

## Chap. 9 Undecidability

### 9.1 A Language that is Not recursively enumerable

#### Code for Turing Machine

TM  $M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, \{q_2\})$

$$Q = \{q_1, q_2, \dots, q_k\}$$

$$\Gamma = \{X_1, X_2, \dots, X_m\} \quad X_1 = 0, X_2 = 1, X_3 = B.$$

$$L = D_1, R = D_2.$$

$$\delta(q_i, X_j) = (q_k, X_l, D_m) \quad i, j, k, l, m \in \mathbb{N}.$$

$$\leftrightarrow 0^i 10^j 10^k 10^l 10^m$$

$$\delta \leftrightarrow \delta_1 11 \delta_2 11 \dots 11 \delta_n.$$

$$(M, w) \leftrightarrow \text{code}(M) 111 \text{code}(w)$$

number of Turing machines is **countable**.

We can **enumerate** TM  $M_i$  for  $i \in \mathbb{N}$ .

## **Diagonalization Language: $L_d$ .**

*Since both of TM's and strings in  $\Sigma^*$  are **countable**,  
we can consider  $(M_i, w_i)$  pair for  $i \in \mathbb{N}$ .*

*Consider  $L_d = \{w_i \in \Sigma^* \mid w_i \notin L(M_i)\}$*

**Theorem 9.2**  *$L_d$  is **not** recursively enumerable.*

**proof** *Suppose  $L_d = L(M)$  for some TM  $M$ .*

*Since  $M$  is a TM,  $\exists i \in \mathbb{N} . \exists. M = M_i$ .*

*If  $w_i \in L_d$ ,  $w_i \notin L_d$  by definition of  $L_d$ .  $\therefore M$  does **not** accept  $w_i$ .*

*If  $w_i \notin L_d$ ,  $w_i \in L_d$  by definition of  $L_d$ .  $\therefore M$  **accepts**  $w_i$ .*

**Contradiction!**  $\therefore M$  does **not** exist.

$\therefore L_d$  is **not** recursively enumerable.

**TM's are countable whereas languages are uncountable!**

**The halting problem**

```

program halt(P: program, I: input)
  if P(I) will stop then print “halts”
  else print “loops forever” fi

```

Assume the program halt exists and consider a program *H*

```

program H(P: program)
  if halt(P, P) = “halts” then loops forever
  else stop fi

```

Consider *H*(*H*)

if *H*(*H*) loops forever  $\Rightarrow$  halt(*H*, *H*) prints “stop”  
 $\Rightarrow$  But *H*(*H*) must stop (**definiton** of *H*).

if *H*(*H*) stop  $\Rightarrow$  halt(*H*, *H*) prints “no stop”  
 $\Rightarrow$  But *H*(*H*) must loops forever (**definiton** of *H*).

$\therefore$  Contradiction! halt does **not** exist.

$\therefore$  **halting problem** does **not** exist.

Languages(sets) that is **not** RE(no TM, no **program**)

$$L_d = \{w_i \in \Sigma^* \mid w_i \notin L(M_i)\}$$

halting problem

power set of integer is **uncountable**

Cantor's diagonal arguments

Russel's paradox

$$S = \{x \mid x \notin x\} \quad x \in x, \text{ iff } x \notin x. \quad \text{But } S \in S, \text{ iff } S \notin S!$$

Some similar examples in the world

A barber who shave everybody who can **not** shave himself.

Shall the barber shave **himself**?

An adjective is heterological, if the adjective does **not** possess  
the property it describes.(monosyllabic, polysyllabic)

Is the adjective “**heterological**” **heterological**?

There is a sign that “It is written by **me**(**liar**)”.Did you(**liar**) write it?

