

5.A Examples of Context-Free Grammars

3.1 and 3.2 of Du & Ko's book (pp. 91 - 108)

Example 3.1 Consider $S \rightarrow \varepsilon \mid aSb$. What is $L(S)$?

Solution $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \dots \Rightarrow a^n S b^n \Rightarrow a^n b^n$.

$$\therefore L(S) = \{a^n b^n \mid n \geq 0\}$$

Example 3.2 Consider $S \rightarrow ABA$, $A \rightarrow a \mid bb$, $B \rightarrow bS \mid \varepsilon$. What is $L(S)$?

Solution Derivation of the above grammar are:

$$\begin{aligned} S &\Rightarrow ABA \Rightarrow \underline{A}b\underline{S}A \Rightarrow \underline{A}b\underline{A}B\underline{A}A \Rightarrow \underline{A}b\underline{A}b\underline{S}A A \Rightarrow \underline{A}b\underline{A}b\underline{A}B\underline{A}A A \\ &= A(bA)^2 BA^3 \Rightarrow \dots \Rightarrow A(bA)^n BA^{n+1} \Rightarrow \dots \end{aligned}$$

$$\therefore L(S) = \{(a+bb)(ba + bbb)^n (a+bb)^{n+1} \mid n \geq 0\}$$

Example 3.3 Find a cfg for $L = \{a^n b^{2n} \mid n \geq 0\}$

Solution 1 $S \rightarrow aSbb$ always generates **matching** pair of a and bb .

$$S \Rightarrow aSbb \Rightarrow aaSbbbb \Rightarrow \dots \Rightarrow a^n S (bb)^n \Rightarrow a^n b^{2n}.$$

Solution 2 **Recursively** express longer string in L by shorter string in L .

$$a^{n+1} b^{2(n+1)} = a(a^n b^{2n})bb. \quad \text{But fails!}$$

If $S \Rightarrow^* a^n b^{2n}$, we **need** rule $S \rightarrow aSbb$ to get $a(a^n b^{2n})bb$.

The rule $S \rightarrow \varepsilon$ gets $S \Rightarrow^* a^n b^{2n}$ for $n \geq 0$.

$$\therefore S \rightarrow \varepsilon \mid aSbb$$

How about $a^{n+1} b^{2(n+1)} = a^n a b b b^{2n}$?

Example 3.4 $L = \{x \in (a+b)^* \mid x = x^R\}$ **palindromes**

If $y \in L$, $y = aya$ or $byb \in L$. $S \rightarrow aSa \mid bSb$ for $|x| \geq 2$.

$x = a$ or $b \in L$ for $|x| = 1$, and $x = \varepsilon$, for $|x| = 0$.

$$\therefore S \rightarrow \varepsilon \mid a \mid b \mid aSa \mid bSb.$$

Example 3.5 Find a CFG for

$L = \{x \in (a+b)^* \mid \text{prefix of } x \text{ has at least as many } a\text{'s as } b\text{'s}\}$

Solution

(1) $\varepsilon \in L$.

basis

(2) If $y \in L$, $ay \in L$ but $by \notin L$.

recursive expression

(3) If $y \notin L$, then y has a prefix which has **one more** b than a .

Let u be the **shortest such** ($\#_b(u) = \#_a(u) + 1$) prefix of y

where $\#_a(x)$ denotes the occurrence of $a \in \Sigma$ in $x \in \Sigma^*$.

Then $u = wb$ and $w \in L$. ($\#_a(w) = \#_b(w)$)

Then $\#_a(awb) = \#_b(awb)$ and if $z \in L$, $awbz \in L$.

\therefore If $w, z \in L$, $awbz \in L$.

recursive expression

(4) Nothing else is in L .

$\therefore S \rightarrow \varepsilon \mid aS \mid aSbS$.

$S \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSbS \Rightarrow ababS \Rightarrow ababaS \Rightarrow ababa$.

Example 3.6 $L = \{x \in (a+b)^* \mid x \text{ has as many } a\text{'s as } b\text{'s}\}$

Let $\#_a(x)$ denotes the number of occurrences of symbol a in x .

Solution 1. Let $d(x) = \#_a(x) - \#_b(x)$. $L = \{x \in (a+b)^* \mid d(x) = 0\}$

Consider u , the **nonempty shortest** prefix of $x \in L$. $\exists. d(u) = 0. (x = uz)$
and assume u **ends** with b , then u **starts** with a and.

proof of claim Let's assume $|u| = k$, $1 \leq \forall i \leq k$, u_i be the prefix of u , $|u_i| = i$.

Consider (u_1, u_2, \dots, u_k) sequence.

$d(u_i)$ and $d(u_{i-1})$ differs by **exactly one**, for $1 < \forall i \leq k$.

