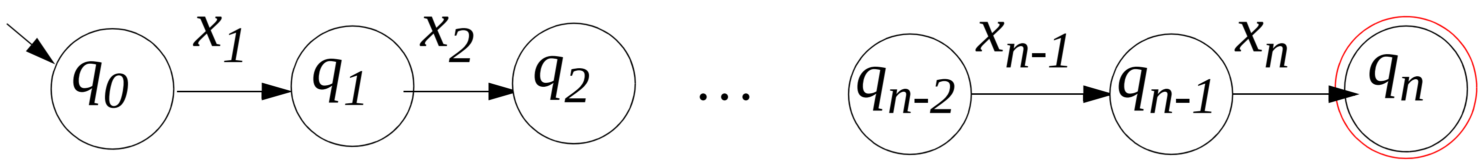
5-B Finite Automata and Regular Gramars

Consider a **finite automaton** $A = (Q, T, \delta, q_0, F)$ where $\delta: Q \times T^* \to 2^Q$.

Assume $1 \le {}^\forall i \le n$: $q_i \in \delta(q_{i-1}, x_i)$, $q_i \in Q$, $x_i \in T^*$, $q_n \in F$. Then

$q_1 \in \delta(q_0, x_1), q_2 \in \delta(q_1, x_2), \ldots, q_n \in \delta(q_{n-1}, x_n), q_n \in F$ or

$q_n \in \delta(\delta(\ldots\delta(\delta(q_0, x_1), x_2), \ldots, x_{n-1}), x_n) = \delta^n(q_1, x_1x_2\ldots x_n) \in F$.



Consider a **grammar** $G = (N, T, P, S)$.

Assume $1 \le {}^\forall i \le n$: $A_{i-1} \to x_iA_i \in P$, $A_i \in N$, $x_i \in T^*$, $A_n \to \varepsilon \in P$. Then

$A_0 \Rightarrow x_1A_1 \Rightarrow \ldots \Rightarrow x_1x_2\ldots x_{n-1}A_{n-1} \Rightarrow x_1x_2\ldots x_{n-1}x_nA_n \Rightarrow x_1x_2\ldots x_{n-1}x_n$.

$A_0 \Rightarrow^n x_1x_2\ldots x_{n-1}x_nA_n \Rightarrow^{A_n \to \varepsilon} x_1x_2\ldots x_{n-1}x_n = x \in T^*$.

Consider *a grammar G = (N, T, P, S) where*

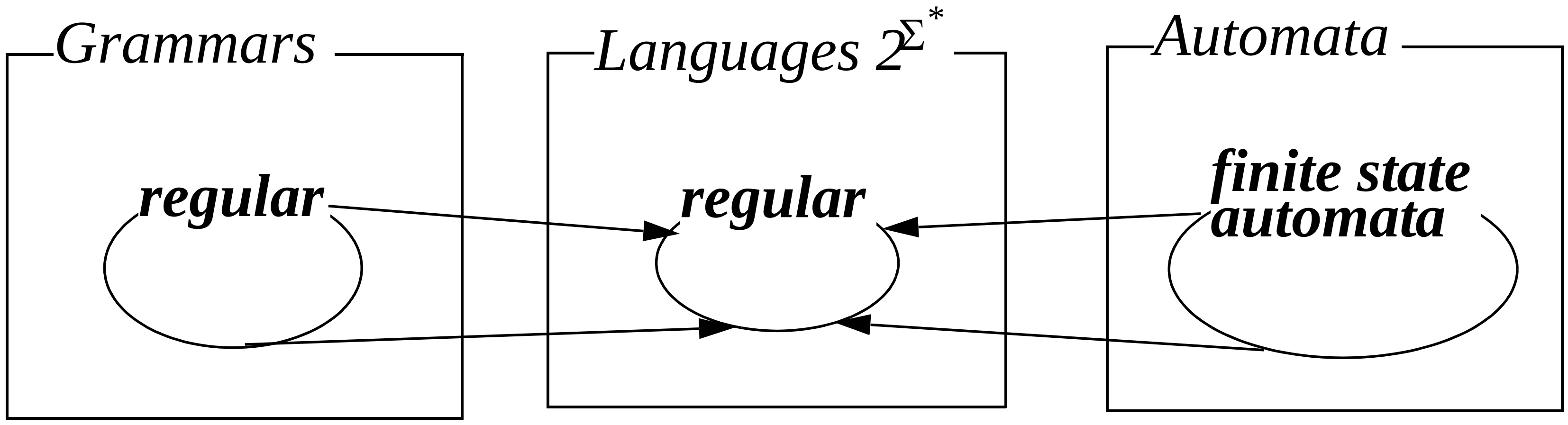
$A \rightarrow xB$ or $A \rightarrow \varepsilon \in P$ where $A, B \in N$ and $x \in T^*$.

