Chap. 3 Regular Expressions and Languages

3.1 Regular Expressions *over* ***some alphabet*** Σ.

Basis:

1. *The* **constant** ε *and* $\emptyset$ *are* **regular expressions**, *denoting the languages* $\{\varepsilon\}$ *and* $\emptyset$, *respectively*, *i.e.*, $L(\varepsilon) = \{\varepsilon\}$, $L(\emptyset) = \emptyset$.
2. *If* $a \in \Sigma$, *then* **a** *is a* **regular expression**, *denoting the languages* $\{a\}$, *i.e.*, $L(\mathbf{a}) = \{a\}$.

Induction:

1. *If E and F are* **regular expressions**, *then* $E + F$ *is a* **regular expression**, *denoting* **union** *of* $L(E)$ *and* $L(F)$, *i.e.*, $L(E + F) = L(E) \cup L(F)$.
2. *If E and F are* **regular expressions**, *then* EF *is a* **regular expression**, *denoting* **concatenation** *of* $L(E)$ *and* $L(F)$, *i.e.*, $L(EF) = L(E)L(F)$.

*3. If E is a **regular expression**, then E^* is a **regular expression**,*
*denoting **closure** of L(E), i.e., $L(E^*) = (L(E))^*$.*
*4. If E is a **regular expression**, then (E) is a **regular expression**,*
denoting L(E), i.e., L((E)) = L(E).

Example 3.2 in p.89

***Precedence** of operators*

1. *parenthesis*
2. *closure*
3. *concatenation*
4. *union*

Example 3.3 in p.91

***Equivalence** of regular expresssions*

Let R, S be regular expressions. We say R = S, if L(R) = L(S).

3.2 Finite Automata and Regular Expressions

3.2.1 From DFA's to Regular Expressions

Theorem 3.4 *If L = L(D) for some DFA D, then there is a* ***regular expressions*** *R such that L = L(R).*

Proof *Let us suppose D's states are {1, 2, …, n} for some n.*

Let us R_{ij}^{k} *be a* ***regular expression*** *such that*

$$L(R_{ij}^{k}) = \{w \in \Sigma^{*} \mid \delta^{|w|}(i, w) = j, 1 \leq {}^{\forall}m \leq |w|-1, \delta^{m}(i, m{:}w) \leq k\}$$

The RE R_{ij}^{k} *denotes the set of strings that take fa D from state i to state j without going through any state numbered higher than k.*

When k=n, no ***restriction****.*

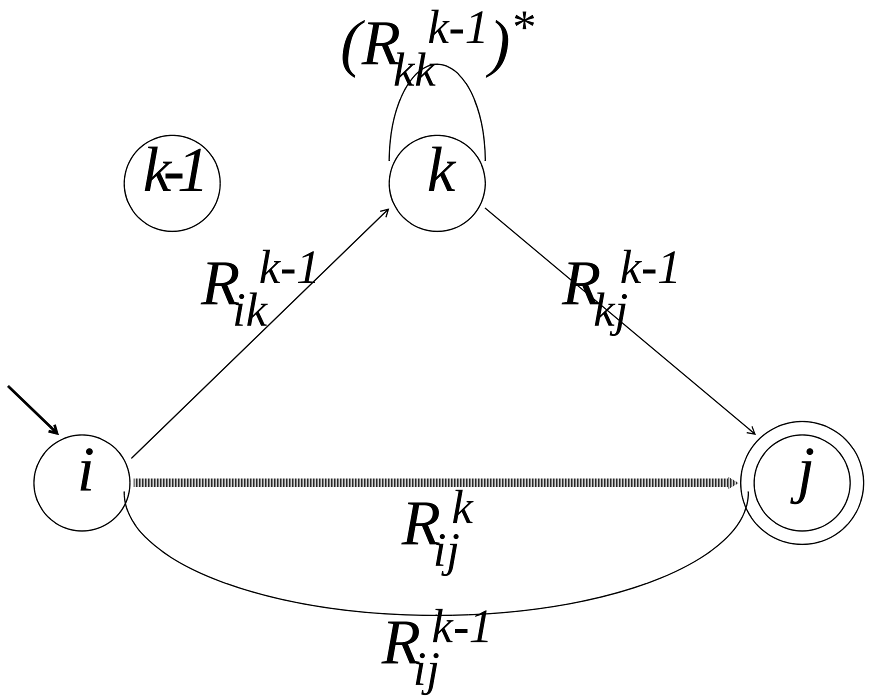
We can construct R_{ij}^{k} $1 \leq^{\forall} i \leq n$, $1 \leq^{\forall} j \leq n$, *and* $0 \leq^{\forall} k \leq n$ *by* ***induction*** *on k.*

basis*:* $k = 0$, $1 \leq {}^{\forall}i \leq n$, $1 \leq {}^{\forall}j \leq n$.