$d(u_k) = 0$, $d(u_{k-1}) = 1$, $d(u_i) \geq 1$ for $1 \leq \forall i \leq k-2$ (shortest). $\therefore u_1 = a$.

$u = ayb$, $x = aybz$. Since $d(x) = d(u) = 0$, $d(y) = d(z) = 0$.

\therefore If $y, z \in L$, $aybz \in L$, $byaz \in L$, and $\varepsilon \in L$.

$\therefore S \rightarrow \varepsilon \mid aSbS \mid bSaS$.

$S \Rightarrow aSbS \Rightarrow^n a^n (Sb)^n S \Rightarrow^* a^n b^n S \Rightarrow^m a^n b^n b^m (aS)^m \Rightarrow^* a^n b^{n+m} a^m$.

Solution 2. Let $L(A_k) = \{x \in (a+b)^* \mid d(x) = k\}$. Then $L = L(A_0)$

If $x \in L(A_0)$. Then

$$(1) x = \varepsilon,$$

$$(2) x = ay \text{ where } y \in L(A_{-1}), \text{ and}$$

$$(3) x = bz \text{ where } z \in L(A_1).$$

$$\therefore A_0 \rightarrow \varepsilon \mid aA_{-1} \mid bA_1$$

If $x \in L(A_{-1})$. Then

$$(1) x = ay \text{ where } y \in L(A_{-2}),$$

$$(2) x = bz \text{ where } z \in L(A_0).$$

If $z = z_1z_2 \in L(A_{-2})$, then $z_1, z_2 \in L(A_{-1})$.

$$\therefore A_0 \rightarrow \varepsilon \mid aA_{-1} \mid bA_1, \quad S \rightarrow \varepsilon \mid aA \mid bB,$$

$$A_1 \rightarrow aA_0 \mid bA_1A_1, \quad A \rightarrow aS \mid bAA,$$

$$A_{-1} \rightarrow bA_0 \mid aA_{-1}A_{-1}. \quad B \rightarrow bS \mid aBB.$$

Example 3.12 $L = \{x \in (a+b)^* \mid x \text{ has twice as many } b\text{'s as } a\text{'s}\}$

Solution Let $d(x) = 2\#_a(x) - \#_b(x)$.

$$A_0 \rightarrow \varepsilon \mid aA_{-1}A_{-1} \mid bA_1,$$

$$A_1 \rightarrow aA_{-1} \mid bA_2,$$

$$A_{-1} \rightarrow aA_{-1}A_{-1}A_{-1} \mid bA_0,$$

$$A_2 \rightarrow aA_0 \mid bA_1A_2 \mid bA_2A_1.$$

Example 3.13 $L = \{x \in (a+b)^* \mid \#_b(x) = 2\#_a(x) + 3\}$

Solution

Add a new start symbol A_3 and a rule

$$A_3 \rightarrow A_1A_2 \mid A_2A_1.$$

Example 3.14 $L = \{a^m b^n c^p d^q \mid m + n = p + q\}$

solution

1. In stage S , match a and d , $S \rightarrow aSd$. Then
if $m \geq q$, move to stage A , if $m \leq q$, move to stage B , $S \rightarrow A \mid B$,
2. In stage $A(m \geq q)$ more a , match a and c , then move to C , $A \rightarrow aAc \mid C$,
3. In stage $B(m \leq q)$ more d , match b and d , then move to C , $B \rightarrow bBd \mid C$,
4. In stage C , no more a nor d , match b and c , then finish. $C \rightarrow bCc \mid \varepsilon$.

$$S \rightarrow aSd \mid A \mid B,$$

$$A \rightarrow aAc \mid C,$$

$$B \rightarrow bBd \mid C,$$

$$C \rightarrow bCc \mid \varepsilon.$$

$$\begin{aligned} S &\Rightarrow^i a^i S d^i \Rightarrow a^i A d^i \Rightarrow^j a^i a^j A c^j d^i \Rightarrow^{k+1} a^{i+j} b^k C c^k c^j d^i \Rightarrow a^{i+j} b^k c^{k+j} d^i. \\ &\Rightarrow a^i B d^i \Rightarrow^j a^i b^j B d^j d^i \Rightarrow^{k+1} a^i b^j b^k C c^k d^j d^i \Rightarrow a^i b^{j+k} c^k d^{j+i}. \end{aligned}$$

