

Chap. 3 Regular Expressions and Languages

3.1 Regular Expressions over some alphabet Σ .

Basis:

1. The **constant** ε and \emptyset are **regular expressions**, denoting the languages $\{\varepsilon\}$ and \emptyset , respectively, i.e., $L(\varepsilon) = \{\varepsilon\}$, $L(\emptyset) = \emptyset$.
2. If $a \in \Sigma$, then **a** is a **regular expression**, denoting the languages $\{a\}$, i.e., $L(\mathbf{a}) = \{a\}$.

Induction:

1. If E and F are **regular expressions**, then $E + F$ is a **regular expression**, denoting **union** of $L(E)$ and $L(F)$, i.e., $L(E + F) = L(E) \cup L(F)$.
2. If E and F are **regular expressions**, then EF is a **regular expression**, denoting **concatenation** of $L(E)$ and $L(F)$, i.e., $L(EF) = L(E)L(F)$.

3. If E is a **regular expression**, then E^* is a **regular expression**, denoting **closure** of $L(E)$, i.e., $L(E^*) = (L(E))^*$.
4. If E is a **regular expression**, then (E) is a **regular expression**, denoting $L(E)$, i.e., $L((E)) = L(E)$.

Example 3.2 in p.89

Precedence of operators

1. *parenthesis*
2. *closure*
3. *concatenation*
4. *union*

Example 3.3 in p.91

Equivalence of regular expressions

Let R, S be regular expressions. We say $R = S$, if $L(R) = L(S)$.

3.2 Finite Automata and Regular Expressions

3.2.1 From DFA's to Regular Expressions

Theorem 3.4 *If $L = L(D)$ for some DFA D , then there is a **regular expressions** R such that $L = L(R)$.*

Proof *Let us suppose D 's states are $\{1, 2, \dots, n\}$ for some n .*

*Let us R_{ij}^k be a **regular expression** such that*

$$L(R_{ij}^k) = \{w \in \Sigma^* \mid \delta^{|w|}(i, w) = j, 1 \leq \forall m \leq |w|-1, \delta^m(i, m:w) \leq k\}$$

The RE R_{ij}^k denotes the set of strings that take fa D from state i to state j without going through any state numbered higher than k .

*When $k=n$, no **restriction**.*

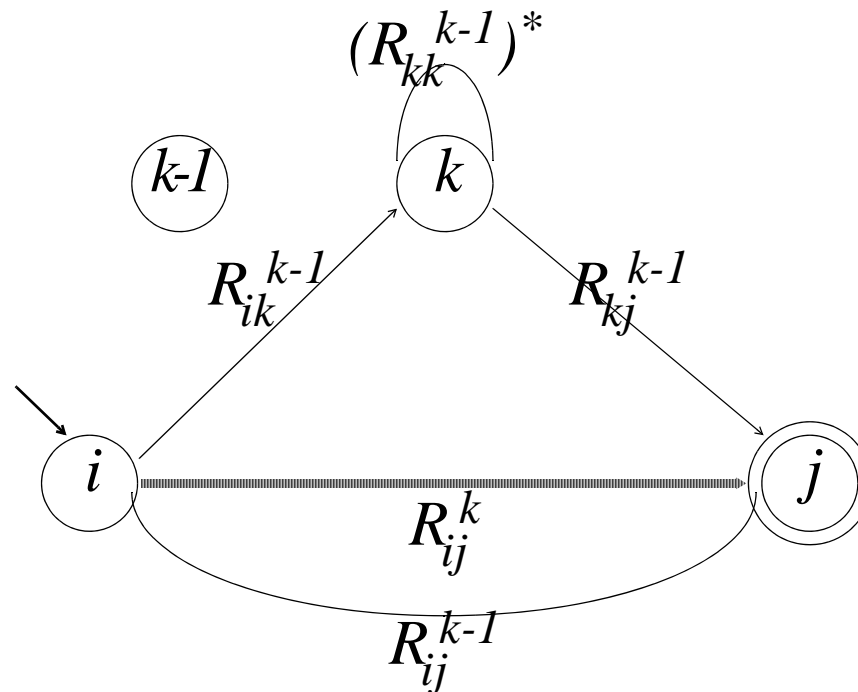
*We can construct R_{ij}^k $1 \leq \forall i \leq n$, $1 \leq \forall j \leq n$, and $0 \leq \forall k \leq n$ by **induction** on k .*

basis: $k = 0$, $1 \leq \forall i \leq n$, $1 \leq \forall j \leq n$.

