1.A Sets, Relations, Graphs, and Functions

1.A.1 Set *a **collection** of objects(**element**)*

Let A be a set and a be an elements in A, then we write $a \in A$.

How to specify sets

1. *to **enumerate all** of the elements*
2. *to state the **properties** that characterizes the elements.*

   $A = \{x \mid p(x)\}$

   *p(x) is a **predicate***

   *p(x) is either **true** or **false** depending on x*

   $A = \{x \in U \mid p(x)\}$

   $A \subseteq U$, *U is the **universe** of **discourse***

   $x \in U$ *U is the **type** of x in A*

   p(x) ***attribute** of x*
3. ***automata**, **grammars**, **programs***

Three cases for two sets A and B

Case 1. ***subset***

$A \subseteq B$ *or* $B \subseteq A$

$\Leftrightarrow \quad A - B = \emptyset$ *or* $B - A = \emptyset$

$\Leftrightarrow \quad A \cap \overline{B} = \emptyset$ *or* $B \cap \overline{A} = \emptyset$

Case 2. ***disjoint***

$A \cap B = \emptyset$

*Case 3. in general(**incomparable**, neither subset nor disjoint)*

not$(A \subseteq B$ *or* $B \subseteq A)$ *and not* $(A \cap B = \emptyset)$

$\Leftrightarrow \quad A \not\subseteq B$ *and* $B \not\subseteq A$ *and* $A \cap B \neq \emptyset$.

$\Leftrightarrow \quad A \cap \overline{B} \neq \emptyset$ *and* $\overline{A} \cap B \neq \emptyset$ *and* $A \cap B \neq \emptyset$.

Venn diagram

$\overline{A} \cap \overline{B} \stackrel{?}{=} \emptyset$

Cartecian product *of two sets, A and B*

$A \times B = \{(a, b) | a \in A, b \in B\}.$

$(a, b) \in A \times B$ ***ordered pair***.

$|A \times B| = |A| \times |B|.$

1.A.2 Binary relation *R from A to B.*

$R \subseteq A \times B.$ $a \in A, b \in B, (a, b) \in R$ *or* $a\, R\, b$.

$|R| \leq |A \times B|.$

Inverse *of a relation R,* $R^{-1} = \{(b, a) \in B \times A | (a, b) \in R\}$

Composition(Product) *of two relations R and S*

where $R \subseteq A \times \underline{B}$ *and* $S \subseteq \underline{B} \times C$.

$R \cdot S = \{(a, c) | (a, b) \in R, (b, c) \in S\}$

Binary relation R ***on*** *A* $R \subseteq A \times A.$

Identity relation *R on A* $id_A = \{(a, a) | a \in A\}$

$\forall R \subseteq A \times A,\ R \cdot id_A = id_A \cdot R = R$

Repeated composition(product) *of a binary relation R.*

Let $R \subseteq A \times A$. *We define*

$R^2 = R \cdot R$, $R^3 = R \cdot R \cdot R$, … $R^n = R \cdot R \cdot \ldots \cdot R$, *and*

$R = R^1$. *Then we can* ***define***

$$R^n R^m = R^{n+m}, \text{ for } (\forall n, m \in \mathbb{N}),\ n, m \geq 1.$$

$R^0 = ?$ *If we define* $R^0 = id_A$. *Then we can* ***extend*** *the definition*

$$R^n R^m = R^{n+m}, \text{ for } n, m \geq 0.$$

*Another (**recursive**) definition for **repeated product** of binary relations*

$R^0 =_B id_A$. ***basis***

$R^n =_R R \cdot R^{n-1}$, $n \geq 1$. ***recursion***

ex) $R^3 =_R R \cdot R^2 =_R R \cdot R \cdot R^1 =_R R \cdot R \cdot R \cdot R^0 =_B R \cdot R \cdot R \cdot id_A = R \cdot R \cdot R$

1.A.3 *A* ***directed graph*** *$G = (V, E)$ is*

V: a set of ***vertices****,*

$E \subseteq V \times V$: a set of ***edges****,*

E: a ***binary relation*** *on V*

Some properties of the binary relations

1) R is ***reflexive****, if $\forall a \in A$, $a R a$.* $id_A \subseteq R$

R is ***irreflexive****, if $\forall a \in A$, $a \not R a$.* $R \cap id_A = \varnothing$