$R_{ij}^{0} = \mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_p$ *if* $i \neq j$, *and* $1 \leq {}^{\forall}q \leq p$, $\delta(i, a_q) = j$,
$= \mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_p + \varepsilon$ *if* $i = j$ *and* $1 \leq {}^{\forall}q \leq p$, $\delta(i, a_q) = j$,
$= \emptyset$ *otherwise.*

induction: $1 \leq {}^{\forall}i \leq n$, $1 \leq {}^{\forall}j \leq n$, ***all*** R_{ij}^{k-1}*'s are known.*

$$R_{ij}^{k} = R_{ik}^{k-1} (R_{kk}^{k-1})^{*} R_{kj}^{k-1} + R_{ij}^{k-1}.$$



*Let s be the **start** state, and $F = \{f_1, \ldots, f_g\}$ be final states.*

$$R = R_{sf_1}^{\,n} + R_{sf_2}^{\,n} + \ldots + R_{sf_g}^{\,n} \text{ such that } L(R) = L.$$

***Example 3.5**(pp 95) Figure 3.4*

$R_{11}^{\,0} = \varepsilon + \mathbf{1}$ $\quad R_{12}^{\,0} = \mathbf{0}$ $\quad R_{21}^{\,0} = \emptyset$ $\quad R_{22}^{\,0} = \varepsilon + \mathbf{0} + \mathbf{1}$

$$R_{ij}^{\,k} = R_{ik}^{\,k-1} (R_{kk}^{\,k-1})^* R_{kj}^{\,k-1} + R_{ij}^{\,k-1}$$

$k=1$ $\quad R_{ij}^{\,1} = R_{i1}^{\,0} (R_{11}^{\,0})^* R_{1j}^{\,0} + R_{ij}^{\,0}.$

$R_{11}^{\,1} = R_{11}^{\,0} (R_{11}^{\,0})^* R_{11}^{\,0} + R_{11}^{\,0} = (\varepsilon + \mathbf{1})(\varepsilon + \mathbf{1})^*(\varepsilon + \mathbf{1}) + \varepsilon + \mathbf{1}$

$\quad = (\varepsilon + \mathbf{1})\mathbf{1}^*(\varepsilon + \mathbf{1}) + \varepsilon + \mathbf{1} = \mathbf{1}^* + \varepsilon + \mathbf{1} = \mathbf{1}^*.$

$R_{12}^{\,1} = R_{11}^{\,0} (R_{11}^{\,0})^* R_{12}^{\,0} + R_{12}^{\,0} = (\varepsilon + \mathbf{1})(\varepsilon + \mathbf{1})^*\mathbf{0} + \mathbf{0} = \mathbf{1}^*\mathbf{0} + \mathbf{0} = \mathbf{1}^*\mathbf{0}.$

$R_{21}^{\,1} = R_{21}^{\,0} (R_{11}^{\,0})^* R_{11}^{\,0} + R_{21}^{\,0} = \emptyset(\varepsilon + \mathbf{1})^*(\varepsilon + \mathbf{1}) + \emptyset = \emptyset.$

$R_{22}^{\,1} = R_{21}^{\,0} (R_{11}^{\,0})^* R_{12}^{\,0} + R_{22}^{\,0} = \emptyset(\varepsilon + \mathbf{1})^*\mathbf{0} + \varepsilon + \mathbf{0} + \mathbf{1} = \varepsilon + \mathbf{0} + \mathbf{1}.$

$k=2 \qquad R_{ij}^{2} = R_{i2}^{1} (R_{22}^{1})^{*} R_{2j}^{1} + R_{ij}^{1}.$

$R_{11}^{2} = R_{12}^{1} (R_{22}^{1})^{*} R_{21}^{1} + R_{11}^{1} = \mathbf{1}^{*}\mathbf{0}(\varepsilon + \mathbf{0} + \mathbf{1})^{*}\varnothing + \mathbf{1}^{*} = \mathbf{1}^{*}.$

$R_{12}^{2} = R_{12}^{1} (R_{22}^{1})^{*} R_{22}^{1} + R_{12}^{1}$

$= \mathbf{1}^{*}\mathbf{0}(\varepsilon + \mathbf{0} + \mathbf{1})^{*}(\varepsilon + \mathbf{0} + \mathbf{1}) + \mathbf{1}^{*}\mathbf{0} = \mathbf{1}^{*}\mathbf{0}(\mathbf{0} + \mathbf{1})^{*}.$

