Chap. 5 Context-Free Grammars

5.1 Context-Free Grammars

5.1.1 An Informal Example

madamimadam *"Madam, I'm Adam"*

A string w is ***palindrome****, if and only if* $w = w^R$.

Palindromes over {0, 1}

basis*:* ε, *0, and 1 are palindromes.*

induction*:* *If w is a palindrome, so are 0w0 and 1w1.*
No other string is palindrome.

Context-Free Grammar

1. $P \rightarrow \varepsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$

5.1.2 Definition of Context-Free Grammars

A guadruple $G = (N, T, P, S)$ is a ***context-free grammar****, if*

1. N is a ***finite*** *set of* ***variables(nonterminals****,* ***syntactic categories****),*

2. T is a ***finite*** *set of* ***terminal*** *symbols,*

where $N \cap T = \emptyset$.

3. P is a ***finite*** *set of* ***productions****(or* ***rules****),*

where each production is a pair $(A, \alpha) \in P$, written $A \rightarrow \alpha \in P$,

$A \in N$ ***left part(head)*** *of production*

$\alpha \in (N \cup T)^*$, ***right part(body)*** *of production*

4. $S \in N$ is a ***distinguished variable****, called* ***start(axiom) symbol****.*

Example 5.2

$G_{pal} = (\{P\}, \{0, 1\}, \{P \rightarrow \varepsilon, P \rightarrow 0, P \rightarrow 1, P \rightarrow 0P0, P \rightarrow 1P1, P\})$

We write $A \rightarrow \alpha_1 \mid \ldots \mid \alpha_n \in P$ ***instead*** *of $A \rightarrow \alpha_1, \ldots, A \rightarrow \alpha_n \in P$.*

Example 5.2' ***Compact Notations for Productions***

$G_{pal} = (\{P\}, \{0, 1\}, \{P \rightarrow \varepsilon \mid 0 \mid 1 \mid 0P0 \mid 1P1\}, P)$

Example 5.3' regular expressions *over* {**a**, **b**, **0**, **1**}

$E \rightarrow E + E \mid EE \mid E^* \mid (E) \mid B$ ***induction***

$B \rightarrow \underline{\varepsilon} \mid \underline{\varnothing} \mid \mathbf{a} \mid \mathbf{b} \mid \mathbf{0} \mid \mathbf{1}$ ***basis***

Note that $\underline{\varepsilon}$ *is not the* ***empty string***

but a ***symbol*** *denoting empty string for regular expression.*

$N = \{E, B\}$ $T = \{\underline{\varepsilon}, \underline{\varnothing}, \mathbf{a}, \mathbf{b}, \mathbf{0}, \mathbf{1}, +, ^*, (,)\}$

Example 5.3' regular expression *revisited*

$E \rightarrow E + E \mid EE \mid E^* \mid (E) \mid \underline{\varepsilon} \mid \underline{\varnothing} \mid \mathbf{a} \mid \mathbf{b} \mid \mathbf{0} \mid \mathbf{1}$

Example 5.3 식 *(Expressions in the text p. 174)*

1. 식 → 식 + 식
2. 식 → 식 * 식
3. 식 → (식)
4. 식 → 문자

4'. 식 → 수

5. 문자 → **a**
6. 문자 → **b**
7. 수 → **0**
8. 수 → **1**

5.1.3 Derivation Using a Grammar

Let $\alpha, \gamma \in (N \cup T)^*$ *and* $B \in N$, *and* $B \to \beta \in P$ *be a* ***production***.
We say string $\alpha B \gamma$ ***directly derives*** $\alpha\beta\gamma$ *in CFG G, written*

$\alpha B \gamma \Rightarrow_G \alpha\beta\gamma$, *we may omit G when it is understood,* $\alpha B \gamma \Rightarrow \alpha\beta\gamma$.

Note that $\to \subseteq N \times (N \cup T)^*$. *But we can extend* $\to \subseteq (N \cup T)^* \times (N \cup T)^*$.

$\to, \Rightarrow \subseteq (N \cup T)^* \times (N \cup T)^*$ $\to$ *and* $\Rightarrow$ *are* ***binary relation on*** $(N \cup T)^*$.

$\to \subseteq \Rightarrow$ $\Rightarrow$ *is an* ***induced*** *binary relation from* $\to$.

Note that $\to$ *is* ***finite*** *but* $\Rightarrow$ *is* ***infinite***. $\Rightarrow$ *is an* ***extension*** *of* $\to$.