Example 3.15 $L = \{a^m b^n c^p \mid m + 2n \geq p\}$

solution Consider $L_1 = \{a^m b^n c^p \mid m + 2n = p\}$

$$S \rightarrow aSc \mid A, \quad A \rightarrow bAcc \mid \varepsilon.$$

Now consider $L_2 = \{a^m b^n c^p \mid m + 2n > p\}$

$$\text{Extra } a \text{ or } b \quad S \rightarrow aS \quad \text{or } A \rightarrow bAc \mid bA$$

$$\therefore S \rightarrow aSc \mid aS \mid A, \quad A \rightarrow bAcc \mid bAc \mid bA \mid \varepsilon.$$

Example 3.16 $L = \{a^m b^n \mid 3m \leq 5n \leq 4m, m, n \in \mathbb{N}\}$

solution Consider $L_0 = \{a^m b^n \mid 3m \leq 5n \leq 4m, m \equiv 0 \pmod{5}\}$

$$m = 5p, 3p \leq n \leq 4p, a^{5p} b^n \in L_0.$$

$$L_0 = \{a^{5p} b^{3p+k} \mid 0 \leq k \leq p\}$$

$$A \rightarrow a^5 A b^3 \mid a^5 A b^4 \mid \varepsilon.$$

$$A \Rightarrow^k a^{5k} A b^{4k} \Rightarrow^{p-k} a^{5k} a^{5(p-k)} A b^{3(p-k)} b^{4k} \Rightarrow a^{5p} b^{3p+k}.$$

$$L_i = \{a^m b^n \mid 3m \leq 5n \leq 4m, m \equiv i \pmod{5}\}$$

$$m = 5p+4, 3p+3 \cdot 4/5 \leq n \leq 4p+4 \cdot 4/5 \text{ or } 3p+3 \leq n \leq 4p+3$$

$$a^{5p+4} b^n \in L_4 \text{ where } 3p+3 \leq n \leq 4p+3.$$

$$\text{if } p = 0, m = 4; 3 \leq n \leq 3, n = 0, a^4 b^3 \in L_4.$$

$$L_4 = \{a^{5p+4} b^{3p+k+3} \mid 0 \leq k \leq p\}.$$

$$S_4 \rightarrow a^4 A b^3. \quad \text{initial condition}$$

$$m = 5p+3, 3p + 3 \cdot 3/5 \leq n \leq 4p + 4 \cdot 3/5 \text{ or } 3p + 2 \leq n \leq 4p + 2.$$

$$\text{if } p = 0, 2 \leq n \leq 2, n = 2, a^3b^2 \in L_3.$$

$$S_3 \rightarrow a^3Ab^2.$$

$$m = 5p+2, 3p + 3 \cdot 2/5 \leq n \leq 4p + 4 \cdot 2/5 \text{ or } 3p + 2 \leq n \leq 4p + 1$$

$$\text{if } p = 0, m = 2; 2 \leq n \leq 1, \text{ no solution.}$$

$$\text{if } p = 1, m = 7; 5 \leq n \leq 6, n = 5 \text{ or } n = 6. a^7b^5, a^7b^6 \in L_2.$$

$$S_2 \rightarrow a^7Ab^5.$$

$$m = 5p+1, 3p+3 \cdot 1/5 \leq n \leq 4p+4 \cdot 1/5 \text{ or } 3p + 1 \leq n \leq 4p$$

$$\text{if } p = 0, m = 1; 1 \leq n \leq 0, \text{ no solution.}$$

$$\text{if } p = 1, m = 6; 4 \leq n \leq 4, n = 4. a^6b^4 \in L_1.$$

$$S_1 \rightarrow a^6Ab^4.$$

$$S \rightarrow A \mid a^4Ab^3 \mid a^3Ab^2 \mid a^6Ab^4 \mid a^7Ab^5,$$

$$A \rightarrow a^5Ab^3 \mid a^5Ab^4 \mid \varepsilon.$$

Regular grammar

A context-free grammar $G = (N, \Sigma, P, S)$ is a **regular**, if all of the productions are of the form:

$$A \rightarrow aB \in P \text{ or } A \rightarrow a \in P \text{ where } A, B \in V, a \in \Sigma. \\ \text{except } S \rightarrow \varepsilon \in P.$$

Extended regular grammar

A context-free grammar $G = (N, \Sigma, P, S)$ is a **extended regular**, if all of the productions are of the form:

$$A \rightarrow xB \in P \text{ or } A \rightarrow x \in P \text{ where } A, B \in N, x \in \Sigma^*.$$

Equivalence of regular and extended regular grammars

Let $G(N, \Sigma, P, S)$ be an extended regular grammar, Then there is a regular grammar $G' = (N', \Sigma, P', S')$ such that $L(G) = L(G')$.

Proof See Chomsky normal form of context-free grammar.

Equivalence of regular grammar and finite automata

Let $A = (Q, \Sigma, \delta, q_0, F)$ be an **extended** finite automata. Then there is a regular grammar $G = (Q, \Sigma, P, q_0)$ such that $L(A) = L(G)$.