$R_{21}^{2} = R_{22}^{1} (R_{22}^{1})^{*} R_{21}^{1} + R_{21}^{1} = (\varepsilon + \mathbf{0} + \mathbf{1})(\varepsilon + \mathbf{0} + \mathbf{1})^{*}\varnothing + \varnothing = \varnothing.$

$R_{22}^{2} = R_{22}^{1} (R_{22}^{1})^{*} R_{22}^{1} + R_{22}^{1}$

$= (\varepsilon + \mathbf{0} + \mathbf{1})(\varepsilon + \mathbf{0} + \mathbf{1})^{*}(\varepsilon + \mathbf{0} + \mathbf{1}) + \varepsilon + \mathbf{0} + \mathbf{1} = (\mathbf{0} + \mathbf{1})^{*}.$

$R = R_{12}^{2} = \mathbf{1}^{*}\mathbf{0}(\mathbf{0} + \mathbf{1})^{*}.$

3.2.2 Converting DFA's to Regular Expressions by Eliminating States

Previous construction

n^3 equations

$O(4^n)$ symbols in the regular expression

Eliminating states

If we eliminate state s, all paths that went though s no longer exists.

*labels: symbols → possibly **infinite** strings*

*→ **regular expression(closure)***

***simultaneous** equations(연립방정식)*

n-equations and n-variables

eliminating variables

Simultaneous equations *for each state with regular expressions*

Let $A = (Q, \Sigma, \delta, q_0, F)$ *be a XFA and* $^{\forall}q \in Q$, R_q *is a* ***regular equation***

$L(R_q) = \{x \in \Sigma^* | \delta^*(q, w) \cap F \neq \varnothing\}$. *Then*

(q is assumed to be an intial state)

$1 \leq {}^{\forall}i \leq n,\ R_{q_i} = r_{i1}R_{q_1} + r_{i2}R_{q_2} + \ldots + r_{in}R_{q_n} + s_i.$

where $1 \leq {}^{\forall}j \leq n,\ {}^{\exists}p \geq 0,\ r_{ij} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_p,$

for $1 \leq {}^{\forall}k \leq p,\ q_j \in \delta(q_i, x_k)$ *where* $x_k \in \Sigma^*$ *and*

$s_i = \varepsilon$, *if* $q_i \in F$, *and* $s_i = \varnothing$, *if* $q_i \notin F$.

Note r_{ij}*'s and* s_i*'s are* ***constant regular expressions***

and R_{q_i}*'s are unknown* ***variables***.

Then n ***states (variables)*** *and n* ***equations***.

We can solve the ***linear*** *simultaneous equation*

1. eliminate ***variables(states)*** *by* ***substitution(*** *대입)*

2. eliminate ***recursive variable*** *by* ***closure****.*

Let $q = \alpha q + \beta$ *where* α *and* β *are* ***regular equation*** *with* ***variables****.*

$$q = \alpha^{*} \beta.$$

strange solution

$$q \neq 1/(1-\alpha)\,\beta$$
$$= (1 + \alpha + \alpha^{2} + \ldots)\,\beta$$
$$\neq \alpha^{*} \beta.$$

Example 3.6 *Figure 3.12(pp 101)*

$$A = (\mathbf{0} + \mathbf{1})A + \mathbf{1}B$$
$$B = (\mathbf{0} + \mathbf{1})C$$
$$C = (\mathbf{0} + \mathbf{1})D + \varepsilon$$
$$D = \varepsilon$$

$$C = (\mathbf{0} + \mathbf{1})\varepsilon + \varepsilon = \mathbf{0} + \mathbf{1} + \varepsilon$$
$$B = (\mathbf{0} + \mathbf{1})(\mathbf{0} + \mathbf{1} + \varepsilon) = (\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1})$$
$$A = (\mathbf{0} + \mathbf{1})A + \mathbf{1}((\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1}))$$
$$= (\mathbf{0} + \mathbf{1})^*\mathbf{1}(\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1})^*\mathbf{1}(\mathbf{0} + \mathbf{1}).$$

Example 3.5 revisited*(pp 95)*

$$A = \mathbf{1}A + \mathbf{0}B \qquad A = \mathbf{1}A + \mathbf{0}(\mathbf{0} + \mathbf{1})^* = \mathbf{1}^*\mathbf{0}(\mathbf{0} + \mathbf{1})^*.$$
$$B = (\mathbf{0} + \mathbf{1})B + \varepsilon \qquad B = (\mathbf{0} + \mathbf{1})^*\varepsilon = (\mathbf{0} + \mathbf{1})^*.$$

