

Chap. 6 Pushdown Automata

6.1 Definition of Pushdown Automata

Example 6.1 $L = \{wcw^R \mid w \in (0+1)^*\}$

$P \rightarrow c \mid 0P0 \mid 1P1$

1. Start at state q_0 , push input symbol onto stack, and stay in q_0 .
2. If input symbol is c in state q_0 , go to state q_1 .
3. If input symbols is same as top of the stack, pop it and stay in q_1 .
4. If no more input symbol and empty stack in state q_1 , accept.

A pushdown automaton (PDA) $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is

1. Q is a **finite set of states**,
2. Σ is a **finite set of input symbols**,
3. Γ is a **finite stack alphabet**,
4. δ is a **transition function**.

$$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^Q \times \Gamma^*.$$

5. $q_0 \in Q$ is an **initial state**,
6. $Z_0 \in \Gamma$ is an **initial stack content**,
7. $F \subseteq Q$ is a **set of final states**.

Example $P = (\{q_0, q_1, q_2\}, \{0, 1, c, \$\}, \{Z, O, Z_0\}, \delta, q_0, Z_0, \{q_2\})$

$$\delta(q_0, 0, Z_0) = \{(q_0, ZZ_0)\}$$

$$\delta(q_0, 0, Z) = \{(q_0, ZZ)\}$$

$$\delta(q_0, 0, O) = \{(q_0, ZO)\}$$

If see 0, then push Z in q_0 .

$$\delta(q_0, 1, Z_0) = \{(q_0, OZ_0)\}$$

$$\delta(q_0, 1, Z) = \{(q_0, OZ)\}$$

$$\delta(q_0, 1, O) = \{(q_0, OO)\}$$

If see 1, then push O in q_0 .

$$\delta(q_0, c, Z_0) = \{(q_1, Z_0)\}$$

$$\delta(q_0, c, Z) = \{(q_1, Z)\}$$

$$\delta(q_0, c, O) = \{(q_1, O)\}$$

If see c, go to q_1 .

$$\delta(q_1, 0, Z) = \{(q_1, \varepsilon)\}$$

If see 0, then pop Z in q_1 .

$$\delta(q_1, 1, O) = \{(q_1, \varepsilon)\}$$

If see 1, then pop O in q_1 .

$$\delta(q_1, \$, Z_0) = \{(q_2, \varepsilon)\}$$

If see \$, go to the final state q_2 .

$\$$: end of string marker

Instantaneous description of PDA

(current state, remained input string, stack contents)

$$(q, x, \gamma) \in Q \times \Sigma^* \times \Gamma^*.$$

$$\vdash_P \subseteq (Q \times \Sigma^* \times \Gamma^*) \times (Q \times \Sigma^* \times \Gamma^*)$$

$$(q, ax, X\beta) \vdash_P (p, x, \gamma\beta), \text{ if } (p, \gamma) \in \delta(q, a, X)$$

$$(q, x, X\beta) \vdash_P (p, x, \gamma\beta), \text{ if } (p, \gamma) \in \delta(q, \varepsilon, X)$$

We may use \vdash instead of \vdash_P if P is understood.

\vdash is a binary relation on $(Q \times \Sigma^* \times \Gamma^*)$.

\vdash^* , reflexive transitive closure of \vdash .

Recursive definition of \vdash^ .*

$$\forall I \in Q \times \Sigma^* \times \Gamma^*, I \vdash^* I.$$

$$\text{If } I \vdash J \text{ and } J \vdash^* K, I \vdash^* K.$$

Theorem 6.5 If $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a PDA and

$(q, xy, \alpha) \vdash^* (p, y, \beta)$ for $q, p \in Q, x, y \in \Sigma^*$ and $\alpha, \beta \in \Gamma^*$. Then
 $(q, xyw, \alpha\gamma) \vdash^* (p, yw, \beta\gamma)$ for any $w \in \Sigma^*$ and $\gamma \in \Gamma^*$.

