F)$ for some PDA P_F.*

Then there is a PDA P_N such that $L = N(P_N)$.

Let $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_F, F)$.

$P_N = (Q \cup \{q_0', q_E\}, \Sigma, \Gamma \cup \{Z_N\}, \delta_N, q_0', Z_N, \emptyset)$

where $q_0', q_E \notin Q$, $Z_F \notin \Gamma$.

δ_N: 1. $\delta_N(q_0', \varepsilon, Z_N) = \{(q_0, Z_F Z_N)\}$ — *push old stack* ***bottom*** Z_F.

2. $\delta_N \supseteq \delta_F$, — ***simulate*** P_F *with* δ_F.

3. $\forall f \in F, \forall Z \in \Gamma \cup \{Z_N\}$, $\delta_N(f, \varepsilon, Z) \supseteq \{(q_E, \varepsilon)\}$.

If ***final*** *state, pop stack symbol and go to the* ***empty*** *state* q_E.

4. $\forall Z \in \Gamma \cup \{Z_N\}$, $\delta_N(q_E, \varepsilon, Z) = \{(q_E, \varepsilon)\}$. — ***Empty*** *stack in* q_E.

6.3.1 From Context-free Grammar to Pushdown Automata

Theorem 6.13 *If $G = (N, \Sigma, P, S)$ is a cfg. Then $^{\exists}$PDA P .∍. $L(G) = N(P)$.*

Construct $P = (\{q\}, \Sigma, N \cup \Sigma, \delta, q, S, \emptyset)$

$^{\forall}A \in N, \delta(q, \varepsilon, A) = \{(q, \alpha) \mid A \rightarrow \alpha \in P\}$ **guess** A as $\alpha (A \rightarrow \alpha \in P)$.

$^{\forall}a \in \Sigma, \delta(q, a, a) = \{(q, \varepsilon)\}$ **verify** $a \in \Sigma$.

guess and verify *parser*

Proof $A \Rightarrow_{lm}^{*} x\alpha$ *if and only if* $(q, x, A) \vdash^{*} (q, \varepsilon, \alpha)$, $x \in \Sigma^{*}$, $\alpha \in (N \cup \Sigma)^{*}$.

(If) If $(q, x, A) \vdash^{i} (q, \varepsilon, \alpha)$, then $A \Rightarrow_{lm}^{*} x\alpha$.

basis $i = 0$, $x = \varepsilon$, $\therefore (q, \varepsilon, A) \vdash^{0} (q, \varepsilon, A)$. $\therefore A \Rightarrow_{lm}^{*} A$.

induction *Let $i \geq 1$, and consider the next-to-last step.*

i) $(q, x, A) \vdash^{i-1} (q, \varepsilon, B\gamma) \vdash_{Guess} (q, \varepsilon, \beta\gamma) = (q, \varepsilon, \alpha)$

$\therefore A \Rightarrow_{lm}^{*} xB\gamma$ by IH and $B \rightarrow \beta \in P$ by construction of δ_{Guess}.

$\therefore A \Rightarrow_{lm}^{*} xB\gamma \Rightarrow_{lm} x\beta\gamma = x\alpha$.

ii) $(q, x, A) = (q, ya, A) \vdash^{i-1} (q, a, a\alpha) \vdash_{Verify} (q, \varepsilon, \alpha)$

$(q, y, A) \vdash^{i-1} (q, \varepsilon, a\alpha)$ (Thm 6.6; $(q, \underline{\varepsilon}, a\alpha)$)

$\therefore A \Rightarrow_{lm}^{*} ya\alpha = x\alpha$ by IH

(Only if) If $A \Rightarrow_{lm}^{i} x\alpha$, then $(q, x, A) \vdash^{*} (q, \varepsilon, \alpha)$.

basis $i = 0$, $A \Rightarrow_{lm}^{0} A$. $(q, \varepsilon, A) \vdash^{0} (q, \varepsilon, A)$.

induction Let $i \geq 1$, and consider the next-to-last step.

$A \Rightarrow_{lm}^{i-1} yB\gamma \Rightarrow_{lm} y\beta\gamma = y\underline{y'\gamma'}\gamma = x\alpha$ where $\beta = y'\gamma'$, $y' \in \Sigma^{*}$, $\gamma' \in (N \cup \Sigma)^{*}$.

$(q, y, A) \vdash^{*} (q, \varepsilon, B\gamma)$ by IH, $(q, yy', A) \vdash^{*} (q, y', B\gamma)$ (by T.6.5)

$\vdash_{G} (q, y', \beta\gamma) = (q, y', y'\gamma'\gamma) \vdash_{V}^{|y'|} (q, \varepsilon, \gamma'\gamma) = (q, \varepsilon, \alpha)$

$\therefore A \Rightarrow_{lm}^{*} x\alpha$ **if and only if** $(q, x, A) \vdash^{*} (q, \varepsilon, \alpha)$.

If $A = S$, $\alpha = \varepsilon$, $S \Rightarrow_{lm}^{*} x$ **if and only if** $(q, x, S) \vdash^{*} (q, \varepsilon, \varepsilon)$

$\therefore L(G) = N(P)$.

6.3.2 From Pushdown Automata to Context-free Grammar

Theorem 6.14 *If a PDA* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$.

