

## Chap. 2 Finite Automata

### 2.1 An Informal Picture of Finite Automata ([See 2nd Edition](#))

*A man with a wolf, goat, and cabbage is on the left bank of a river*

*A boat carries one man and only one of the other three.*

*The wolf eats the goat **without the man**.*

*The goat eats the cabbage **without the man**.*

*Is it possible to cross the river without the goat or cabbage being eaten?*

*states: occupants of each bank after a crossing*

*16 subsets of the man( $m$ ), wolf( $W$ ), goat( $G$ ), and cabbage( $C$ ).*

*16 states:  $mWGC-\emptyset$ ,  $WC-mG$ , ...,  $\emptyset-mWGC$*

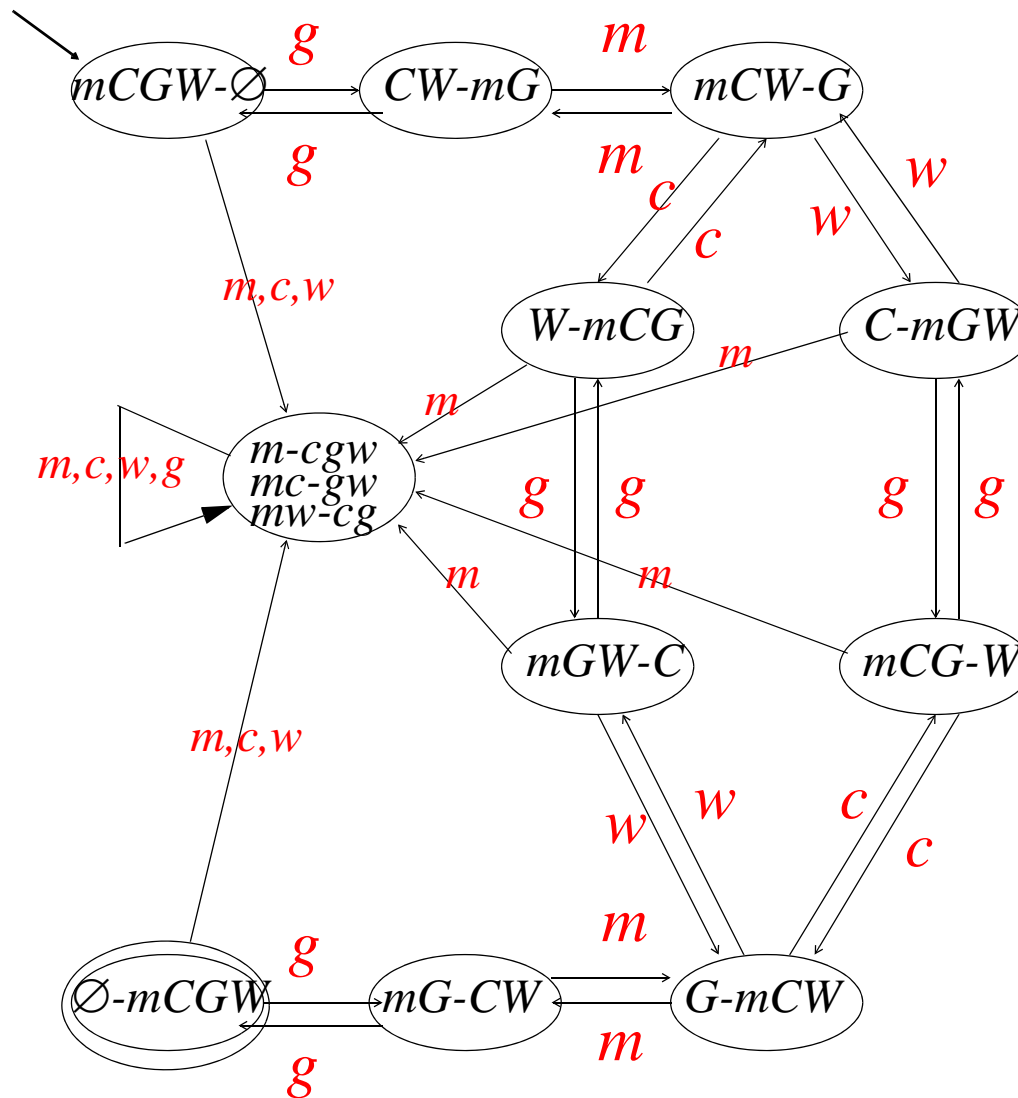
*input symbol: four ways of crossing the river*