Definition *A grammar G = (N, T, P, S) is* ***regular****, if*

$A \rightarrow xB$ or $A \rightarrow x \in P$ where $A, B \in N$ and $x \in T^*$.

Theorem Equivalance *of* ***finite automata*** *and* ***regular grammars****.*

proof Lem. 1 *and* ***Lem. 2****.*



Lem*. 1** *Let* $A = (Q, T, \delta, q_0, F)$ *be a* ***finite automaton*(XFA) with* $\delta: Q \times \Sigma^* \to 2^Q$ *and* $G = (N, T, P, S)$ *be a* ***regular grammar*** *where*

(1) $N = \{[q] \mid q \in Q\} \leftrightarrow Q$,

(2) $T = T$,

(3) $P = \{[q] \to x[p] \mid p \in \delta(q, x)\} \cup \{[f] \to \varepsilon \mid f \in F\}$, *and*

(4) $S = [q_0]$. *Then* $L(A) = L(G)$.

proof $p \in \delta^*(q, x)$, *iff* $[q] \Rightarrow_G^* x[p]$ *is* ***trivial***$({}^? p \in \delta(q, x) \leftrightarrow [q] \to x[p])$.

If $\delta^*(q_0, x) \in F$, *then* $[q_0] \Rightarrow_G^* x[f]$, $f \in F$.

$\therefore [f] \to \varepsilon \in P$.

$\therefore S = [q_0] \Rightarrow_G^* x[f] \Rightarrow_G^{[f]\to\varepsilon} x$. $\therefore L(A) \subseteq L(G)$.

If $S = [q_0] \Rightarrow_G^* x[p] \Rightarrow_G^{[p]\to\varepsilon} x$., *then*

$p \in \delta^*(q_0, x)$ *and* $p \in F$. $\therefore \delta^*(q_0, x) \in F$. $\therefore L(G) \subseteq L(A)$.

$\therefore L(A) = L(G)$.

Lem. 2 *Let* $G_{rg} = (N, T, P, S)$ *be a* ***regular grammar*** *and* $A = (Q, \Sigma, \delta, q_0, F)$ *be a* ***finite automaton*** *where*

(1) $Q = \{[A] \in Q \mid A \in N\} \cup \{[Ax] \in Q \mid A \to x \in P\}$,

(2) $T = T$,

(3) $\delta = \{[B] \in \delta([A], x) \mid A \to xB \in P\}$
$\cup \{[Ax] \in \delta(q, x) \mid A \to x \in P\}$,

(4) $q_0 = [S] (q_0 \leftrightarrow [S])$, and

(5) $F = \{[Ax] \in Q \mid A \to x \in P\}$. Then $L(G) = L(A)$.

proof $A \Rightarrow_{rg}^{*} xB$, iff $[B] \in \delta^{*}([A], x)$ is ***trivial***(?1st part of δ).

If $S \Rightarrow_{rg}^{*} x$, then $\exists k \geq 0$.э. $x = (|x|-k):x \cdot x:k$, $[Ax:k] \in \delta^{*}(q_0, |x|-k:x)$ and
$\delta([Ax:k], x:k) \in F$. $\therefore \delta^{*}(q_0, x) \in F$. $\therefore L(G) \subseteq L(A)$.

If $\delta^{*}(q_0, x) \in F$, then $\exists k \geq 0$: $x = (|x|-k):x \cdot x:k$, $S \Rightarrow^{*} |x|-k:xA \Rightarrow^{A \to x:k} x$.
$\therefore L(A) \subseteq L(G)$. $\therefore L(G) = L(A)$.

