Repeated product(composition) *of relation* ***revisited***

Let $R \subseteq A \times B$ *and* $S \subseteq B \times C$. *The* ***product(composition)*** *of R and S, denoted* $R \circ S \subseteq A \times C$, *is*

$$R \circ S = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$$

Repeated composition of relations

Let $R \subseteq A \times A$. *Then we have defined,* $R^n \subseteq A \times A$ $(n \geq 0)$.

$R^0 = id_A$, $n = 0$, R^0 *is an* ***identity*** *relation*

$R^n = R^{n-1} \circ R$ $n \geq 1$.

We also have defined, $R^*, R^+ \subseteq A \times A$.

$R^* = \cup_{i \in N_0} R^i$, ***reflexive transitive*** *closure of R*

$R^+ = \cup_{i \in N_+} R^i$, ***transitive*** *closure of R.*

Repeated composition of function

*Let f: A → B and g: B → C. The **product**(**composition**) of R and S, denoted f ∘ g: A → C, is*

$$f \circ g = \{(a, c) \in A \times C \mid (a, b) \in f, (b, c) \in g\}$$

$$\text{or } = \{f \circ g(a) = c \mid f(a) = b, g(b) = c\}$$

$$\text{or } \quad f \circ g(a) = g(f(a))$$

*We **extend** $f^n: A \to A$ for $n \geq 0$.*

$f^0 = id_A,$ $\quad n = 0,$ $\quad$ *f^0 is an identity **function***

$f^n = f^{n-1} \circ R$ $\quad n \geq 1.$

We also define, $f^*, f^+: A \to A$.

$$f^* = \cup_{i \in N_0} f^i,$$

$$f^+ = \cup_{i \in N_+} f^i.$$

Repeated composition of function *with the* ***second domain***

Let $f: A \times B \to A$. *We extend* $f \circ f: A \times B^2 \to A$.

$$f \circ f = \{((a_0, b_1b_2), a_2) \in ((A \times B^2) \times A) \mid ((a_0, b_1), a_1) \in f, ((a_1, b_2), a_2) \in f\}$$

or $= \{f \circ f(a_0, b_1b_2) = a_2 \mid f(a_0, b_1) = a_1, f(a_1, b_2) = a_2\}$

or $\quad f \circ f(a_0, b_1b_2) = f(f(a_0, b_1), b_2)$

We ***extend*** $f^n: A \times B^n \to A$ *for* $n \geq 0$. *Let* $a \in A$, $\varepsilon \in B^0$, $w \in B^{n-1}$, $b \in B$.

$f^0(a, \varepsilon) = a$, $\quad n = 0$, $\quad f^0(a, \varepsilon)$ *is an* ***identity*** *function*

$f^n(a, wb) = f^{n-1} \circ f \quad n \geq 1$.

or $= f(f^{n-1}(a, w), b)$

We also define, $f^*: A \times B^* \to A$ *and* $f^+: A \times B^+ \to A$.

$f^* = \cup_{i \in N_0} f^i$,

$f^+ = \cup_{i \in N_+} f^i$.

Repeated composition of function *whose* ***range*** *is a* ***set***

Let $f: A \times B \to 2^A$. *We* ***extend***

$f': 2^A \times B \to 2^A$. Let $A' \subseteq A$ (or $A' \in 2^A$)

$$f'(A', b) = \cup_{a \in A'} f'(a, b).$$

We may write f ***instead*** *of f', since* $f \subseteq f'$, *and*

we also may write f(a) ***instead*** *of f({a}) for short.*

$f: 2^A \times B \to 2^A$.

We ***extend*** $f^n: 2^A \times B^n \to 2^A$ for $n \geq 0$. Let $A' \in 2^A$, $\varepsilon \in B^0$, $w \in B^{n-1}$, $b \in B$.

$f^0(A', \varepsilon) = A'$, $\quad n = 0,$ $\quad f^0(A', \varepsilon)$ *is an* ***identity*** *function*

$f^n(A', wb) = f^{n-1} \circ f \quad n \geq 1.$

or $= f(f^{n-1}(A', w), b)$

We also ***define*** $f^*: 2^A \times B^* \to 2^A$. $\qquad f^+: 2^A \times B^+ \to 2^A$.

$$f^* = \cup_{i \in N_0} f^i. \qquad f^+ = \cup_{i \in N_+} f^i.$$