Self contradiction **denial of self recursion!**

## 9.2 An undecidable problem that is RE

### *Recursive languages*

*If  $w \in L$ ,  $M$  halts and accepts.*

*If  $w \notin L$ ,  $M$  halts and does not accept.*

*subclass of RE languages*

*type 1 in Chomsky's hierarchy*

### *Recursively enumerable languages (RE languages)*

*If  $w \in L$ ,  $M$  halts and accepts.*

*If  $w \notin L$ ,  $M$  halts and does not accept or loops forever.*

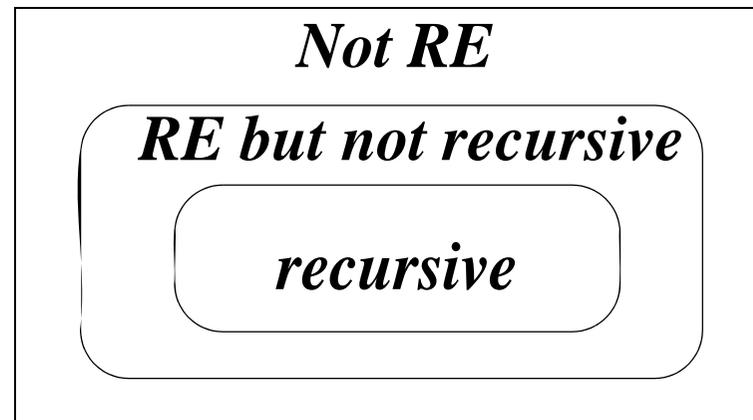
*Problem  $P$  is called **decidable**, if  $P$  is recursive*

*Problem  $P$  is called **undecidable**, if  $P$  is **not** recursive*

*$P$  may be RE or non-RE*

*Three classes of languages(problems)*

<i>recursive</i>	<i>decidable</i>	<i>countable</i>
<i>RE but not recursive</i>	<i>undecidable</i>	<i>countable</i>
<i>not RE</i>	<i>undecidable</i>	<i>uncountable</i>



*Recursive*

*Recursive enumerable*

*Decidable(algorithm)*

*Turing computable*

*partial recursive function*

*programmable(computable)*

## *Languages and problems*

$$L: \Sigma^* \rightarrow \{0, 1\}$$

$$P: \mathbb{N} \rightarrow \{0, 1\}$$

*Both of languages and problems are **uncountable**.*

*But TM(program) are **countable**.*

*There are problems(languages) that is not recursively enumerable.*

*halting problem*

*Russel's paradox*

*Diagonalization languages( $L_d$ )*

$$\text{complement of } L_d \quad \bar{L}_d = \{w_i \in \Sigma^* \mid w_i \in L(M_i)\}$$

*RE(but not recursive)*

## ***Complement of recursive and recursively enumerable languages***

***Theorem 9.3*** *If  $L$  is recursive,  $\bar{L}$  is also recursive.*

***proof*** *Let  $L = L(M)$  for some TM  $M$  that always halts.*

*Consider  $\bar{M}$*

*accept and halt  $\rightarrow$  not accept and halt.*

*not accept and halt  $\rightarrow$  accept and halt.*

*$\exists \bar{M} . \exists . \bar{L} = L(\bar{M})$  and always halts.*

*More detail*

$M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, \{q_2\})$  *no transition from  $F$*

$\bar{M} = (Q \cup \{f\}, \{0, 1\}, \Gamma, \bar{\delta}, q_1, B, \{f\})$

$\bar{\delta} = \delta \cup \{\delta(q, X) = (f, X, S) \mid q \in Q\}$

*no transition in  $M \rightarrow$  transition to  $f$*

**Theorem 9.4** If both of  $L$  and  $\bar{L}$  are RE,  $L$  is **recursive** (so is  $\bar{L}$ ).

**proof** Let  $L = L(M_1)$  and  $\bar{L} = L(M_2)$ .

Consider a **two** tape TM  $M$ .

tape 1 simulates the tape of  $M_1$  and tape 2 simulates the tape of  $M_2$ .  
states and state transitions of  $M$  simulates  $M_1$  and  $M_2$  in **parallel**.

If  $x \in L \rightarrow M_1$  **accept** and halt  $\rightarrow M$  accept  $x$  and **halt**.