2) R is ***symmetric****, if $a R b$* impiles *$b R a$.* $R = R^{-1}$

R is ***antisymmetric****, if $a R b$* and $a \neq b$ implies *$b \not R a$.* $R \cap R^{-1} \subseteq id_A$

R is ***asymmetric****, if $a R b$* implies *$b \not R a$.* $R \cap R^{-1} = \varnothing$

R is ***asymmetric*** $\Rightarrow$ *R is* ***irreflexive****.*

R is ***asymmetric*** $\Rightarrow$ *R is* ***antisymmetric****.*

3) R is ***transitive****, if $a R b$* and *$b R c$* implies *$a R c$.* $R \cdot R \subseteq R$

Let $\mathbb{P}$ = {reflexive, symmetric, transitive}. Then R' be $\mathbb{P}$-closure of R, if

i) R' is $\mathbb{P}$.

ii) $R \subseteq R'$.

iii) R' is the ***smallest*** *set among satisfying i) and ii).*

$\Leftrightarrow {}^{\forall}R''$ *satisfying i) and ii),* $R' \subseteq R''$.

reflexive closure *of R,* $R' = R \cup id_A$.

symmetric closure *of R,* $R'' = R \cup R^{-1}$.

transitive closure *of R,*

$R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots = \cup_{i \in N_1} R^i$ *where* $N_1 = \{1, 2, 3, \ldots\}$.

reflexive-transitive closure *of R,*

$R^* = R^0 \cup R^1 \cup R^2 \cup R^3 \cup \ldots = \cup_{i \in N_0} R^i$ *where* $N_0 = \{0, 1, 2, \ldots\}$.

What is the ***reflexive(-symmetric)-transitive closure*** *of R*
in the graph (A, R)?

Let A be a set and $A_1, A_2, \ldots, A_n \subseteq A$. $\{A_1, A_2, \ldots, A_n\}$ is called a ***partion*** *of A, written Par(A), if $\cup_{i\in\{1, 2, \ldots, n\}} A_i = A$, $1 \le i \ne j \le n$: $A_i \cap A_j = \emptyset$.*

Power set *of a set A,*

$P(A) = 2^A = \{B \mid B \subseteq A\}$ $\quad B \subseteq A \Leftrightarrow B \in 2^A$.

$|2^A| = 2^{|A|}$.

$par(A) \subseteq 2^A$.

A binary relation R on A is ***equivalence****,*

if R is ***reflexive****,* ***symmetric****, and* ***transitive****.*

Par(A) *partion of A*

A binary relation R on A is ***(ir)reflexive partial order****,*

*if R is (****ir****)****reflexive****,* ***antisymmetric****, and* ***transitive****.*

A: partially-ordered set(po set)

Let $R \subseteq A \times A$ be an ***equivalence***,

$[a]_R = \{b \in A \mid a\ R\ b\}$ ***equivalence class***,

if $a\ R\ b$, $[a]_R = [b]_R$.

$\{[a]_R \mid a \in A\}$ *equivalence* ***partition***.

$\cup_{a \in A} [a]_R = A$, if $a \not R\ b$. $[a]_R \cap [b]_R = \emptyset$.

Let be $\leq$ a partial order on A. $\leq \subseteq A \times A$

Then (A, $\leq$) is called as partially ordered set or ***poset*** *for short.*

Let (A, $\leq$) be a poset. We define binary operator on A,

$\vee, \wedge : A \times A \rightarrow 2^A$

$a \vee b = min \{c \in A | a \leq c \text{ and } b \leq c\}$ *least upper bound*

$a \wedge b = max \{c \in A | c \leq a \text{ and } c \leq b.$ *greatest lower bound*

If a ***unique*** *lub and a* ***unique*** *glb,*

$\vee, \wedge : A \times A \rightarrow A.$ *(A, $\leq$) is called as a* ***lattice*** *and*

(A, $\vee$, $\wedge$) is called a ***algebra*** *induced by the lattice (A, $\leq$).*

Boolean *algebra, ({f, t}, $\vee$, $\wedge$), is induced by the lattice ({f, t}, {f $\leq$ t}).*

Let A be a sets. Then

Set *algebra on A, (2^A, $\cup$, $\cap$), is induced by the lattice (2^A, $\subseteq$).*

Singleton set *algebra, ($2^{\{a\}}$, $\cup$, $\cap$), is* ***isomorphic*** *to*

boolean *algebra, ({f, t}, $\vee$, $\wedge$) with respect to* ***bijection*** *g.*

What is the ***bijective*** *function g?*

Let A be a set and ⊕ be a binary operation on A.