Recursive *definition of* $\Rightarrow^n$ *for* $n \in \mathbb{N}_0$.

1. $\alpha \Rightarrow^0 \alpha, \forall \alpha \in (N \cup T)^*$. ***basis***

2. *For* $n \geq 1$, *if* $\alpha \Rightarrow^{n-1} \beta$, *and* $\beta \Rightarrow \gamma$, *then* $\alpha \Rightarrow^n \gamma$. ***recursion***

Definition of $\Rightarrow^*$.

$\Rightarrow^* = \cup_{i \in \mathbb{N}_0} \Rightarrow^i$. ***reflexive transitive closure*** *of* $\Rightarrow$.

We say α ***derives*** β, *if* $\alpha \Rightarrow^* \beta$ *for some* $\alpha, \beta \in (N \cup T)^*$.
We say α *is a* ***sentential form****(5.1.6)of G, if* $S \Rightarrow^* \alpha$ *for some* $\alpha \in (N \cup T)^*$.
We say w is a ***sentence*** *of G, if* $S \Rightarrow^* w$ *for some* $w \in T^*$.

The ***language*** *of G(5.1.5), denoted L(G), is* $L(G) = \{w \in T^* \mid S \Rightarrow^* w\}$.
A language L is ***context-free****, if there is a cfg G such that* $L = L(G)$.

Notational conventions for CFG

$a, b, c, \ldots \in T$	***terminal symbols***
$A, B, C, \ldots \in N$	***variable symbols***
$X, Y, Z, \ldots \in N \cup T$	***general symbols***
$x, y, z, \ldots \in T^*$	***terminal strings***
$\alpha, \beta, \gamma, \ldots \in (N \cup T)^*$	***general strings***

Derivation** of CFG is **nondeterministic

1. *Which variable to be replaced*
   - ***leftmost** vs. **rightmost***
2. *Which right hand side of the rule to be replaced*
   - $B \to \beta \mid \beta'$.

*5.1.4 **Leftmost** and **Rightmost Derivations***

***Leftmost derivation**, $\Rightarrow_{lm}$, to replace **leftmost** variable*

$$S \Rightarrow_{lm}^{*} xB\gamma \Rightarrow_{lm} x\beta\gamma \Rightarrow_{lm}^{*} xy\gamma \Rightarrow_{lm}^{*} xyz$$

where $x, y, z \in T^{*}$, $\gamma \in (N \cup T)^{*}$, $B \to \beta \in P$.

***Rightmost derivation**, $\Rightarrow_{rm}$, to replace **rightmost** variable*

$$S \Rightarrow_{rm}^{*} \alpha Bz \Rightarrow_{rm} \alpha\beta z \Rightarrow_{rm}^{*} \alpha yz \Rightarrow_{rm}^{*} xyz$$

where $x, y, z \in T^{*}$, $\alpha \in (N \cup T)^{*}$, $B \to \beta \in P$.

*Note that $\Rightarrow_{lm}, \Rightarrow_{rm} \subseteq \Rightarrow$. $\Rightarrow_{lm}$ and $\Rightarrow_{rm}$ are **reductions** of $\Rightarrow$.*

5.2 Parse Trees

5.2.1 Constructing Parse Trees

Let $G = (N, \Sigma, P, S)$ be a cfg. The ***parse tree*** *for G are trees*

1. *Each* ***interior*** *node is labelled by a* ***variable*** $A \in N$
2. *Each* ***leaf*** *node is labelled by a* ***terminal*** $a \in T$ *or* ε.
3. *If an interior node is labelled A and its children are labelled*
   $X_1, X_2, \ldots, X_n$ *from left to right*
   $A \rightarrow X_1X_2\ldots X_n \in P$.

Recursive(Top Down) definition(construction) *of (rooted)* ***parse tree***.

Basis *$(\{S\}, \varnothing, \underline{S})$ is a parse tree.*

Recursion *Let $(V, E, \underline{S})$ be parse trees and $A \rightarrow X_1X_2\ldots X_n \in P$.*

If the variable A is a node of $(V, E, \underline{S})$. Then

A subtree with root A is added to $(V, E, \underline{S})$. $(V', E', \underline{S})$ is a new parse tree,

$V' = V \cup \{X_1, X_2, \ldots, X_n\}$ *and* $E' = E \cup \{(A, X_i) | \ 1 \leq \forall i \leq n\}$.

