

Repeated product(composition) of relation revisited

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The **product(composition)** of R and S , denoted $R \circ S \subseteq A \times C$, is

$$R \circ S = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$$

Repeated composition of relations

Let $R \subseteq A \times A$. Then we have defined, $R^n \subseteq A \times A$ ($n \geq 0$).

$$R^0 = id_A, \quad n = 0, \quad R^0 \text{ is an identity relation}$$

$$R^n = R^{n-1} \circ R \quad n \geq 1.$$

We also have defined, $R^*, R^+ \subseteq A \times A$.

$$R^* = \bigcup_{i \in N_0} R^i,$$

reflexive transitive closure of R

$$R^+ = \bigcup_{i \in N_+} R^i,$$

transitive closure of R .

Repeated composition of function

Let $f: A \rightarrow B$ and $g: B \rightarrow C$. The **product (composition)** of R and S , denoted $f \circ g: A \rightarrow C$, is

$$f \circ g = \{(a, c) \in A \times C \mid (a, b) \in f, (b, c) \in g\}$$

$$\text{or } = \{f \circ g(a) = c \mid f(a) = b, g(b) = c\}$$

$$\text{or } \quad f \circ g(a) = g(f(a))$$

We **extend** $f^n: A \rightarrow A$ for $n \geq 0$.

$$f^0 = id_A, \quad n = 0,$$

f^0 is an **identity function**

$$f^n = f^{n-1} \circ f \quad n \geq 1.$$

We also define, $f^*, f^+: A \rightarrow A$.

$$f^* = \cup_{i \in N_0} f^i,$$

$$f^+ = \cup_{i \in N_+} f^i.$$

Repeated composition of function with the second domain

Let $f: A \times B \rightarrow A$. We extend $f \circ f: A \times B^2 \rightarrow A$.

$$f \circ f = \{((a_0, b_1b_2), a_2) \in ((A \times B^2) \times A) \mid ((a_0, b_1), a_1) \in f, ((a_1, b_2), a_2) \in f\}$$

$$\text{or} = \{f \circ f(a_0, b_1b_2) = a_2 \mid f(a_0, b_1) = a_1, f(a_1, b_2) = a_2\}$$

$$\text{or} \quad f \circ f(a_0, b_1b_2) = f(f(a_0, b_1), b_2)$$

We extend $f^n: A \times B^n \rightarrow A$ for $n \geq 0$. Let $a \in A$, $\varepsilon \in B^0$, $w \in B^{n-1}$, $b \in B$.

$$f^0(a, \varepsilon) = a, \quad n = 0, \quad f^0(a, \varepsilon) \text{ is an **identity** function}$$

$$f^n(a, wb) = f^{n-1} \circ f \quad n \geq 1.$$

$$\text{or} = f(f^{n-1}(a, w), b)$$

We also define, $f^*: A \times B^* \rightarrow A$ and $f^+: A \times B^+ \rightarrow A$.

$$f^* = \cup_{i \in N_0} f^i,$$

$$f^+ = \cup_{i \in N_+} f^i.$$

Repeated composition of function whose range is a set

Let $f: A \times B \rightarrow 2^A$. We extend

$f': 2^A \times B \rightarrow 2^A$. Let $A' \subseteq A$ (or $A' \in 2^A$)

$$f'(A', b) = \cup_{a \in A'} f(a, b).$$

We may write f **instead** of f' , since $f \subseteq f'$, and
we also may write $f(a)$ **instead** of $f(\{a\})$ for short.

$$f: 2^A \times B \rightarrow 2^A.$$

We extend $f^n: 2^A \times B^n \rightarrow 2^A$ for $n \geq 0$. Let $A' \in 2^A$, $\varepsilon \in B^0$, $w \in B^{n-1}$, $b \in B$.

$$f^0(A', \varepsilon) = A', \quad n = 0, \quad f^0(A', \varepsilon) \text{ is an identity function}$$

$$f^n(A', wb) = f^{n-1} \circ f \quad n \geq 1.$$

$$\text{or } = f(f^{n-1}(A', w), b)$$

We also define $f^*: 2^A \times B^* \rightarrow 2^A$.

$$f^* = \cup_{i \in N_0} f^i.$$

$$f^+: 2^A \times B^+ \rightarrow 2^A.$$

$$f^+ = \cup_{i \in N_+} f^i.$$