Theorem 6.6 If $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a PDA and

$(q, xyw, \alpha) \vdash^* (p, yw, \beta)$ for $q, p \in Q, x, y, w \in \Sigma^*$ and $\alpha, \beta \in \Gamma^*$. Then
 $(q, xy, \alpha) \vdash^* (p, y, \beta)$.

6.2 The language of a PDA

$$L(P) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (f, \varepsilon, \alpha), f \in F\}$$

language accepted by final state

$$N(P) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (f, \varepsilon, \varepsilon)\}$$

language accepted by null stack

6.2.3 From Empty Stack to Final State

Theorem 6.9 If $L = N(P_N)$ for some PDA P_N .

Then there is a PDA P_F such that $L = L(P_F)$.

Let $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_N, \emptyset)$.

$P_F = (Q \cup \{q_0', q_F\}, \Sigma, \Gamma \cup \{Z_F\}, \delta_F, q_0', Z_F, \{q_F\})$

where $q_0', q_F \notin Q, Z_F \notin \Gamma$.

δ_F : 1. $\delta_F(q_0', \varepsilon, Z_F) = \{(q_0, Z_N Z_F)\}$.

push old stack **bottom** Z_N .

2. $\delta_F \supseteq \delta_N$,

simulate P_N with δ_N .

3. $\forall q \in Q, \delta_F(q, \varepsilon, Z_F) = \{(q_F, Z_F)\}$.

If stack is **empty**(Z_F), go to the **final** state q_F

6.2.4 From Final State to Empty Stack

Theorem 6.11 If $L = L(P_F)$ for some PDA P_F

Then there is a PDA P_N such that $L = N(P_N)$.

Let $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_F, F)$.

$P_N = (Q \cup \{q_0', q_E\}, \Sigma, \Gamma \cup \{Z_N\}, \delta_N, q_0', Z_N, \emptyset)$

where $q_0', q_E \notin Q, Z_F \notin \Gamma$.

- δ_N :
1. $\delta_N(q_0', \varepsilon, Z_N) = \{(q_0, Z_F Z_N)\}$ *push old stack **bottom** Z_F*
 2. $\delta_N \supseteq \delta_F$ *simulate P_F with δ_F*
 3. $\forall f \in F, \forall Z \in \Gamma \cup \{Z_N\}, \delta_N(f, \varepsilon, Z) \supseteq \{(q_E, \varepsilon)\}$.

*If final state, **pop** stack symbol and go to the **empty** state q_E .*

4. $\forall Z \in \Gamma \cup \{Z_N\}, \delta_N(q_E, \varepsilon, Z) = \{(q_E, \varepsilon)\}$. **Empty** stack in q_E .

6.3.1 From Context-free Grammar to Pushdown Automata

Theorem 6.13 If $G = (V, T, P, S)$ is a cfg. Then \exists PDA P .s.t. $L(G) = N(P)$.

Construct $P = (\{q\}, T, V \cup T, \delta, q, S, \emptyset)$

$\forall A \in V, \delta(q, \varepsilon, A) = \{(q, \alpha) \mid A \rightarrow \alpha \in P\}$ **guess** A as $\alpha (A \rightarrow \alpha \in P)$.

$\forall a \in T, \delta(q, a, a) = \{(q, \varepsilon)\}$ **verify** $a \in T$.

guess and verify parser

Proof $A \Rightarrow_{lm}^* x\alpha$ if and only if $(q, x, A) \vdash^* (q, \varepsilon, \alpha)$, $x \in T^*$, $\alpha \in (V \cup T)^*$.