*Two **futures** of a **new** leaf nodes X's in the parse tree $(V, E, \underline{S})$.*

*i) $X \in T \rightarrow$ the node X remains as a **leaf** node.*

*ii) $X \in N \rightarrow$ the node X will be an **interior** node(**root** of a **subtree**).*

***Recursive**(Bottum Up) **definition2**(construction) of **parse tree**.*

Basis $\forall a \in T$: $(\{a\}, \varnothing, a)$ *are parse trees(**forest**).*

Recursion *Let* $A \rightarrow X_1X_2\ldots X_n \in P$ *where* $1 \leq \forall i \leq n$: $X_i \in N \cup T$ *and*

$(V_1, E_1, X_1), (V_2, E_2, X_2), \ldots, (V_n, E_n, X_n)$ *be parse trees(**forest**).*

*Then (V, E, A) is a **new** parse tree where*

$V = \{A\} \cup V_1 \cup V_2 \cup \ldots \cup V_n$ *and*

$E = \{(A, X_i) \mid 1 \leq i \leq n\} \cup E_1 \cup E_2 \cup \ldots \cup E_n$.

5.2.2 The Yield of a Parse Tree

Concatenation** of **leaves** of a tree **from left to right

***Sentential form**$(N \cup T)^*$ or **sentence**(T^*).*

5.2.3 Inference, Derivation, and Parse Tree

Following four statements are ***equivalent*** *for some* ***terminal*** *string* $x \in T^*$.

(1) $A \Rightarrow^* x$, ***Derivation***

(2) $A \Rightarrow_{lm}^* x$, ***Leftmost derivation***

(3) $A \Rightarrow_{rm}^* x$, ***Rightmost derivation***

(4) *There is a parse tree with* ***root*** *A and* ***yield*** *x.*

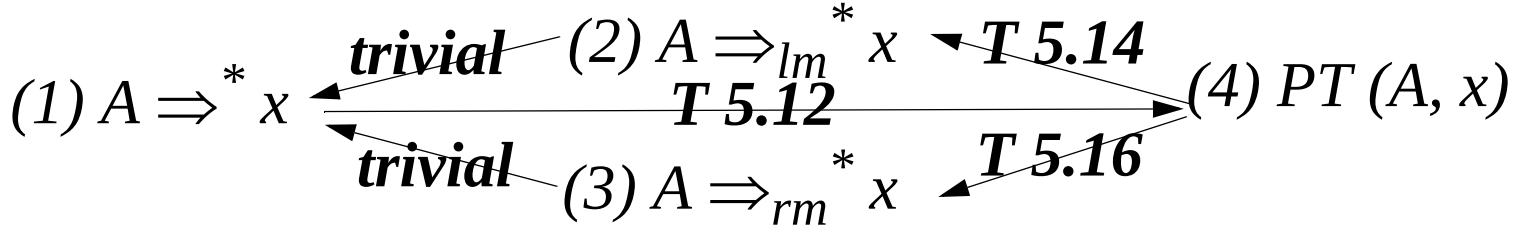
Proof $(2) \Rightarrow (1), (3) \Rightarrow (1)$ *are trivial* $(\Rightarrow_{lm}, \Rightarrow_{rm} \subseteq \Rightarrow)$.

5.2.4 From Inference to Trees

$(1) \Rightarrow (4)$: ***Thm. 5.12***

5.2.5 From Trees to Derivations

$(4) \Rightarrow (2), (4) \Rightarrow (3)$: ***Thm 5.14*** *and* ***5.16***



Theorem 5.12 *If* $A \Rightarrow^* x$, *then there is a parse tree with* ***root*** A *and* ***yield*** x

Proof *Induction on* ***number*** *of* ***derivations***

Basis $A \Rightarrow^1 x, A \rightarrow x \in P$. $\therefore$ *Parse tree in Fig. 5.8(p187).*

Induction *Assume* $A \rightarrow X_1X_2\ldots X_n \in P, 1 \leq^{\forall} i \leq n: X_i \Rightarrow^{k_i} x_i$, *and* $x = x_1x_2\ldots x_n$

1) If $X_i \in \Sigma$, $X_i = x_i$, $X_i \Rightarrow^{k_i} (\Rightarrow^0; =) x_i$. $\therefore$ ***parse tree*** *with* ***leaf*** $x_i \in \Sigma$.

2) If $X_i \in N$, $X_i \Rightarrow^{k_i} x_i$. $\therefore$ ***parse tree*** *with* ***root*** X_i *and* ***yield*** x_i*.(IH)*

$\therefore A \Rightarrow X_1X_2\ldots X_n \Rightarrow^{k_1} x_1X_2\ldots X_n \Rightarrow^{k_2} x_1x_2\ldots X_n \Rightarrow^{k_3} \ldots \Rightarrow^{k_n} x_1x_2\ldots x_n$.