Proof $P = \{q \rightarrow xp \mid p \in \delta(q, x), x \in \Sigma^*\} \cup \{f \rightarrow \varepsilon \mid f \in F\}$.

Prove $q \Rightarrow^* xp$, if and only if, $p \in \delta^*(q, x)$.

$\therefore q_0 \Rightarrow^* x$, if and only if, $\delta^*(q_0, x) \in F$.

Let $G = (N, T, P, S)$ be a regular grammar. There is a finite automata

$A = (N \cup \{f\}, T, \delta, S, \{f\})$ such that $L(G) = L(A)$.

Proof $\delta = \{B \in \delta(A, x) \mid A \rightarrow xB \in P, x \in \Sigma^*\} \cup \{f \in \delta(A, x) \mid A \rightarrow x \in P\}$.

Prove $B \in \delta^*(A, x)$, if and only if, $A \Rightarrow^* xB$.

$\therefore \delta^*(S, x) \in F$, if and only if, $S \Rightarrow^* x$.

Consider a **derivation** in regular grammar $G = (N, \Sigma, P, S)$ and **state transition** in the corresponding finite automaton $M = (Q, \Sigma, \delta, q_S, F)$.

Let $x, y, z \in \Sigma^*$, $A, B \in N$, $q_A, q_B \in Q$, $q_F \in F$, and $A \leftrightarrow q_A$, $B \leftrightarrow q_B$. Then

$$S \Rightarrow^* xA \Rightarrow xyB \Rightarrow^* xyz. \quad A \rightarrow yB \in P$$

$$\delta(q_S, xyz) \stackrel{*}{=} \delta(q_A, yz) = \delta(q_B, z) \stackrel{*}{=} q_F \in F. \quad q_B \in \delta(q_A, y)$$

<i>Grammars</i>	generating terminal strings
<i>Automata</i>	consuming terminal strings

Let $G(N, \Sigma, P, S)$ be an extended regular grammar, Then there is a regular grammar $G' = (N', \Sigma, P', [S])$ such that $L(G) = L(G')$.

proof $N' = \{[Ax] \mid A \rightarrow xyB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, y \in \Sigma^*\} \cup \{[\varepsilon]\}$

$P' = \{[A] \rightarrow [B] \mid A \rightarrow B \in P, B \in N\} \cup$

$\{[Ax] \rightarrow a[Axa] \mid A \rightarrow xayB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, y \in \Sigma^+, a \in \Sigma\} \cup$

$\{[Ax] \rightarrow a[B] \mid A \rightarrow xaB \in P, B \in N, x \in \Sigma^*, a \in \Sigma\} \cup$

$\{[Ax] \rightarrow a \mid A \rightarrow xa \in P, x \in \Sigma^*, a \in \Sigma\}.$

$\{[Ax] \rightarrow a[B] \mid A \rightarrow xaB \in P, B \in N \cup \{\varepsilon\}, x \in \Sigma^*, a \in \Sigma\} \cup$

$\{[\varepsilon] \rightarrow \varepsilon\}.$

Assume $A \rightarrow a_1 \dots a_n B \in P (n \geq 0)$. Then n -rules,

$[A] \rightarrow a_1 [A a_1]$, $[A a_1] \rightarrow a_2 [A a_1 a_2]$, ..., $[A a_1 \dots a_{n-2}] \rightarrow a_{n-1} [A a_1 \dots a_{n-1}]$,
and $[A a_1 \dots a_{n-1}] \rightarrow a_n [B]$.

Assume $A \rightarrow a_1 \dots a_n \in P (n \geq 0)$. Then n -rules,

$[A] \rightarrow a_1 [A a_1]$, $[A a_1] \rightarrow a_2 [A a_1 a_2]$, ..., $[A a_1 \dots a_{n-2}] \rightarrow a_{n-1} [A a_1 \dots a_{n-1}]$,
and $[A a_1 \dots a_{n-1}] \rightarrow a_n [\varepsilon]$.

$A \Rightarrow^* xB$ in G , if and only if, $[A] \Rightarrow^* x[B]$ in G' .

basis If $A \Rightarrow^0 xB$ in G , $x = \varepsilon$, $A = B$. Then $[A] \Rightarrow^0 [A] = x[B]$

recursion If $A \Rightarrow^n xB$ in G , Then $[A] \Rightarrow^* x[B]$ in G' .

$A \Rightarrow^{n-1} \Rightarrow yC \Rightarrow xB$ in G , Then $C \rightarrow zB \in P$, $x = yz \in \Sigma^*$, $B \in N \cup \{\varepsilon\}$.

$[A] \Rightarrow^* x[B]$