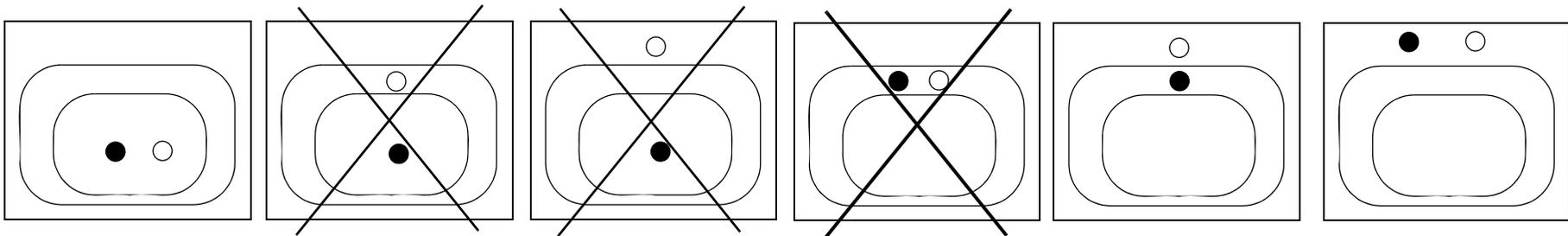
If  $x \notin L \rightarrow M_2$  **accept** and halt  $\rightarrow M$  **not** accept  $x$  and **halt**.

$\therefore L$  is **recursive**.

Only **three** case for the **complement** of language (among 6 cases)

Thm 9.3: If  $L$  is **recursive**,  $\bar{L}$  is **recursive** and vice versa.

Thm 9.4:  $L$  and  $\bar{L}$  are **not** both RE and **not recursive**.



**Universal language:**  $L_u$ . complement of  $L_d$ .

$$L_u = \{w_i \in \Sigma^* \mid w_i \in L(M_i)\} = \overline{L_d}$$

**Theorem 9.6**  $L_u$  is RE but not recursive.

**proof** Let  $U$ , **universal TM**, be a multi tape TM such that  $L(U) = L_u$ .

tape 1:  $(M_i, w_i)$

tape 2: simulate the tape of  $M_i$ .

If  $M_i$  accept  $w_i$ ,  $U$  accept  $M_i$ .

$\therefore L_u$  is **RE**. (simplified version of 9.2.3)

Since  $L_d$  is **not RE**.

$\therefore L_u$  is **not recursive**. (Three cases for the complement)

$\therefore L_u$  is RE but **not recursive**.

$L_d = \overline{L_u}$  is not RE. (the second case in the diagram of p8)

### 9.3 Undecidable Problems About Turing Machines

$P$  is a *decision problem* on the domain  $D$ , if  $\forall d \in D$ ,  $P(d)$  is *yes* or *no*.

$P: D \rightarrow \{\text{yes}, \text{no}\}$

$Y_P, N_P \subseteq D$  is called *yes(no) instances* of  $P$ , if

$Y_P = \{d \in D \mid P(d) = \text{yes}\}$

$N_P = \{d \in D \mid P(d) = \text{no}\}$ , respectively.

A *problem*  $P$  is *decidable*,

if there exists a *decider* (program, algorithm, TM) that always tells *yes* or *no* correctly.

*Undecidable*, otherwise.

Assume  $D$  is countable.

Then  $|\{P(D)\}|$  is uncountable.

But decider is countable.

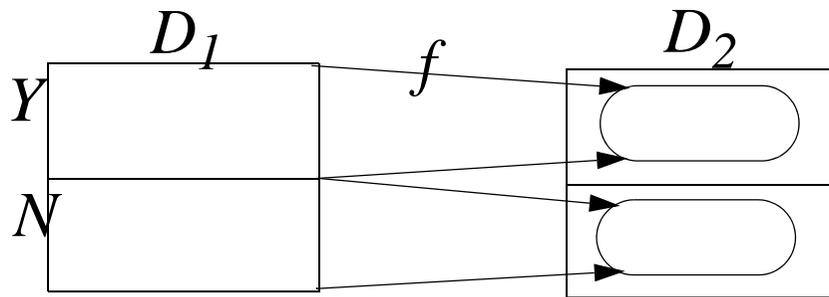
$\therefore \exists P, \exists P$  is undecidable.