$$\begin{aligned}
 R_{ij}^0 &= \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_p && \text{if } i \neq j, \text{ and } 1 \leq \forall q \leq p, \delta(i, a_q) = j, \\
 &= \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_p + \varepsilon && \text{if } i = j \text{ and } 1 \leq \forall q \leq p, \delta(i, a_q) = j, \\
 &= \emptyset && \text{otherwise.}
 \end{aligned}$$

induction: $1 \leq \forall i \leq n, 1 \leq \forall j \leq n$, all R_{ij}^{k-1} 's are known.

$$R_{ij}^k = R_{ik}^{k-1} (R_{kk}^{k-1})^* R_{kj}^{k-1} + R_{ij}^{k-1}.$$



Let s be the **start** state, and $F = \{f_1, \dots, f_g\}$ be final states.

$$R = R_{sf_1}^n + R_{sf_2}^n + \dots + R_{sf_g}^n \text{ such that } L(R) = L.$$

Example 3.5(pp 95) Figure 3.4

$$R_{11}^0 = \varepsilon + \mathbf{1} \quad R_{12}^0 = \mathbf{0} \quad R_{21}^0 = \emptyset \quad R_{22}^0 = \varepsilon + \mathbf{0} + \mathbf{1}$$

$$R_{ij}^k = R_{ik}^{k-1} (R_{kk}^{k-1})^* R_{kj}^{k-1} + R_{ij}^{k-1}$$

$$k=1 \quad R_{ij}^1 = R_{i1}^0 (R_{11}^0)^* R_{1j}^0 + R_{ij}^0.$$

$$\begin{aligned} R_{11}^1 &= R_{11}^0 (R_{11}^0)^* R_{11}^0 + R_{11}^0 = (\varepsilon + \mathbf{1})(\varepsilon + \mathbf{1})^*(\varepsilon + \mathbf{1}) + \varepsilon + \mathbf{1} \\ &= (\varepsilon + \mathbf{1})\mathbf{1}^*(\varepsilon + \mathbf{1}) + \varepsilon + \mathbf{1} = \mathbf{1}^* + \varepsilon + \mathbf{1} = \mathbf{1}^*. \end{aligned}$$

$$R_{12}^1 = R_{11}^0 (R_{11}^0)^* R_{12}^0 + R_{12}^0 = (\varepsilon + \mathbf{1})(\varepsilon + \mathbf{1})^*\mathbf{0} + \mathbf{0} = \mathbf{1}^*\mathbf{0} + \mathbf{0} = \mathbf{1}^*\mathbf{0}.$$

$$R_{21}^1 = R_{21}^0 (R_{11}^0)^* R_{11}^0 + R_{21}^0 = \emptyset(\varepsilon + \mathbf{1})^*(\varepsilon + \mathbf{1}) + \emptyset = \emptyset.$$

$$R_{22}^1 = R_{21}^0 (R_{11}^0)^* R_{12}^0 + R_{22}^0 = \emptyset(\varepsilon + \mathbf{1})^*\mathbf{0} + \varepsilon + \mathbf{0} + \mathbf{1} = \varepsilon + \mathbf{0} + \mathbf{1}.$$

$$k=2 \quad R_{ij}^2 = R_{i2}^1 (R_{22}^1)^* R_{2j}^1 + R_{ij}^1.$$

$$R_{11}^2 = R_{12}^1 (R_{22}^1)^* R_{21}^1 + R_{11}^1 = \mathbf{1}^* \mathbf{0} (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1})^* \emptyset + \mathbf{1}^* = \mathbf{1}^*.$$

$$\begin{aligned} R_{12}^2 &= R_{12}^1 (R_{22}^1)^* R_{22}^1 + R_{12}^1 \\ &= \mathbf{1}^* \mathbf{0} (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1})^* (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1}) + \mathbf{1}^* \mathbf{0} = \mathbf{1}^* \mathbf{0} (\mathbf{0} + \mathbf{1})^*. \end{aligned}$$

$$R_{21}^2 = R_{22}^1 (R_{22}^1)^* R_{21}^1 + R_{21}^1 = (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1}) (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1})^* \emptyset + \emptyset = \emptyset.$$

$$\begin{aligned} R_{22}^2 &= R_{22}^1 (R_{22}^1)^* R_{22}^1 + R_{22}^1 \\ &= (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1}) (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1})^* (\boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1}) + \boldsymbol{\varepsilon} + \mathbf{0} + \mathbf{1} = (\mathbf{0} + \mathbf{1})^*. \end{aligned}$$

$$R = R_{12}^2 = \mathbf{1}^* \mathbf{0} (\mathbf{0} + \mathbf{1})^*.$$

3.2.2 Converting DFA's to Regular Expressions by Eliminating States

Previous construction

n^3 equations

$O(4^n)$ symbols in the regular expression

Eliminating states

If we eliminate state s , all paths that went through s no longer exist.