$\therefore$ *If* $A \Rightarrow^m x_1x_2\ldots x_n = x (m \geq 1)$ *where* $\Sigma_n k_i = m-1$, *then*

parse tree *with* ***root*** A *and* ***yield*** x*.(Fig. 5.9; p188)*

Theorem 5.14 *If there is a parse tree with* ***root*** *A and* ***yield*** *x,* $A \Rightarrow_{lm}^{*} x$.

Proof *Induction on* ***height*** *of a tree*

Basis *Parse tree with height 1, in Fig. 5.8.* $A \to x \in P$. $A \Rightarrow_{lm} x$.

Induction *Consider a* ***parse tree*** *with* ***root*** *A, height m, and sons* $X_1, X_2, \ldots, X_n$ *from left to right, and* ***yield*** $x = x_1x_2\ldots x_n$*.(Fig. 5.9)*

1) If $X_i \in \Sigma$, $X_i = x_i$. $X_i \Rightarrow_{lm}^{0} x_i$.

2) If $X_i \in N$, $X_i \Rightarrow_{lm}^{+} x_i$*.(IH; with height is less than m)*

$A \Rightarrow_{lm} X_1X_2\ldots X_n \Rightarrow_{lm}^{*} x_1X_2\ldots X_n \Rightarrow_{lm}^{*} x_1x_2\ldots X_n \Rightarrow_{lm}^{*} x_1x_2\ldots x_n$.

Theorem 5.16 *If there is a parse tree with* ***root*** *A and* ***yield*** *x,* $A \Rightarrow_{rm}^{*} x$.

Proof *Induction on* ***height*** *of a tree(similar to leftmost derivation)*

$A \Rightarrow_{rm} X_1X_2\ldots X_n \Rightarrow_{rm}^{*} X_1X_2\ldots x_n \Rightarrow_{rm}^{*} X_1x_2\ldots x_n \Rightarrow_{rm}^{*} x_1x_2\ldots x_n$.

5.3 Application of Context-free Grammars

Context-free grammars

N. Chomsky late 1950's

***Syntax** of natural languages(English, French, ...)*

***Syntax** of programming languages*

ALGOL 60, Pascal, C, Java, Python, ML, ...

XML

***5.3.1 Parsers**(Two grammars) read text*

5.3.2 The Yacc Parser-generator

read 프로젝트 참고자료 (lex and yacc)

***5.3.3 Markup Languages** read text*

***5.3.4 XML and Document-Type Languages** read text*

5.4 Ambiguity in Grammars and Languages

G_{Aexp}: $E \rightarrow E + E \mid E * E \mid a \mid (E)$

$E \Rightarrow_{lm} E + E \Rightarrow_{lm} a + E \Rightarrow_{lm} a + E * E \Rightarrow_{lm} a + a * E \Rightarrow_{lm} a + a * a.$

$E \Rightarrow_{lm} E * E \Rightarrow_{lm} E + E * E \Rightarrow_{lm} a + E * E \Rightarrow_{lm} a * E \Rightarrow_{lm} a + a * a.$

A grammar G is said to be ***ambiguous****, if* $^{\exists}x \in L(G)$ *.э.*

x has more than one ***parse trees(syntactic structure)****,*

(x has more that one ***leftmost(rightmost)*** *derivation sequences)*

otherwise, ***unambiguous****.*

Derivation revisited

We may write $\alpha A\beta \Rightarrow^{r} \alpha\gamma\beta$, *if* $r = A \rightarrow \gamma \in P$.

Recursive extension of derivation with rules(rule string)

$\alpha \Rightarrow^{\varepsilon} \alpha$, *for* $\alpha \in (N \cup \Sigma)^{*}$, *and*

$\alpha \Rightarrow^{\pi r} \gamma$, *if* $\alpha \Rightarrow^{\pi} \beta$, $\beta \Rightarrow^{r} \gamma$ *for* $\pi \in P^{*}$, $r \in P$, $\alpha,\beta,\gamma \in (N \cup \Sigma)^{*}$.

Parser *of a grammar G,*

$^{\forall}x \in T^{*}$, *if* $x \in L(G)$ ***syntactic structure****(parse tree),*
otherwise say NO.

Is the parser for G is ***deterministic****?*

Not always!