$\oplus: A \times A \rightarrow A.$

i) $\forall a, b \in A, a \oplus b \in A.$ ***closed*** ***algebraic system***

ii) $\forall a, b, c \in A, a \oplus (b \oplus c) = (a \oplus b) \oplus c$ ***associative*** ***semi-group***

binary operation ⇒ n-ary operation

iii) $\exists e \in A$.∍. $\forall a \in A, e \oplus a = a \oplus e = a$ ***identity*** ***monoid***

Let (A, ⊕, e) and (B, ⊗, ε) be two monoids.

If i) h: A → B is a onto function, $|A| \geq |B|$

ii) h(a⊕b) = h(a)⊗h(b), and *preserve operation*

iii) h(e) = ε. *preserve identity*

Then h is called a ***homomorphism****, and the monoid (B, °, ε) is called a* ***homomrphic*** *to the monoid (A, ⊕, e) w.r.t. h.*

(A, ⊕, e) is called ***concretization*** *of (B, ⊗, ε) and (B, °, ε) is called* ***abstract interpretation*** *of (A, ⊕, e).*

If f is one-to-one and onto, f is called ***isomorphism****.*

1. A. 4 *A binary **relation** from A to B is a **function** from A to B, if*

1) $\forall a \in A, \exists (a, b) \in f,$ ***total***

2) $\forall a \in A, \exists_1 (a, b) \in f.$ ***unique***

$f: A \to B$ $(a, b) \in f$ or $a\ f\ b$ or $f(a) = b$ or $f\ a = b$.

*Three **faces** of a binary **relation***

i) $R \subseteq A \times B.$ $(a, b) \in R.$ *i) a **set of** (ordered) **pairs***

ii) $R: A \times B \to \{$***false, true***$\}.$

$a\ R\ b$, *iff* $(a, b) \in R.$ *ii) a **relational operator***$(<, =, \leq)$

iii) $R: A \to 2^B.$

$R(a) = \{b_1, b_2, \ldots, b_n\}$, *iff* $(a, b_1), (a, b_2), \ldots, (a, b_n) \in R.$

$\forall a \in A, \exists_1 \{b_1, b_2, \ldots, b_n\} \subseteq B$ *or* $\exists_1 \{b_1, b_2, \ldots, b_n\} \in 2^B.$

$\therefore R: A \to 2^B.$ *iii) a **set valued function***

Let f: A → B. Is f^{-1}: B → A a function?

No!!!

Function f: A → B is ***onto***(***surjection***; ***correspondence***), *if*

$\forall b \in B, \exists a \in A$.϶. $f(a) = b$. $|A| \geq |B|$

If f is onto, f^{-1} is ***total*** *but* ***not*** *unique function.*

Function f: A → B is ***one-to-one***(***injection***, *1-1), if*

$\forall a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$.

if $\exists b \in B$.϶. $f(a) = b$, $\exists_1 a \in A$. $|A| \leq |B|$

If f is 1-1, f^{-1} is ***unique*** *but* ***not*** *total function.*

Function f: A → B is ***bijective***,

if f is both ***1-1*** *and* ***onto(1-1 correpondence***).

$\forall b \in B$, *if* $\exists_1 a \in A$.϶. $f(a) = b$.. $|A| = |B|$

If f is 1-1 onto, f^{-1} is both ***total*** *and* ***unique***, *so is a* ***function***.

1.B Set isomorphism and infinite sets

*If there exists a **bijection**(짝짓기 , **1-1 onto**) f from A to B,*

*two sets A and B have same **cardinality**, written |A| = |B|, and*

*two sets A and B are said to be **isomorphic** w.r.t. f, written $A \cong_f B$.*

*A set is said to be **countable**(**enumerable**),*

*if it has the **same cardinality** with a **subset** of ℕ,*

*either **finite** or **infinite***

*and **uncountable** (**infinite**), otherwise.*

*A set is **countably infinite**, if it has the **same cardinality** with ℕ.*

*the **cardinality** of ℕ is denoted as ℵ, |ℕ| = ℵ.*

*Let A be **countable**. Then we can **enumerate** the set in **numeric** order.*

$A = \{a_0, a_1, \ldots, a_n\}$ ***finite** for some $n \geq 0$.*

$A = \{a_0, a_1, \ldots \}$ ***infinite**(**countable**, **enumerable**)*

Consider

$N_1 = \{1, 2, 3, \ldots\}$ $\quad |N_1| = |\mathbb{N}| = \aleph$, but $N_1 \subset \mathbb{N}$.