A ***Rewriting system*** *(or* ***Semi-Thue system****) $R = (V^*, \rightarrow)$ where*
(1) V is a set of ***configurations*** *(or* ***instantaneous descriptions****; ID), and*
(2) $\rightarrow$ is a ***finite*** *set of* ***relation on*** *a free* ***monoid*** *V^* $(\rightarrow \subseteq V^* \times V^*)$.*

A pair $(\omega_1, \omega_2) \in \rightarrow$ is called a ***rule*** *(or* ***production****) of R, and*

denoted by $\omega_1 \rightarrow \omega_2$.

The string ω_1 is called the ***left-hand*** *side(좌변) and*

and ω_2 the ***right-hand*** *side(우변) of the rule $\omega_1 \rightarrow \omega_2$.*

If γ is a string in V^* that can be ***decomposed*** *as $\alpha\omega_1\beta$ where ω_1 is left-hand side of the rule, then $\gamma = \alpha\omega_1\beta$ can be* ***written*** *as $\alpha\omega_2\beta$ where ω_2 is* ***right****-hand side of the rule,* ***denoted*** *as $\alpha\omega_1\beta \Rightarrow_R^{\omega_1 \rightarrow \omega_2} \alpha\omega_2\beta$.*

Let R = (V, P) be a rewriting system. If $r = \omega_1 \rightarrow \omega_2 \in P$. Then we define

*$\Rightarrow_R^r$ (or $\Rightarrow^r$ for short) on V^* by ($\rightarrow^r$ is unique(****finite****) but $\Rightarrow^r$ is* ***infinite****)*

$\Rightarrow_R^{\,r} = \Rightarrow_R^{\omega_1 \to \omega_2} = \{(\alpha\omega_1\beta, \alpha\omega_2\beta) \in V^* \times V^* \mid \alpha, \beta \in V^*\} \subseteq V^* \times V^*$.

If $\gamma_1, \gamma_2 \in V^*$, $\gamma_1 \Rightarrow_R^{\,r} \gamma_2$, *then we say that*

in R γ_1 ***derives*** γ_2 *using* ***rule r*** *and that*

rule r *is* ***applicable*** *to* γ_1*(or* ***can be applied to*** γ_1*).*

Let $\gamma \in V^*$*. Then we define* $\Rightarrow_R^{\pi}$ *(or* $\Rightarrow^r$ *for short)* ***recursively*** *as follows*

basis*:* $\Rightarrow_R^{\varepsilon}$ id_{V^*} *;* *Note that* $\Rightarrow_{G'}^{\varepsilon} = id_{V^*}$ *for any* $G' = (V, P')$*.*

recursion*:* $\Rightarrow_R^{\pi} = \Rightarrow_R^{\,r} \Rightarrow_R^{\pi'} \in P^+$ *where* $\pi = r\pi'$ *for* $r \in P$, $\pi' \in P^*$.

*We **can** specify **initial** and **final configurations** of rewriting systems as*

$R = (V^*, \rightarrow, \iota, \Phi)$

(3) $\iota \in V^*$ *is an **initial configuration**, and*

(4) $\Phi \subseteq V^*$ *is a set of **final configurations**.*

$$L(R) = \{\phi \in \Phi \mid \iota \Rightarrow_R^* \phi \in \Phi\}.$$

*A **rewriting system** for a finite automaton* $A = (Q, T, \delta, q_0, F)$ *is*

$R_A = (Q \times T^*, \rightarrow, (q_0, x), (F \times \Sigma^*))$ *where*

$$\rightarrow = \{(q, x) \rightarrow (p, \varepsilon) \mid p \in \delta(q, x)\}$$

$$L(R_A) = \{x \in T^* \mid (q_0, x) \Rightarrow_{R_A}^* (f, \varepsilon) \in (F \times \Sigma^*)\}.$$