It is ***undecidable*** *whether G is* ***ambiguous*** *or not.*

If G is ***unambiguous****, the parser for G may be* ***deterministic or not****.*
If G is ***ambiguous****, the parser for G is* ***nondeterministic****.*
If the parser for G is ***deterministic****, G is* ***unambiguous****.*

	parser	***structure***
regular	***deterministic****(FA)*	***linear***
context-free	***nondeterministic****(PDA=FA+****stack****)*	***hierarchical***

Deterministic parsing** of **context-free languages

*If $S \Rightarrow_{lm}^{\pi} x$ for $x \in T^*$, $\pi \in P^*$ is called the **left parse** of the sentence x.*

*If $S \Rightarrow_{rm}^{\pi} x$ for $x \in T^*$, $\pi^R \in P^*$ is called the **right parse** of the sentence x.*

parse tree** ⇔ **left parse** ⇔ **right parse** ⇔ **syntactic structure

***left(right) parser**: **left(right)** parse*

*LL(k): **Left-to-right scan** in **Leftmost derivation** with k-lookahead symbols*
*LR(k): **Left-to-right scan** in **Rightmost derivation** with k-lookahead symbol*

Left parser *same direction in scan and derivation*
***normal order, top-down parsing**(LL parsing)*
Right parser *different direction in scan and derivation*
***reversed order, bottom-up parsing**(LR, LALR, SLR parsing)*

Removing ambiguity in the grammar

G_{Aexp}: $E \rightarrow E + E \mid E * E \mid a \mid (E)$ $\quad |G_{Aexp}| = 4+4+2+4 = 14$

*Assume that precedence of * is higher than that of +, and*
*+ and * are left associative.*

G_{Uexp}: $E \rightarrow E + T \mid T * F \mid a \mid (E)$

$T \rightarrow T * F \mid a \mid (E)$

$F \rightarrow a \mid (E)$ $\quad |G_{Uexp}| = 14+10+6 = 30$

$G_{Uexp'}$: $E \rightarrow E + T \mid T$

$T \rightarrow T * F \mid F$

$F \rightarrow a \mid (E)$ $\quad |G_{Uexp'}| = 6+6+6 = 18$

$|G| = \Sigma_{A\rightarrow\alpha\in P} |A| + |\alpha| = \Sigma_{A\rightarrow\alpha\in P} |\alpha| + 1$ ***Size** of a grammar*

$A \rightarrow B$ *is called* ***unit production****, if* $A, B \in N$.

***reduces** the **size** of grammar but **increases** the **height** of the P.T.!*

*in yacc, G_{Aexp} is used with **precenece** and **associativity** of operators*

*A **context free language** L is said to be **inherently ambiguous**,*
*if **every** cfg G .ə. L = L(G) is **ambiguous**.*

$L = \{a^n b^n c^m d^m \mid n, m \geq 1\} \cup \{a^n b^m c^m d^n \mid n, m \geq 1\}$ *is **inherently ambiguous**.*

Consider G:

$S \rightarrow AB \mid C$
$A \rightarrow aAb \mid ab$
$B \rightarrow cBd \mid cd$
$C \rightarrow aCd \mid aDd$
$D \rightarrow bDc \mid bc$

and the sentence $a^n b^n c^n d^n$.

$S \Rightarrow_{lm} AB \Rightarrow_{lm}^{A\rightarrow aAb} aAbB \Rightarrow_{lm}^{(A\rightarrow aAb)^{n-2}} a^{n-1}Ab^{n-1}B \Rightarrow_{lm}^{A\rightarrow ab} a^n b^n B$

$\Rightarrow_{lm}^{B\rightarrow cBd} a^n b^n cBd \Rightarrow_{lm}^{(B\rightarrow cBd)^{n-2}} a^n b^n c^{n-1}Bd^{n-1} \Rightarrow_{lm}^{B\rightarrow cd} a^n b^n c^n d^n.$

$S \Rightarrow_{lm} C \Rightarrow_{lm}^{C\rightarrow aCd} aCd \Rightarrow_{lm}^{(C\rightarrow aCd)^{n-2}} a^{n-1}Cd^{n-1} \Rightarrow_{lm}^{C\rightarrow ad} a^n Dd^n$

$\Rightarrow_{lm}^{D\rightarrow bDc} a^n bDcd^n \Rightarrow_{lm}^{(D\rightarrow bDc)^{n-2}} a^n b^{n-1}Dc^{n-1}d^n \Rightarrow_{lm}^{D\rightarrow bc} a^n b^n c^n d^n.$