$E = \{e \in \mathbb{N} \mid e = 2i, i \in \mathbb{N}\}$ $\quad |E| = |\mathbb{N}| = \aleph$, but $E \subset \mathbb{N}$.

$I = \mathbb{N} \cup \{-i \mid i \in \mathbb{N}\}$ $\quad |I| = |\mathbb{N}| = \aleph$, but $\mathbb{N} \subset I$.

$Q = \mathbb{N} \times \mathbb{N}$ $\quad |Q| = |\mathbb{N}| = \aleph$.

enumerate *(i, j)* $\in \mathbb{N} \times \mathbb{N}$ *in* (**natural**) ***numeric*** *order*

$\mathbb{N} \times \mathbb{N} = \{\underline{(0, 0)}, \underline{(1, 0), (0, 1)}, \underline{(2, 0), (1, 1), (0, 2)}, \underline{(3, 0), (2, 1),} \ldots \}$

$f(i, j) = 1 + 2 + 3 + \ldots + (i+j) + j$ $\quad f(0, 0) = 0$

$= (i+j)(i+j+1)/2 + j$

$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is one-to-one and onto

$\therefore |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph$.

Prove $|\Sigma^*| = \aleph$.

Proof Canonical(lexicographical) order *for* Σ^*.

in order of ***size*** *and if same size,* ***alphabetic order***.

Let $|\Sigma| = k$. *Then we can alphabetic order* $\Sigma = \{a_0, a_1 \ldots, a_{k-1}\}$,

and we can order $x \in \Sigma^*$ *where* $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots = \cup_{i \in N_0} \Sigma^i$.

$\Sigma^* = \{\underline{\varepsilon}, \underline{a_0, a_1 \ldots, a_{k-1}}, \underline{a_0a_0, a_0a_1, \ldots, a_0a_{k-1}, a_1a_0, \ldots, a_{k-1}a_{k-1}}, \underline{a_0a_0a_0, \ldots, a_{k-1}a_{k-1}a_{k-1}}, \ldots\}$

If $x = a_1a_2 \ldots a_n$, *the order of x, f(x), is n-digit k-ary number plus base.*

$$f(x) = k^0 + k^1 \ldots + k^{n-1} + a_1k^{n-1} + a_2k^{n-2} + \ldots + a_nk^0$$

$$= (k^n - 1)/(k-1) + a_1k^{n-1} + a_2k^{n-2} + \ldots + a_nk^0.$$

We ***enumerate*** $x = a_1a_2 \ldots a_n \in \Sigma^*$ *in* ***numeric order***

$\therefore f:\Sigma^* \rightarrow \mathbb{N}$ *is one-to-one onto. Q.E.D.*

Consider $\{0, 1\}^{\mathbb{N}}$*:* ***infinite*** *binary strings (See pp.12)*

and $2^{\mathbb{N}}$*:* ***power*** *set of natural numbers (Note that* $2^A = \{B \mid B \subseteq A\}$*)*

Cantor's diagonal argument

Assume $2^{\mathbb{N}}$ *is* ***countable****.*

We can ***enumerate*** $|2^{\mathbb{N}}|$ *subsets of* $\mathbb{N}$*, in numeric order as follows,*

$2^{\mathbb{N}} = \{a_0, a_1, \ldots, a_i, \ldots\}$ *where*

$\forall i \in \mathbb{N}, a_i \leftrightarrow (b_{i0}, b_{i1}, \ldots b_{ii}, \ldots) \in 2^{\mathbb{N}}$

$\forall j \in \mathbb{N}$, *if* $b_{ij} = 1$ *then* $j \in a_i \in 2^{\mathbb{N}}$,

if $b_{ij} = 0$ *then* $j \notin a_i \in 2^{\mathbb{N}}$.