DFA in Figure 2.14 (pp 63) revisited

$A = \mathbf{1}A + \mathbf{0}B \qquad A = \mathbf{1}A + \mathbf{0}(\mathbf{0} + \mathbf{10})^*(\mathbf{11}A + \mathbf{1})$

$= (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})A + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1}$

$= (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})^*\mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1}$

$B = \mathbf{0}B + \mathbf{1}C \qquad B = \mathbf{0}B + \mathbf{11}A + \mathbf{10}B + \mathbf{1} = (\mathbf{0} + \mathbf{10})B + \mathbf{11}A + \mathbf{1}$

$= (\mathbf{0} + \mathbf{10})^*(\mathbf{11}A + \mathbf{1})$

$C = \mathbf{1}A + \mathbf{0}B + \varepsilon$

NFA Figure 2.9(pp 56)

$C = \varepsilon \qquad B = \mathbf{1}C = \mathbf{1}\varepsilon = \mathbf{1}$

$A = (\mathbf{0} + \mathbf{1})A + \mathbf{0}B = (\mathbf{0} + \mathbf{1})A + \mathbf{01} = (\mathbf{0} + \mathbf{1})^*\mathbf{01}$

$\therefore (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})^*\mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1} = (\mathbf{0} + \mathbf{1})^*\mathbf{01}$

DFA in Figure 2.14 (pp 63) revisited

$C = \underline{\mathbf{0}B + \mathbf{1}A} + \varepsilon = A + \varepsilon \qquad B = \mathbf{0}B + \mathbf{1}C = \underline{\mathbf{0}B + \mathbf{1}A} + \mathbf{1}\varepsilon = A + \mathbf{1}$

$A = \underline{\mathbf{0}B + \mathbf{1}A} = \mathbf{1}A + \mathbf{0}A + \mathbf{01} = (\mathbf{0} + \mathbf{1})A + \mathbf{01} = (\mathbf{0} + \mathbf{1})^*\mathbf{01}$

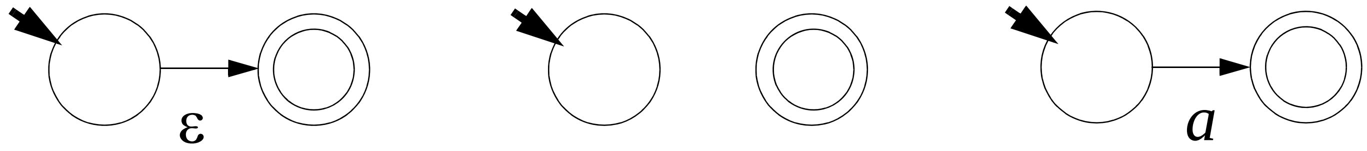
3.2.3 Converting Regular Expressions to Automata

Theorem 3.7 *Every language* ***defined*** *by a* ***regular expression*** *is also* ***defined*** *by a* ***finite automaton****.*

Proof *Suppose L = L(R) for some regular expression R.*

We show that L = L(E) for some ε-NFA E.

