is easy to see that M_3 accepts a word if and only if $L_1 \neq L_2$. Hence, by Theorem 3.7, there is an algorithm to determine if $L_1 = L_2$. □

3.4 THE MYHILL-NERODE THEOREM AND MINIMIZATION OF FINITE AUTOMATA

Recall from Section 1.5 our discussion of equivalence relations and equivalence classes. We may associate with an arbitrary language L a natural equivalence relation R_L; namely, $xR_L y$ if and only if for each z, either both or neither of xz and yz is in L. In the worst case, each string is in an equivalence class by itself, but there may be fewer classes. In particular, the *index* (number of equivalence classes) is always finite if L is a regular set.

There is also a natural equivalence relation on strings associated with a finite automaton. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. For x and y in Σ^* let $xR_M y$ if and only if $\delta(q_0, x) = \delta(q_0, y)$. The relation R_M is reflexive, symmetric, and transitive, since "=" has these properties, and thus R_M is an equivalence relation. R_M divides the set Σ^* into equivalence classes, one for each state that is reachable from q_0. In addition, if $xR_M y$, then $xzR_M yz$ for all z in Σ^*, since by Exercise 2.4,

$$\delta(q_0, xz) = \delta(\delta(q_0, x), z) = \delta(\delta(q_0, y), z) = \delta(q_0, yz).$$

An equivalence relation R such that xRy implies $xzRyz$ is said to be *right invariant* (*with respect to concatenation*). We see that every finite automaton induces a right invariant equivalence relation, defined as R_M was defined, on its set of input strings. This result is formalized in the following theorem.

Theorem 3.9 (*The Myhill-Nerode theorem*). The following three statements are equivalent:

1) The set $L \subseteq \Sigma^*$ is accepted by some finite automaton.
2) L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3) Let equivalence relation R_L be defined by: $xR_L y$ if and only if for all z in Σ^*, xz is in L exactly when yz is in L. Then R_L is of finite index.

Proof

(1) → (2) Assume that L is accepted by some DFA $M = (Q, \Sigma, \delta, q_0, F)$. Let R_M be the equivalence relation $xR_M y$ if and only if $\delta(q_0, x) = \delta(q_0, y)$. R_M is right invariant since, for any z, if $\delta(q_0, x) = \delta(q_0, y)$, then $\delta(q_0, xz) = \delta(q_0, yz)$. The index of R_M is finite, since the index is at most the number of states in Q. Furthermore, L is the union of those equivalence classes that include a string x such that $\delta(q_0, x)$ is in F, that is, the equivalence classes corresponding to final states.

(2) → (3) We show that any equivalence relation E satisfying (2) is a *refinement* of R_L; that is, every equivalence class of E is entirely contained in some equivalence class of R_L. Thus the index of R_L cannot be greater than the index of E and so is

finite. Assume that xEy. Then since E is right invariant, for each z in Σ^*, $xzEy$ and thus yz is in L if and only if xz is in L. Thus $xR_L y$, and hence the equivalence class of x in E is contained in the equivalence class of x in R_L. We conclude that each equivalence class of E is contained within some equivalence class of R_L.

$(3) \rightarrow (1)$ We must first show that R_L is right invariant. Suppose $xR_L y$, and let w be in Σ^*. We must prove that $xwR_L yw$; that is, for any z, xwz is in L exactly when ywz is in L. But since $xR_L y$, we know by definition of R_L that for any v, xv is in L exactly when yv is in L. Let $v = wz$ to prove that R_L is right invariant.

Now let Q' be the finite set of equivalence classes of R_L and $[x]$ the element of Q' containing x. Define $\delta'([x], a) = [xa]$. The definition is consistent, since R_L is right invariant. Had we chosen y instead of x from the equivalence class $[x]$, we would have obtained $\delta'([x], a) = [ya]$. But $xR_L y$, so xz is in L exactly when yz is in L. In particular, if $z = az'$, xaz' is in L exactly when yaz' is in L, so $xaR_L ya$, and $[xa] = [ya]$. Let $q_0' = [\epsilon]$ and let $F' = \{[x] \mid x \text{ is in } L\}$. The finite automaton $M' = (Q', \Sigma, \delta', q_0', F')$ accepts L, since $\delta'(q_0', x) = [x]$, and thus x is in $L(M')$ if and only if $[x]$ is in F'. □

Example 3.7 Let L be the language **0*10***. L is accepted by the DFA M of Fig. 3.2. Consider the relation R_M defined by M. As all states are reachable from the start state, R_M has six equivalence classes, which are

$$C_a = (\mathbf{00})^*, \qquad C_d = (\mathbf{00})^*\mathbf{01},$$
$$C_b = (\mathbf{00})^*\mathbf{0}, \qquad C_e = \mathbf{0}^*\mathbf{100}^*,$$
$$C_c = (\mathbf{00})^*\mathbf{1}, \qquad C_f = \mathbf{0}^*\mathbf{10}^*\mathbf{1}(\mathbf{0} + \mathbf{1})^*$$

L is the union of three of these classes, C_c, C_d, and C_e.

The relation R_L for L has $xR_L y$ if and only if either

i) x and y each have no 1's,

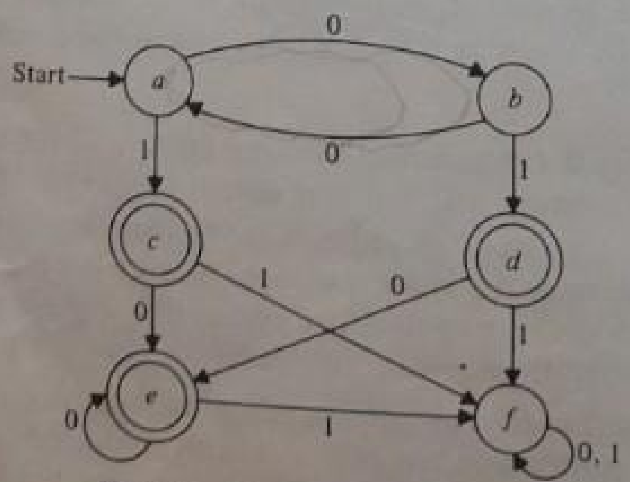

Fig. 3.2 DFA M accepting L.