## Reducing one problem to another

We say a problem  $P_1$  on  $D_1$  **reduces to**  $P_2$  on  $D_2$ , if

$$\exists f: D_1 \rightarrow D_2, \text{ s.t. } \forall d_1 \in Y_{P_1}(D_1), f(d_1) \in Y_{P_2}(D_2)$$

$$\forall d_1 \in N_{P_1}(D_1), f(d_1) \in N_{P_2}(D_2)$$



If  $P_1$  (on  $D_1$ ) **reduces to**  $P_2$  (on  $D_2$ ),

$P_2$  is at least as **hard** as  $P_1$ . ( $P_1 \leq P_2$ )

$P_2$  is **not easier** than  $P_1$ .

**Theorem 9.7** *If there is a reduction from  $P_1$  to  $P_2$  ( $P_1 \leq P_2$ ), then:*

a) *If  $P_1$  is **undecidable**, then so is  $P_2$ . (If  $P_2$  is **decidable**, so is  $P_1$ .)*

b) *If  $P_1$  is **non-RE**, then so is  $P_2$ . (If  $P_2$  is **RE**, so is  $P_1$ .)*

**proof** a) *Suppose  $P_2$  is **decidable**. Then*

$\exists D_2 . \exists . \forall x \in P_1, f(x) \in P_2, D_2$  halts “yes”,  $D_1$  halts “yes”.

$\forall x \notin P_1, f(x) \notin P_2, D_2$  halts “no”,  $D_1$  halts “no”.

$\therefore P_1$  is **decidable**. (Figure 9.7)

b) *Assume  $P_2$  is **RE**. Then*

$\exists M_2 . \exists . \forall x \in P_1, f(x) \in P_2, M_2$  halts “yes”,  $M_1$  halts “yes”.

$\forall x \notin P_1, f(x) \notin P_2, M_2$  halts “no” or loops forever,

$M_1$  halts “no” or loops forever.

$\therefore P_1$  is **RE**.

*If  $L_u$  reduces to  $P$ ,  $P$  is **not recursive**. (RE or not RE)*

### 9.3.2 Turing Machine that Accepts the Empty Language

$$L_e = \{M \mid L(M) = \emptyset\}$$

$$L_{ne} = \{M \mid L(M) \neq \emptyset\}$$

**Theorem 9.8**  $L_{ne}$  is recursively enumerable.

*proof* Consider a NTM  $M_{ne}$

1. Guess a TM  $M$  and an input string  $w$
2. A TM  $U$  test if  $M$  accepts  $w$ . ( $U$  simulates  $M$  for  $w$ )
3. If  $M$  accepts  $w$  then  $M_{ne}$  accepts  $M$ .

$$\therefore L(M_{ne}) = L_{ne}.$$

But it is not so easy to find a TM  $M_e$   $\exists$ .  $L(M_e) = L_e$ .

Actually there is no TM  $M_e$   $\exists$ .  $L(M_e) = L_e$ .

We shall prove that in Thm. 9.9 and 9.10.

**Theorem 9.9**  $L_{ne}$  is not recursive.

**proof** Reduce  $L_u$  to  $L_{ne}$ . ( $L_u \leq L_{ne}$ )

Consider a TM  $M_{ne}$ .

1.  $U$  simulates  $M$  for  $w$ . (guess  $(M, w)$  pair)
2. If  $U$  accepts  $w$  ( $w \in L(M)$ ), then code for  $M \in L_{ne}$ .
3. If  $U$  does not accept  $w$  ( $w \notin L(M)$ ), then code for  $M \notin L_{ne}$ .

Transform  $(M, w)$  pair to  $M_{ne}$ .  $\exists. L(M_{ne}) = \{M \mid w \in L(M)\}$

If  $w \in L(M)$ ,  $M \in L_{ne}$ .

If  $w \notin L(M)$ ,  $M \notin L_{ne}$ .

$\therefore$  We Reduced  $L_u$  to  $L_{ne}$ .

$\therefore L_{ne}$  is not recursive.