*A **rewriting system** for a grammar* $G = (N, T, P, S)$ *is*

$R_G = ((N \cup T)^*, P, S, T^*)$ *where*

$$L(R_G) = \{x \in T^* \mid S \Rightarrow_{R_G}^+ x \in T^*\}.$$

***Rewritings** in **finite automata** A = (Q, Σ, δ, q_0, F) is*

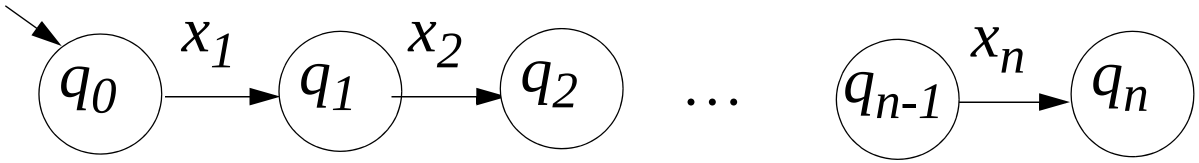
$R_{fa} = (Q \times \Sigma^*, \rightarrow_{fa}, (q_0, x), \{(f, \varepsilon) | f \in F\})$

where $(q, x) \rightarrow_{fa} (p, \varepsilon)$, if $p \in \delta(q, x)$.

Let $x = x_1x_2 \ldots x_n \in \Sigma^*$ for $n \geq 0$ and $1 \leq {}^\forall i \leq n$: $q_i \in \delta(q_{i-1}, x_i)$, i.e.,

$q_n \in \delta(\ldots \delta(\delta(q_0, x_1), x_2), \ldots), x_n)$ or

$q_1 \in \delta(q_0, x_1), q_2 \in \delta(q_1, x_2), \ldots, q_n \in \delta(q_{n-1}, x_n)$, if and only if,.



$1 \leq {}^\forall i \leq n$: $r_i = (q_{i-1}, \mathbf{x_i}) \rightarrow_{fa} (q_i, \varepsilon)$

$(q_0, x_1x_2\ldots x_n) \Rightarrow_{fa}^{(q_0, \mathbf{x_1})\rightarrow(q_1, \varepsilon)} (q_1, x_2\ldots x_n) \Rightarrow_{fa}^{(q_1, \mathbf{x_2})\rightarrow(q_2, \varepsilon)} \ldots$

$\ldots \Rightarrow_{fa} (q_{n-1}, x_n) \Rightarrow_{fa}^{(q_{n-1}, \mathbf{x_n})\rightarrow(q_n, \varepsilon)} (q_n, \varepsilon), q_n \in \boldsymbol{F}.$

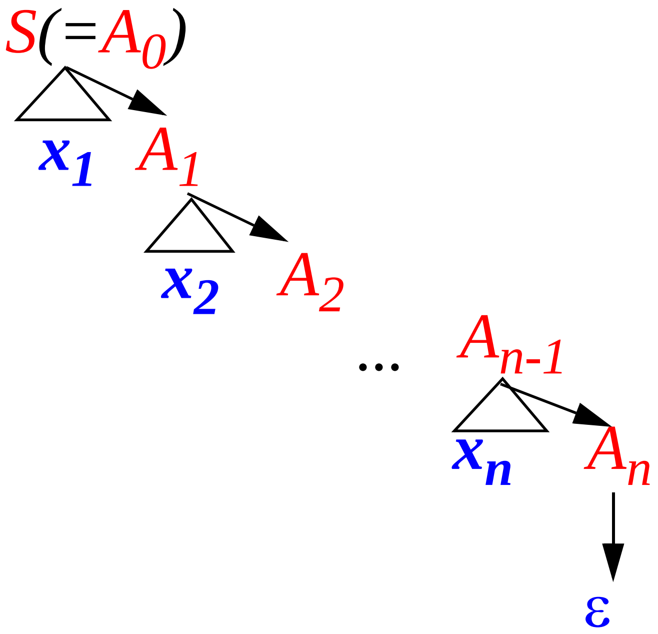
$(q_0,\mathbf{x_1}x_2\ldots x_n) \xrightarrow{(q_0,\mathbf{x_1})\rightarrow(q_1,\varepsilon)} (q_1,\mathbf{x_2}\ldots x_n) \xrightarrow{(q_1,\mathbf{x_2})\rightarrow(q_2,\varepsilon)} \ldots (q_{n-1},x_n) \xrightarrow{(q_{n-1},\mathbf{x_n})\rightarrow(q_n,\varepsilon)} (q_n,\varepsilon)$