(If) If $(q, x, A) \vdash^i (q, \varepsilon, \alpha)$, then $A \Rightarrow_{lm}^* x\alpha$.

basis $i = 0, x = \varepsilon, \therefore (q, \varepsilon, A) \vdash^0 (q, \varepsilon, A). \therefore A \Rightarrow_{lm}^* A$.

induction Let $i \geq 1$, and consider the next-to-last step.

i) $(q, x, A) \vdash^{i-1} (q, \varepsilon, B\gamma) \vdash_{Guess} (q, \varepsilon, \beta\gamma) = (q, \varepsilon, \alpha)$

$\therefore A \Rightarrow_{lm}^* xB\gamma$ by IH and $B \rightarrow \beta \in P$ by construction of δ_{Guess} .

$\therefore A \Rightarrow_{lm}^* xB\gamma \Rightarrow_{lm} x\beta\gamma = x\alpha$.

$$\begin{aligned}
\text{ii) } (q, x, A) = (q, ya, A) &\vdash^{i-1} (q, a, a\alpha) \vdash_{\text{Verify}} (q, \varepsilon, \alpha) \\
(q, y, A) &\vdash^{i-1} (q, \varepsilon, a\alpha) \quad (\text{Thm 6.6; } (q, \underline{\varepsilon}, a\alpha)) \\
\therefore A &\Rightarrow_{lm}^* ya\alpha = x\alpha \text{ by IH}
\end{aligned}$$

(Only if) If $A \Rightarrow_{lm}^i x\alpha$, then $(q, x, A) \vdash^* (q, \varepsilon, \alpha)$.

basis $i = 0$, $A \Rightarrow_{lm}^0 A$. $(q, \varepsilon, A) \vdash^0 (q, \varepsilon, A)$.

induction Let $i \geq 1$, and consider the next-to-last step.

$A \Rightarrow_{lm}^{i-1} yB\gamma) \Rightarrow_{lm} y\beta\gamma = yy'\gamma'\gamma = x\alpha$ where $\beta = y'\gamma'$, $y' \in T^*$, $\gamma' \in (V \cup T)^*$.

$$\begin{aligned}
(q, y, A) &\vdash^* (q, \varepsilon, B\gamma) \text{ by IH, } (q, yy', A) \vdash^* (q, y', B\gamma) \text{ (by T.6.5)} \\
&\vdash_G (q, y', \beta\gamma) = (q, y', y'\gamma'\gamma) \vdash_{V^{/y'/}} (q, \varepsilon, \gamma'\gamma) = (q, \varepsilon, \alpha)
\end{aligned}$$

$\therefore A \Rightarrow_{lm}^* x\alpha$ if and only if $(q, x, A) \vdash^* (q, \varepsilon, \alpha)$.

If $A = S$, $\alpha = \varepsilon$, $S \Rightarrow_{lm}^* x$ if and only if $(q, x, S) \vdash^* (q, \varepsilon, \varepsilon)$

$\therefore L(G) = N(P)$.

6.3.2 From Pushdown Automata to Context-free Grammar

Theorem 6.14 If a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$.

Then there is a CFG G such that $L(G) = N(P)$.

proof $G = (Q \times \Gamma \times Q \cup \{S\}, \Sigma, P, S)$

$$P = \{S \rightarrow [q_0, Z_0, q] \mid \forall q \in Q\}$$

$$\cup \{[q, A, p_m] \rightarrow a [p, Y_1, p_1][p_1, Y_2, p_2] \dots [p_{m-1}, Y_m, p_m] \mid \\ (p, Y_1 \dots Y_m) \in \delta(q, a, A), a \in \Sigma \cup \{\varepsilon\}, \forall p_1, \dots, p_m \in Q\} \\ (\text{if } m = 0, [q, A, p] \rightarrow a \in P, a \in \Sigma \cup \{\varepsilon\})$$

$[q, A, p] \Rightarrow_{lm}^* x \in \Sigma^*$, if and only if $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

Nonterminal $[q, A, p]$ derives terminal string x if and only if

x causes PDA P to pop A from stack

starting in the state q and ending in the state p .