$\therefore$ ***Power set*** *of* $\mathbb{N} \leftrightarrow$ ***infinite binary string***

Power set of integers and infinite binary strings are ***isomorphic****.*

$\therefore |2^{\mathbb{N}}| = |\{0, 1\}^{\mathbb{N}}|$.

Consider $a \leftrightarrow (\overline{b}_0, \overline{b}_1, \ldots \overline{b}_i, \ldots)$ *where* $\forall i \in \mathbb{N}$, $\overline{b}_i = 1$, *if* $b_{ii} = 0$,
$\overline{b}_i = 0$, *if* $b_{ii} = 1$.

Since $\forall i \in \mathbb{N}$, $\overline{b}_i \neq b_{ii}$, $\therefore \forall i \in \mathbb{N}$, $a \neq a_i$.

$\therefore a \notin \{a_0, a_1, \ldots, a_i, \ldots\}$.

But $a \in 2^{\mathbb{N}}$ *by the* ***definition*** *of power set.*

$a \notin \{a_0, a_1, \ldots, a_i, \ldots\}$ *but* $a \in 2^{\mathbb{N}}$.

$\therefore$ *Contradiction!!!*

We ***fail*** *to* ***enumerate*** $2^{\mathbb{N}} = \{a_0, a_1, \ldots, a_i, \ldots\}$ *in* ***numeric order***!

$\therefore$ *We* ***conclude*** *that* $|2^{\mathbb{N}}| > |\{a_0, a_1, \ldots, a_i, \ldots\}| = \aleph$.

$2^{\mathbb{N}}$ *is* ***uncountable***.

$\{0, 1\}^*$ vs $\{0, 1\}^{\mathbb{N}}$.

$\{0, 1\}^*$: **all**(**countably** infinite union of **finite** binary strings

$= \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2 \cup \{0, 1\}^3 \cup \ldots$

$= \{\varepsilon\} \cup \{0, 1\} \cup \{00, 01, 10, 11\} \cup \{000, \ldots, 111\} \cup \ldots$

$= \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots, 111, \ldots\}$

$\{0, 1\}^{\mathbb{N}}$: **all**(**uncountaby** infinite union of) **infinite** binary strings

$= \{000\ldots000\ldots,$ $\leftrightarrow \{\}$

$100\ldots000\ldots,$ $\leftrightarrow \{0\}$

$010\ldots000\ldots,$ $\leftrightarrow \{1\}$

$110\ldots000\ldots,$ $\leftrightarrow \{0, 1\}$

$\ldots,$

$111\ldots111\ldots\}$ $\leftrightarrow \{0, 1, 2, \ldots\} = \mathbb{N}$

$|\{0, 1\}^{\mathbb{N}}| = |2^{\mathbb{N}}| > |\{0, 1\}^*| = \aleph.$

Cantor's diagonal argument

Complement of diagonal element

Russel's paradox

$S = \{x \mid x \notin x\}$

$x \in S$, *iff* $x \notin x$. *But* $S \in S$, *iff* $S \notin S$. ***contradictory!***

Halting problem

H(P): ***if*** *halt(P, P) then loop forever*

elses *not halt(P, P) then stop* ***fi***

What happens if H(H) stops or loops forever?

Denial of self recursion

Σ^* *is* ***countable***. ***strings*** *are* ***countable***

But is 2^{Σ^*} ***uncountable***. ***languages*** *are* ***uncountable***

class of languages

N. Chomsky

Finite(countable)

Countably infinite

natural numbers, integers, rational numbers,

***finite** strings …*

Uncountable

Cantor's diagonal argument

***power set** of natural numbers*

***infinite** strings*

real numbers

*Some informal descriptions on **countable** and **uncountable** infiniteness*

$\aleph \pm k = \aleph$ $\aleph \times k = \aleph$ ***countable***

$\aleph \times \aleph = \aleph$ $\aleph^k = \aleph$ ***countable***

But $k \times k \times \ldots = k^{\aleph} > \aleph$ ***uncountable**(k ≥ 2)*

1.C Strings and languages, revisited.

Concatenation *of* ***strings, revisited***

$\cdot : \Sigma^* \times \Sigma^* \to \Sigma^*$.