labels: symbols \rightarrow possibly infinite strings

\rightarrow ***regular expression(closure)***

simultaneous equations(연립방정식)

n-equations and n-variables

eliminating variables

Simultaneous equations for each state with regular expressions

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a XFA and $\forall q \in Q$, R_q is a **regular equation**

$$L(R_q) = \{x \in \Sigma^* \mid \delta^*(q, w) \cap F \neq \emptyset\}. \text{ Then}$$

(q is assumed to be an initial state)

$$1 \leq \forall i \leq n, R_{q_i} = r_{i1}R_{q_1} + r_{i2}R_{q_2} + \dots + r_{in}R_{q_n} + s_i.$$

$$\text{where } 1 \leq \forall j \leq n, \exists p \geq 0, r_{ij} = \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_p,$$

for $1 \leq \forall k \leq p$, $q_j \in \delta(q_i, x_k)$ where $x_k \in \Sigma^*$ and

$s_i = \varepsilon$, if $q_i \in F$, and $s_i = \emptyset$, if $q_i \notin F$.

Note r_{ij} 's and s_i 's are **constant regular expressions**
and R_{q_i} 's are **unknown variables**.

Then n states (variables) and n equations.

We can solve the **linear** simultaneous equation

1. eliminate **variables**(states) by **substitution**(대입)
2. eliminate **recursive variable** by **closure**.

Let $q = \alpha q + \beta$ where α and β are **regular equation with variables**.

$$q = \alpha^* \beta.$$

strange solution

$$\begin{aligned} q &\neq 1/(1-\alpha) \beta \\ &= (1 + \alpha + \alpha^2 + \dots) \beta \\ &\neq \alpha^* \beta. \end{aligned}$$

Example 3.6 Figure 3.12(pp 101)

$$A = (\mathbf{0} + \mathbf{1})A + \mathbf{1}B$$

$$B = (\mathbf{0} + \mathbf{1})C$$

$$C = (\mathbf{0} + \mathbf{1})D + \varepsilon$$

$$D = \varepsilon$$

$$C = (\mathbf{0} + \mathbf{1})\varepsilon + \varepsilon = \mathbf{0} + \mathbf{1} + \varepsilon$$

$$B = (\mathbf{0} + \mathbf{1})(\mathbf{0} + \mathbf{1} + \varepsilon) = (\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1})$$

$$\begin{aligned} A &= (\mathbf{0} + \mathbf{1})A + \mathbf{1}((\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1})) \\ &= (\mathbf{0} + \mathbf{1})^* \mathbf{1}(\mathbf{0} + \mathbf{1})^2 + (\mathbf{0} + \mathbf{1})^* \mathbf{1}(\mathbf{0} + \mathbf{1}). \end{aligned}$$

Example 3.5 revisited(pp 95)

$$A = \mathbf{1}A + \mathbf{0}B$$

$$A = \mathbf{1}A + \mathbf{0}(\mathbf{0} + \mathbf{1})^* = \mathbf{1}^* \mathbf{0}(\mathbf{0} + \mathbf{1})^*.$$

$$B = (\mathbf{0} + \mathbf{1})B + \varepsilon$$

$$B = (\mathbf{0} + \mathbf{1})^* \varepsilon = (\mathbf{0} + \mathbf{1})^*.$$

DFA in Figure 2.14 (pp 63) revisited

$$\begin{aligned}
 A &= \mathbf{1}A + \mathbf{0}B & A &= \mathbf{1}A + \mathbf{0}(\mathbf{0} + \mathbf{10})^*(\mathbf{11}A + \mathbf{1}) \\
 & & &= (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})A + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1} \\
 & & &= (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})^*\mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1}
 \end{aligned}$$