1) If $(q, x, A) \vdash^i (p, \varepsilon, \varepsilon)$, then $[q, A, p] \Rightarrow_{lm}^* x$.

basis $i = 1$, $(q, x, A) \vdash (p, \varepsilon, \varepsilon)$, $(p, \varepsilon) \in \delta(q, x, A)$, $x \in \Sigma \cup \{\varepsilon\}$.

$\therefore [q, A, p] \rightarrow x \in P$ where $x \in \Sigma \cup \{\varepsilon\}$.

induction $(q, x, A) = (q, ay, A) \vdash (p_1, y, Y_1 \dots Y_m) \vdash^{i-1} (p, \varepsilon, \varepsilon)$.

$\therefore \exists p_2, \dots, p_m, p \in Q$ and assume $y = y_1 \dots y_m \in \Sigma^*$.

$(p_1, y_1 \dots y_m, Y_1 \dots Y_m) \vdash^* (p_2, y_2 \dots y_m, Y_2 \dots Y_m) \vdash^* \dots (p_m, y_m, y_m) \vdash (p, \varepsilon, \varepsilon)$.

$1 \leq \forall i \leq m$, $(p_i, y_i, Y_i) \vdash^* (p_{i+1}, \varepsilon, \varepsilon)$. (Thm 6.5 and y_i depends only on Y_i)

$\therefore [p_i, Y_i, p_{i+1}] \Rightarrow_{lm}^* y_i$ by IH.

$\therefore [p_1, Y_1, p_2][p_2, Y_2, p_3] \dots [p_m, Y_m, p] \Rightarrow_{lm}^* y_1 y_2 \dots y_m = y$

Since $(q, ay, A) \vdash (p_1, y, Y_1 \dots Y_m)$, $(p_1, Y_1 \dots Y_m) \in \delta(q, a, A)$.

$\therefore \exists [q, A, p] \rightarrow a [p_1, Y_1, p_2][p_2, Y_2, p_3] \dots [p_m, Y_m, p] \in P. (\forall p_i \in Q)$

$\therefore [q, A, p] \Rightarrow_{lm} a [p_1, Y_1, p_2][p_2, Y_2, p_3] \dots [p_m, Y_m, p] \Rightarrow_{lm}^* ay = x$.

2) If $[q, A, p] \Rightarrow_{lm}^i x$, then $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

basis $i = 1$, $[q, A, p] \rightarrow x \in P$, $(p, \varepsilon) \in \delta(q, x, A)$ where $x \in \Sigma \cup \{\varepsilon\}$.

induction $[q, A, p] \Rightarrow_{lm} a [p_1, Y_1, p_2] \dots [p_m, Y_m, p] \Rightarrow_{lm}^{i-1} x \in \Sigma^*$.

$x = ay_1 \dots y_m$ where $1 \leq \forall i \leq m$, $[p_i, Y_i, p_{i+1}] \Rightarrow_{lm}^i y_i$ where $p_{m+1} = p$.

$\therefore (p_i, y_i, Y_i) \vdash^* (p_{i+1}, \varepsilon, \varepsilon)$ by IH.

Since $[q, A, p] \rightarrow a [p_1, Y_1, p_2] \dots [p_m, Y_m, p] \in P$,

$a [p_1, Y_1, p_2] \dots [p_m, Y_m, p] \in \delta(q, a, A)$.

$\therefore (q, ay_1 \dots y_m, A) \vdash (p_1, y_1 \dots y_m, Y_1 \dots Y_m) \vdash^* \dots \vdash^* (p, \varepsilon, \varepsilon)$.

$[q, A, p] \Rightarrow_{lm}^* x$, $\forall p \in Q$ if and only if $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

$[q_0, Z_0, p] \Rightarrow_{lm}^* x$, $\forall p \in Q$ if and only if $(q_0, x, Z_0) \vdash^* (p, \varepsilon, \varepsilon)$.

$S \Rightarrow_{lm}^* x$, $\forall p \in Q$ if and only if $(q_0, x, Z_0) \vdash^* (p, \varepsilon, \varepsilon)$.

Q.E.D.