**Theorem 9.10**  $L_e$  is not RE.

**proof** Since  $L_e = \overline{L_{ne}}$  and  $L_{ne}$  is RE but not recursive. Case 2 of p7.

## ***Rice's Theorem and Properties of RE Languages***

Consider a **property**  $P$  of a set of languages.

property of being **context free** is set of all CFL's.

property of being **empty** is  $\{\emptyset\}$ .

$$P: 2^{\Sigma^*} \rightarrow \{\text{true}, \text{false}\}$$

$$P = \{L \subseteq \Sigma^* \mid P(L)\} = \{L \in 2^{\Sigma^*} \mid P(L)\}.$$

A **property** is **trivial**, if it is either **empty** or is **all** of the languages.  
**nontrivial** otherwise.

$P = \emptyset$  or  $2^{\Sigma^*}$  are trivial.

But  $P = \{\emptyset\}$  is nontrivial.

$P$  may be represented as set of TM's,  $L_P$

$$P \leftrightarrow L_P = \{M \in \text{TM} \mid L(M) = L\}$$

**Theorem 9.11 (Rice's Theorem)** *Every nontrivial property of the recursively enumerable languages are undecidable.*

Let  $P$  be a nontrivial property of RE languages.

1. Assume  $P(\emptyset) = \text{false}$  (or  $\emptyset \notin P$ ).

$$\therefore \exists L . \exists . P(L) = \text{true} \text{ or } \exists M_L . \exists . M_L \in L_P$$

We shall **reduce**  $L_u$  to  $L_P$ . ( $L_u \leq L_P$ )

1.  $U$  simulate  $w$  for  $M$

2. If  $U$  accepts  $w$ ,  $L = L(M) \leftrightarrow \exists M_L \in L_P$

2.1  $M_L$  simulate for  $x$

2.2 If  $M_L$  accepts  $x$ ,  $M$  accepts  $M_L$ .

3. If  $U$  does not accept  $w$ , do nothing.

2. Assume  $P(\emptyset) = \text{true}$  (or  $\emptyset \in P$ ).

Consider complement property  $\bar{P}$ ,  $L_{\bar{P}} = \bar{L}_P$

$\therefore \bar{P}$  is undecidable(above).  $\therefore P$  is undecidable.(Thm 9.3)

### ***9.3.4 Problems about Turing-Machine Specifications***

- 1. Whether the language accepted by a TM is empty ( $L_e, L_{ne}$ ).*
- 2. Whether the language accepted by a TM is finite.*
- 3. Whether the language accepted by a TM is regular.*
- 4. Whether the language accepted by a TM is context-free.*

*Undecidable!*

*Problem is a language*

<i>language</i>	<i>problem</i>	<i>function</i>
<i>Not R.E.</i>	<i>Not computable</i>	
<i>recursively enumerable</i>	<i>computable</i>	<i>partial function</i>
<i>recursive</i>	<i>decidable(alway halt)</i>	<i>(total) function</i>

*Three class of languages(problems)*

*1. Not recursively enumerable*

*Not computable*

*2. Recursively enumerable but not recursive*

*Computable but **not** decidable (undecidable)*

*partial  $\mu$ -recursive function*

*3. recursive*

*decidable*

*total  $\mu$ -recursive function*

*Following systems are equivalent*

*Turing machine*

*Turing(1930)*

*$\mu$ -recursive function(partial recursive) function*

*Gödel(1934; lecture at Princeton), Herbrand, Kleene(1936)*

*$\lambda$ -calculus*

*Church(1933-41), Kleene(1935), Rosser(1935)*

*combinatory logic*

*Schönfinkel(1924), Curry(1929)*

*Post correspondence system*

*Post(1936)*

*type 0 grammar*

*Chomsky(1959)*

*while programs*

*Meyer, Ritchie(1967)*

*Equivalence of  $\mu$ -recursive function and  $\lambda$ -calculus*

*Church(1936) attributes to Kleene*

*Equivalence of TM and  $\lambda$ -calculus*

*Turing(1937)*

*TM for computable*

*Turing's thesis*

*Turing-Church's thesis*