Rewritings *in the* ***regular grammar*** $G = (N, T, P_{rg}, S)$ *is*

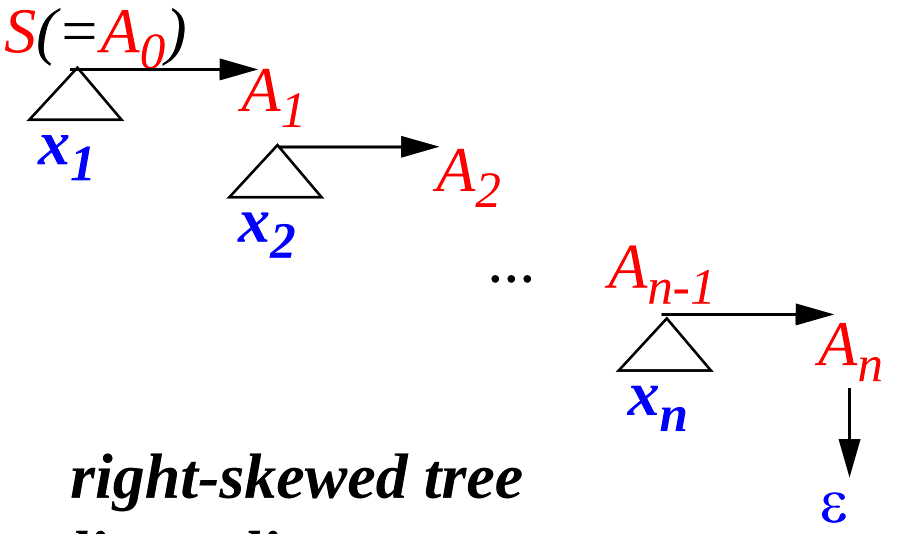
$R_{rg} = ((N \cup T)^*, P_{rg}, S, T^*)\ A \rightarrow_{rg} xB$, *if* $A \in N$, $x \in T^*$, *and* $B \in N \cup \{\varepsilon\}$.

$1 \leq \forall i \leq n: r_i = A_{i-1} \rightarrow_{rg} x_i A_i$ *and* $r_{n+1} = A_n \rightarrow_{rg} \varepsilon$.

$A_0 \Rightarrow_{rg}^{A_0 \rightarrow x_1 A_1} x_1 A_1 \Rightarrow_{rg}^{A_1 \rightarrow x_2 A_2} x_1 x_2 A_2 \ \ldots \Rightarrow_{rg}^{A_{n-2} \rightarrow x_{n-1} A_{n-1}} x_1 x_2 \ldots x_{n-1} A_{n-1}$
$\Rightarrow_{rg}^{A_{n-1} \rightarrow x_n A_n} x_1 x_2 \ldots x_n A_n \Rightarrow_{rg}^{A_n \rightarrow \varepsilon} x_1 x_2 \ldots x_n \in \Sigma^*$.



right-skewed tree
linear list



Compare ***derivations*** *in (1) a regular grammar G = (N, T, P, S),*
(2) ***state transitions*** *in the* ***corresponding*** *finite automaton*

$A = ([N], T, \delta, [S], F)$ where

$\delta = \{[B] \in \delta([A], x) | A \to xB \in P\} \cup \{f \in \delta([A], x) | A \to x \in P, f \in F\}$,

and (3) ***rewritings*** *in* ***rewriting system***

$R_A = ([N] \times T^*, \to, ([S], xyz), F \times \{\varepsilon\})$

where $\to = \{([A], x) \to ([B], \varepsilon) | [B] \in \delta([A], x), [A], [B] \in [N], x \in T^*\}$.

Let $x, y, z \in T^*, A, B \in N, [A], [B] \in [N] = Q, f \in F]$. Then

$S \quad \Rightarrow_G^* \; xA \quad \Rightarrow_G^{A \to yB} \quad xyB \quad \Rightarrow_G^* \; xyz \in T^*$.

grammars *generating terminal strings*

$\delta^*([S], xyz) =_A^x \delta^*([A], yz) =_A^{[B] \in \delta([A], y)} =_A \delta^*([B], z) = f \in F$.