*a **function from** two strings **to** a string*

*a **binary operation** on **strings***

(1) $\forall x, y \in \Sigma^*, xy \in \Sigma^*$. ***closed***

(2) $\exists x, y \in \Sigma^*, xy \neq yx$ ***noncommutative***

(3) $\forall x, y, z \in \Sigma^*, x(yz) = (xy)z$ ***associative***

(4) $\forall x \in \Sigma^*, \varepsilon x = x\varepsilon = x$ *ε is the **identity** element*

$\therefore$ *$(\Sigma^*, \cdot, \varepsilon)$ is a noncommutative **monoid**.*

*Another (**recursive**) definition of Σ^*.*

basis	$\varepsilon \in \Sigma^*$ *and* $\forall a \in \Sigma, a \in \Sigma^*$.
recursion 1	*If* $x, y \in \Sigma^*$, *then* $xy \in \Sigma^*$.
recursion 2	*If* $x \in \Sigma^*$ *and* $a \in \Sigma$, *then* $xa \in \Sigma^*$.

Extend** the domain and range of the concatenation from **strings** to **languages(set of strings)

$\cdot : 2^{\Sigma^*} \times 2^{\Sigma^*} \to 2^{\Sigma^*}$.

Let $L, S \subseteq \Sigma^*$ *(or* $L, S \in 2^{\Sigma^*}$*). Then we define*

$$LS = \{xy \mid x \in L, y \in S\}$$

$$|LS| \le |L| \times |S|$$

$(2^{\Sigma^*}, \cdot, \{\varepsilon\})$ *is a (induced) noncommutative **monoid**.*

power of an alphabet revisited

$\Sigma^{0} = \{\varepsilon\}$ **basis**

$\Sigma^{n} = \Sigma\Sigma^{n-1}$ *for* $n \geq 1$ **recursion**

$|\Sigma^{n}| = |\Sigma|^{n}$.

We can ***extend*** *to the power of* ***languages***

$L^{0} = \{\varepsilon\}$ **basis**

$L^{n} = LL^{n-1}$ *for* $n \geq 1$ **recursion**

$|L^{n}| \leq |L|^{n}$.

$L^{*} = L^{0} \cup L^{1} \cup L^{2} \cup \ldots = \cup_{i \in N_0} L^{i}$.

$L^{+} = L^{1} \cup L^{2} \cup L^{3} \cup \ldots = \cup_{i \in N_1} L^{i}$.

For any $L \subseteq \Sigma^{*}$, $L^{*} \subseteq \Sigma^{*}$.

$\varepsilon \notin \Sigma^{+}$, *but* $\varepsilon \in L^{+}$, *only if* $\varepsilon \in L$.

String, revisited

Let x be a string over Σ*. Then we write* $x \in \Sigma^*$.

Let $|x| = n (n \geq 0)$*. Then*

$x = a_1 a_2 \ldots a_n \in \Sigma^n$ or Σ^* (if $n=0$, $x = a_1 a_2 \ldots a_n = \varepsilon$)

where $1 \leq {}^{\forall} i \leq n$, $a_i \in \Sigma$.

Consider a function $x: \{1, 2, \ldots, n\} \rightarrow \Sigma$.

$1 \leq {}^{\forall} i \leq n$, if $x(i) = a_i$, we can write $x = (a_1, a_2, \ldots, a_n)$

$a_1 a_2 \ldots a_n \leftrightarrow^{1:1} (a_1, a_2, \ldots, a_n)$

The ***strings*** *over* Σ *of length* $n(\Sigma^n)$,

is ***isomorphic*** *to the* ***functions*** *from* $\{1, 2, \ldots, n\}$ *to* Σ *w.r.t. f.*

Let $B^A = \{f \mid f: A \rightarrow B\}$*. Then* $|B^A| = |B|^{|A|}$.

$|\Sigma^n| = |\Sigma^{\{1, 2, \ldots, n\}}| = |\Sigma|^n$.

Let $x = a_1a_2 \ldots a_n$ be a string of length n and $k \geq 0$.

$x^R = a_na_{n-1} \ldots a_2a_1$. ***reversal*** *of x.*

$k{:}x = a_1a_2 \ldots a_k$, *if* $k \leq n$; ***prefix*** *of x with length k.*

$= x$, *otherwise.*

$x{:}k = (k{:}x^R)^R$. ***suffix*** *of x with length k.*