Then there is a CFG G such that $L(G) = N(P)$.

proof $G = (Q \times \Gamma \times Q \cup \{S\}, \Sigma, P, S)$

$$P = \{S \rightarrow [q_0, Z_0, q] \mid {}^{\forall}q \in Q\}$$

$$\cup \{[q, A, p_m] \rightarrow a\, [p, Y_1, p_1][p_1, Y_2, p_2] \ldots [p_{m-1}, Y_m, p_m] \mid$$

$$(p, Y_1 \ldots Y_m) \in \delta(q, a, A), a \in \Sigma \cup \{\varepsilon\}, {}^{\forall}p_1, \ldots, p_m \in Q\}$$

$$(\text{if } m = 0, [q, A, p] \rightarrow a \in P, a \in \Sigma \cup \{\varepsilon\})$$

$[q, A, p] \Rightarrow_{lm}{}^{*}\ x \in \Sigma^*$, *if and only if* $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon)$.

Nonterminal $[q, A, p]$ *derives terminal string* x *if and only if*

x *causes PDA P to pop* A *from stack*

starting in the state q *and ending in the state* p.

1) If $(q, x, A) \vdash^{i} (p, \varepsilon, \varepsilon)$, then $[q, A, p] \Rightarrow_{lm}^{*} x$.

basis $i = 1$, $(q, x, A) \vdash (p, \varepsilon, \varepsilon)$, $(p, \varepsilon) \in \delta(q, x, A)$, $x \in \Sigma \cup \{\varepsilon\}$.

$\therefore [q, A, p] \rightarrow x \in P$ where $x \in \Sigma \cup \{\varepsilon\}$.

induction $(q, x, A) = (q, ay, A) \vdash (p_1, y, Y_1 \ldots Y_m) \vdash^{i-1} (p, \varepsilon, \varepsilon)$.

$\therefore \exists p_2, \ldots, p_m, p \in Q$ and assume $y = y_1 \ldots y_m \in \Sigma^*$.

$(p_1, y_1 \ldots y_m, Y_1 \ldots Y_m) \vdash^{*} (p_2, y_2 \ldots y_m, Y_2 \ldots Y_m) \vdash^{*} \ldots (p_m, y_m, y_m) \vdash (p, \varepsilon, \varepsilon)$.

$1 \le \forall i \le m$, $(p_i, y_i, Y_i) \vdash^{*} (p_{i+1}, \varepsilon, \varepsilon)$. (Thm 6.5 and y_i depends only on Y_i)

$\therefore [p_i, Y_i, p_{i+1}] \Rightarrow_{lm}^{*} y_i$ by IH.

$\therefore [p_1, Y_1, p_2][p_2, Y_2, p_3] \ldots [p_m, Y_m, p] \Rightarrow_{lm}^{*} y_1 y_2 \ldots y_m = y$

Since $(q, ay, A) \vdash (p_1, y, Y_1 \ldots Y_m)$, $(p_1, Y_1 \ldots Y_m) \in \delta(q, a, A)$.

$\therefore \exists [q, A, p] \rightarrow a [p_1, Y_1, p_2][p_2, Y_2, p_3] \ldots [p_m, Y_m, p] \in P$. $(\forall p_i \in Q)$

$\therefore [q, A, p] \Rightarrow_{lm} a [p_1, Y_1, p_2][p_2, Y_2, p_3] \ldots [p_m, Y_m, p] \Rightarrow_{lm}^{*} ay = x$.

2) If $[q, A, p] \Rightarrow_{lm}^{i} x$, then $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon)$.

basis $i = 1$, $[q, A, p] \rightarrow x \in P$, $(p, \varepsilon) \in \delta(q, x, A)$ where $x \in \Sigma \cup \{\varepsilon\}$.

induction $[q, A, p] \Rightarrow_{lm} a\, [p_1, Y_1, p_2] \ldots [p_m, Y_m, p] \Rightarrow_{lm}^{i-1} x \in \Sigma^{*}$.

$x = ay_1 \ldots y_m$ where $1 \le {}^{\forall} i \le m$, $[p_i, Y_i, p_{i+1}] \Rightarrow_{lm}^{i} y_i$ where $p_{m+1} = p$.

$\therefore (p_i, y_i, Y_i) \vdash^{*} (p_{i+1}, \varepsilon, \varepsilon)$ by IH.

Since $[q, A, p] \rightarrow a\, [p_1, Y_1, p_2] \ldots [p_m, Y_m, p] \in P$,

$a\, [p_1, Y_1, p_2] \ldots [p_m, Y_m, p] \in \delta(q, a, A)$.

$\therefore (q, ay_1 \ldots y_m, A) \vdash (p_1, y_1 \ldots y_m, Y_1 \ldots Y_m) \vdash^{*} \ \ldots \vdash^{*} (p, \varepsilon, \varepsilon)$.

$[q, A, p] \Rightarrow_{lm}^{*} x$, ${}^{\forall} p \in Q$ if and only if $(q, x, A) \vdash^{*} (p, \varepsilon, \varepsilon)$.

$[q_0, Z_0, p] \Rightarrow_{lm}^{*} x$, ${}^{\forall} p \in Q$ if and only if $(q_0, x, Z_0) \vdash^{*} (p, \varepsilon, \varepsilon)$.

$S \Rightarrow_{lm}^{*} x$, ${}^{\forall} p \in Q$ if and only if $(q_0, x, Z_0) \vdash^{*} (p, \varepsilon, \varepsilon)$.

Q.E.D.