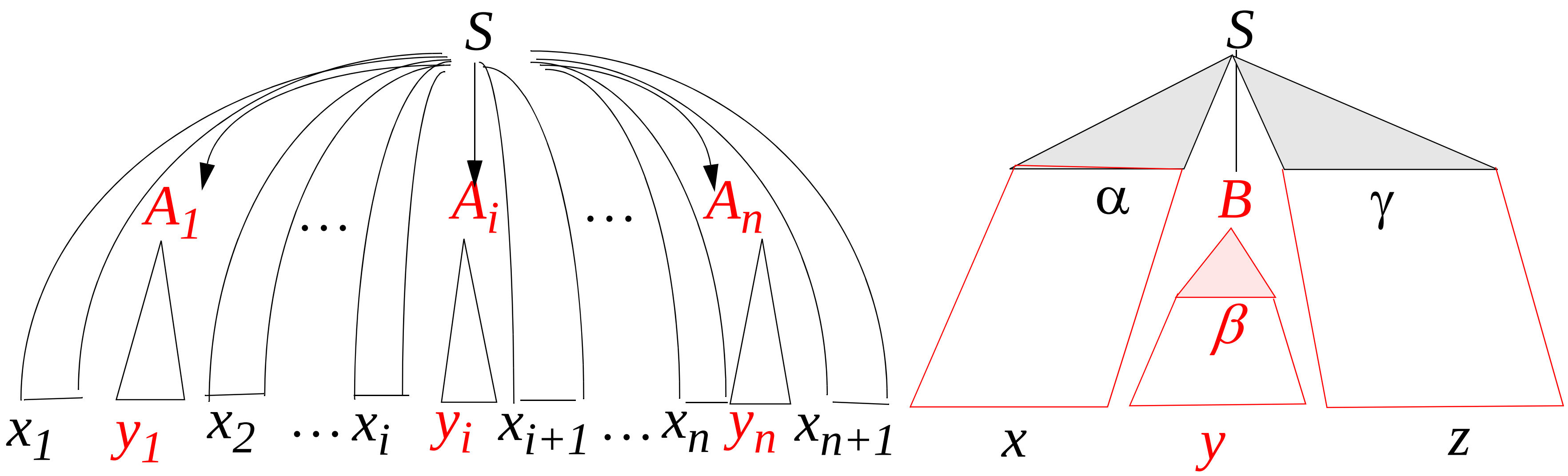
$([S], xyz) \Rightarrow_{R_A}^x ([A], yz) \quad \Rightarrow_{R_A}^{([A], y) \to ([B], \varepsilon)} ([B], z) \Rightarrow_{R_A}^* (f, \varepsilon) \in F \times \{\varepsilon\}$.

automata ***consuming*** *terminal strings*

Let G = (N, T, P, S) be a ***context-free*** *grammar. Then*

$S \Rightarrow^* x_1A_1x_2A_2\ldots x_iA_ix_{i+1}\ldots x_nA_nx_{n+1} \Rightarrow^* x_1y_1x_2y_2\ldots x_iy_ix_{i+1}\ldots x_ny_nx_{n+1}.$

many nonterminals in the ***sentential form****...*



Consider ***two*** *kind of derivations* $\Rightarrow$

leftmost *derivation* $\Rightarrow_{lm} \subset \Rightarrow.$

rightmost *derivation* $\Rightarrow_{rm} \subset \Rightarrow.$

General derivation

$$S \Rightarrow^{*} \alpha B\gamma \Rightarrow^{B\to\beta} \alpha\beta\gamma \Rightarrow^{*} \alpha y\gamma \Rightarrow^{*} xyz.$$

Leftmost derivation

$$S \Rightarrow_{lm}^{*} \mathbf{x}\underline{\mathbf{B}\gamma} \Rightarrow_{lm}^{B\to\beta} x\beta\gamma \Rightarrow_{lm}^{*} xy\gamma \Rightarrow_{lm}^{*} xyz.$$

B is the **leftmost** nonterminal

Rightmost derivation

$$S \Rightarrow_{rm}^{*} \underline{\alpha\mathbf{B}}\mathbf{z} \Rightarrow_{rm}^{B\to\beta} \alpha\beta z \Rightarrow_{rm}^{*} \alpha yz \Rightarrow_{rm}^{*} xyz.$$

B is the **rightmost** nonterminal