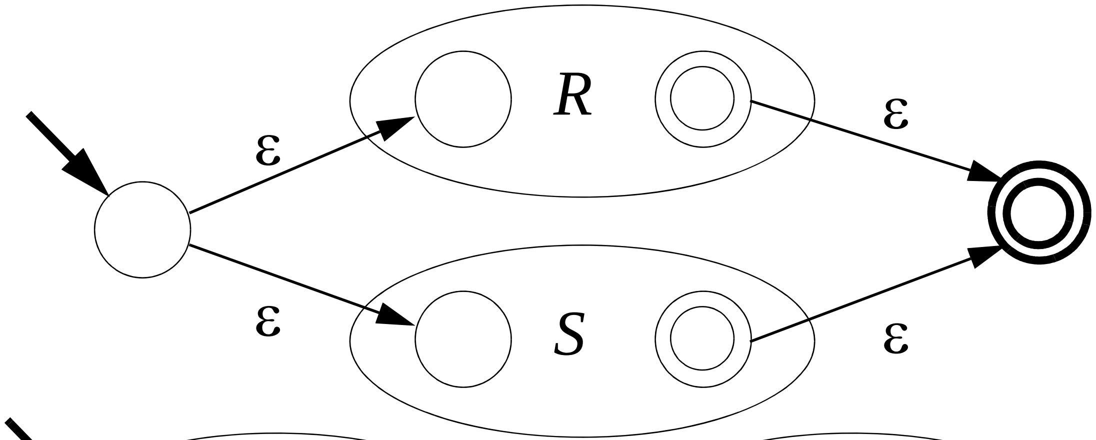
basis*:*



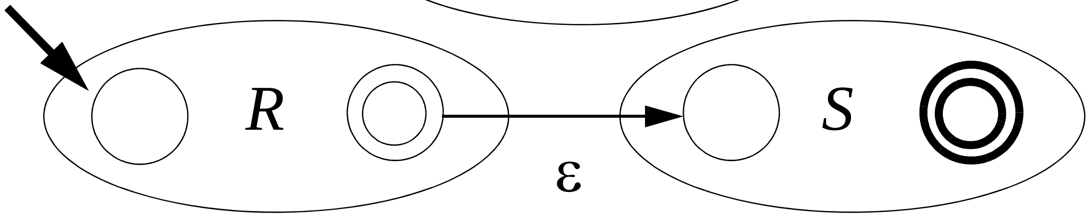
1. ε *2.* ∅ *3.* **a**

induction:

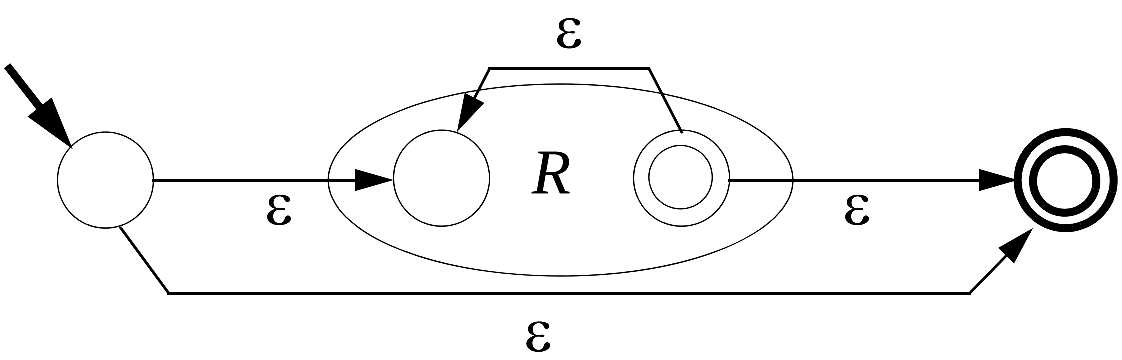
1. R + S



2. R S



3. R^*



3.4 Algebraic Laws for Regular Expressions

Let R, S, T be regular expressions over Σ.

$R + S = S + R$	***union*** *is* ***commutative***
$(R + S) + T = R + (S + T)$	***union*** *is* ***associative***
$RS \neq SR$	***concatenation*** *is* ***non-commutative***
$(RS)T = R(ST)$	***concatenation*** *is* ***associative***
$\varnothing + R = R + \varnothing = R$	*∅ is the* ***identity*** *for* ***union***
$\varepsilon R = R\varepsilon = R$	*ε is the* ***identity*** *for* ***concatenation***
$\varnothing R = R\varnothing = \varnothing$	*∅ is the* ***annihilator*** *for* ***concatenation***
$R(S + T) = RS + RT$	***concatenation distributes*** *over* ***union***
$(S + T)R = SR + TR$	

$R + R = R$ *Union is* ***idempotent***

$(R^*)^* = R^*$ *Closure is* ***idempotent***

$\varnothing^* = \varepsilon \quad \varepsilon^* = \varepsilon$ *but* $\varnothing^+ = \varnothing \quad \varepsilon^+ = \varepsilon.$

$R^+ = RR^* = R^*R.$

$R^* = R^+ + \varepsilon$ *but* $R^+ \neq R^* - \{\varepsilon\}$, *if* $\varepsilon \in R$.

$\Sigma^* = \Sigma^+ + \varepsilon$ *and* $\Sigma^+ = \Sigma^* - \{\varepsilon\}$ *(since* $\varepsilon \notin \Sigma$*).*

Theorem *3.A Any* ***finite*** *language is* ***regular****.*

Proof *Any* ***finite*** *language can be denoted by (finite)* ***regular expression****.*

union *and* ***concatenation****.*

no closure ***infinite***

*Following statements are **logically equivalent***

1. *L is **regular**.*
2. *L = L(D) for some DFA D.*
3. *L = L(N) for some NFA N.*
4. *L = L(E) for some ε-NFA E.*
5. *L = L(X) for some XFA X.*
6. *L = L(R) for some RE R.*

*Following statements are **logically equivalent***

1. *L is **regular**.*
2. *L = L(A) for some **finite automaton** A.*
3. *L = L(R) for some **regular expression** R.*
4. *L = L(G) for some **regular grammar** G. (TBD in Chap. 5)*