$$\begin{aligned}
 B &= \mathbf{0}B + \mathbf{1}C & B &= \mathbf{0}B + \mathbf{11}A + \mathbf{10}B + \mathbf{1} = (\mathbf{0} + \mathbf{10})B + \mathbf{11}A + \mathbf{1} \\
 & & &= (\mathbf{0} + \mathbf{10})^*(\mathbf{11}A + \mathbf{1})
 \end{aligned}$$

$$C = \mathbf{1}A + \mathbf{0}B + \varepsilon$$

NFA Figure 2.9(pp 56)

$$C = \varepsilon \quad B = \mathbf{1}C = \mathbf{1}\varepsilon = \mathbf{1}$$

$$A = (\mathbf{0} + \mathbf{1})A + \mathbf{0}B = (\mathbf{0} + \mathbf{1})A + \mathbf{0}\mathbf{1} = (\mathbf{0} + \mathbf{1})^*\mathbf{0}\mathbf{1}$$

$$\therefore (\mathbf{1} + \mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{11})^*\mathbf{0}(\mathbf{0} + \mathbf{10})^*\mathbf{1} = (\mathbf{0} + \mathbf{1})^*\mathbf{0}\mathbf{1}$$

DFA in Figure 2.14 (pp 63) revisited

$$C = \underline{\mathbf{0}B} + \underline{\mathbf{1}A} + \varepsilon = \mathbf{A} + \varepsilon \quad B = \mathbf{0}B + \mathbf{1}C = \underline{\mathbf{0}B} + \underline{\mathbf{1}A} + \mathbf{1}\varepsilon = \mathbf{A} + \mathbf{1}$$

$$A = \underline{\mathbf{0}B} + \underline{\mathbf{1}A} = \mathbf{1}A + \mathbf{0}A + \mathbf{0}\mathbf{1} = (\mathbf{0} + \mathbf{1})A + \mathbf{0}\mathbf{1} = (\mathbf{0} + \mathbf{1})^*\mathbf{0}\mathbf{1}$$

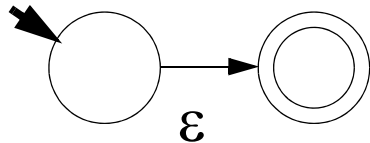
3.2.3 Converting Regular Expressions to Automata

Theorem 3.7 Every language defined by a regular expression is also defined by a finite automaton.

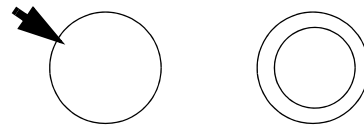
Proof Suppose $L = L(R)$ for some regular expression R .

We show that $L = L(E)$ for some ε -NFA E .

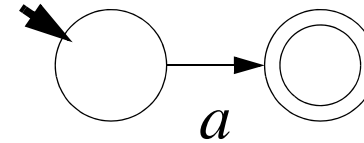
basis:



1. ε



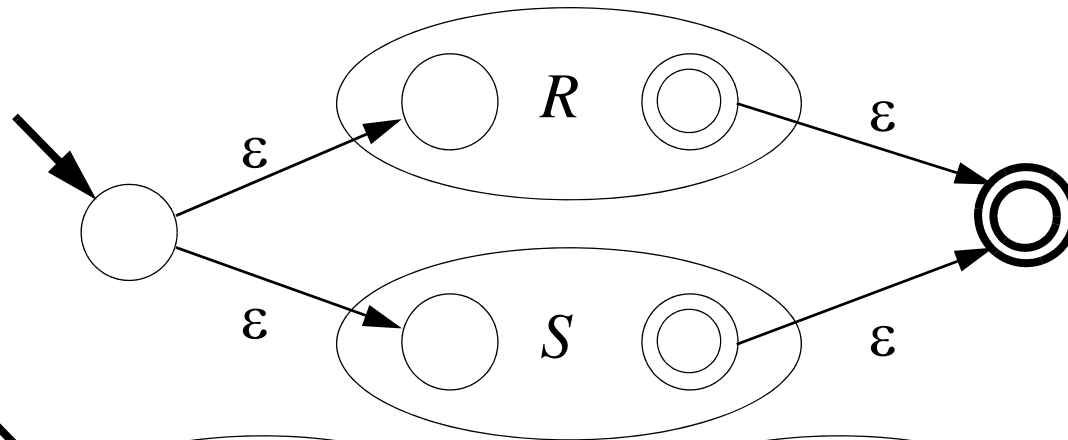
2. \emptyset



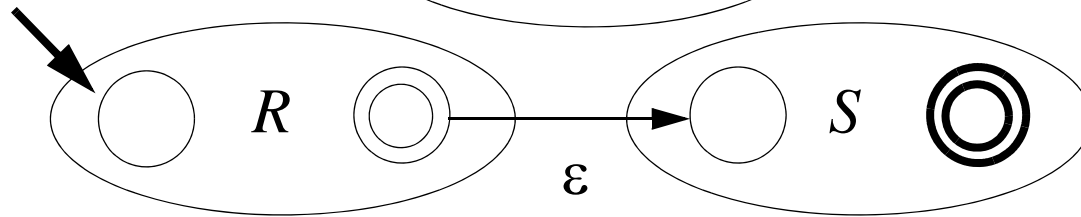
3. **a**

induction:

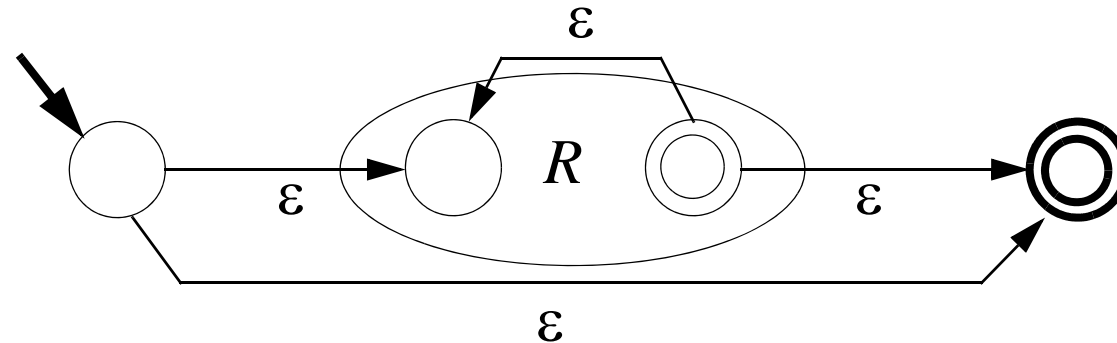
1. $R + S$



2. RS



3. R^*



3.4 Algebraic Laws for Regular Expressions

Let R, S, T be regular expressions over Σ .

$$R + S = S + R$$

union is commutative

$$(R + S) + T = R + (S + T)$$

union is associative

$$RS \neq SR$$

concatenation is non-commutative

$$(RS)T = R(ST)$$

concatenation is associative

$$\emptyset + R = R + \emptyset = R$$

\emptyset is the identity for union

$$\varepsilon R = R\varepsilon = R$$

ε is the identity for concatenation

$$\emptyset R = R\emptyset = \emptyset$$

\emptyset is the annihilator for concatenation

$$R(S + T) = RS + RT$$

concatenation distributes over union

$$(S + T)R = SR + TR$$

$$R + R = R$$

$$(R^*)^* = R^*$$

Union is *idempotent*

Closure is *idempotent*

$$\emptyset^* = \varepsilon \quad \varepsilon^* = \varepsilon$$

but $\emptyset^+ = \emptyset \quad \varepsilon^+ = \varepsilon.$

$$R^+ = RR^* = R^*R.$$

$$R^* = R^+ + \varepsilon$$

but $R^+ \neq R^* - \{\varepsilon\}$, if $\varepsilon \in R$.

$$\Sigma^* = \Sigma^+ + \varepsilon$$

and $\Sigma^+ = \Sigma^* - \{\varepsilon\}$ (since $\varepsilon \notin \Sigma$).

Theorem 3.A Any *finite* language is *regular*.

Proof Any *finite* language can be denoted by (finite) *regular expression*.

union and concatenation.

no closure

infinite

Following statements are logically equivalent

1. L is **regular**.
2. $L = L(D)$ for some DFA D .
3. $L = L(N)$ for some NFA N .
4. $L = L(E)$ for some ε -NFA E .
5. $L = L(X)$ for some XFA X .
6. $L = L(R)$ for some RE R .

Following statements are logically equivalent

1. L is **regular**.
2. $L = L(A)$ for some **finite automaton** A .
3. $L = L(R)$ for some **regular expression** R .
4. $L = L(G)$ for some **regular grammar** G . (TBD in Chap. 5)