*A man crosses alone( $m$ ) with the wolf( $w$ ), goat( $g$ ), or cabbage( $c$ ).*

*input string: a sequence of crossings (input symbols =  $\{m, w, g, c\}$ )*

*state transition: From a state to another state*

*by an **input symbol** (crossing)*



## 2.2 Deterministic Finite Automata(DFA)

### 2.2.1 Definition of a DFA

A *deterministic finite automaton*  $A = (Q, T, \delta, q_0, F)$  consists of:

1. A *finite set of states*, denoted  $Q$ ,
2. A *finite set of input symbols*, denoted  $T$ ,  $\Sigma$  in the text
3. A *transition function*,

$$\delta: Q \times T \rightarrow Q$$

Let  $p, q \in Q$ , and  $a \in T$ , if we write  $\delta(q, a) = p$ , meaning that when **current** state is in  $q$  and **input** symbol is  $a$ , the current state is changed to  $p$ , and input symbol is **changed** to the **next(right)** one.

4. A *start(initial) state*,  $q_0 \in Q$ , and
5. A *set of final or accepting states*,  $F \subseteq Q$ .

A DFA  $A = (Q, T, \delta, q_0, F)$  “five-tuple” notation

## 2.2.2 How a DFA Processes Strings

How a DFA decides whether or not “accept”

a *input string*(sequence of input symbols)

Suppose  $a_1a_2 \dots a_n$  is a sequence of input symbols

1. We start out the DFA with in its **start** state  $q_0$ .
2. We consult the **transition function**,  $\delta$ ,  
with current state and current input symbol,  
assume in the first trial  $\delta(q_0, a_1) = q_1$ ,
3. **Repeat** this step with the **next** state  $q_1$  and the **next** input symbol  $a_2$ .  
Assume  $\delta(q_1, a_2) = q_2, \delta(q_2, a_3) = q_3, \dots, \delta(q_{n-1}, a_n) = q_n$ .
4. **Finally**, check if  $\delta(q_{n-1}, a_n) = q_n \in F$  or not.

If  $q_n \in F$ , **accept**  $a_1a_2 \dots a_n$ . If  $q_n \notin F$  **does not accept**  $a_1a_2 \dots a_n$ .

$\delta(\delta(\dots\delta(\delta(q_0, a_1), a_2), \dots), a_{n-1}), a_n) = q_n$

where  $\delta(q_{i-1}, a_i) = q_i$  for  $1 \leq \forall i \leq n$ .

**function** DFA(

$\underline{D} = (Q: \text{set of states}, T: \text{set of symbols}, \delta: Q \times \Sigma \rightarrow Q, q_0 \in Q, F \subseteq Q);$

$\underline{x} = a_1 a_2 \dots a_n \in T^*$  /\* ( $\underline{D}; \underline{x} \in T^*$ ) are two formal parameters \*/

/\*  $\underline{D}$  has five tuples,  $\underline{D} = (Q, T, \delta, q_0, F)$  \*/

**return** { $\underline{D}$  accept  $\underline{x}$ ,  $\underline{D}$   $\neg$ accept  $\underline{x}$ }

$q \in Q, i \in \mathbb{N};$  /\*  $q$  and  $i$  are two local variables \*/

$q, i := q_0, 1;$

**do**  $i \leq n \rightarrow$  **if**  $\exists \delta(q, a_i) \rightarrow q, i := \delta(q, a_i), i+1$

$/ \neg \exists \delta(q, a_i) \rightarrow$  **return**  $\underline{D}$   $\neg$ accept  $\underline{x}$

**fi**

**od**

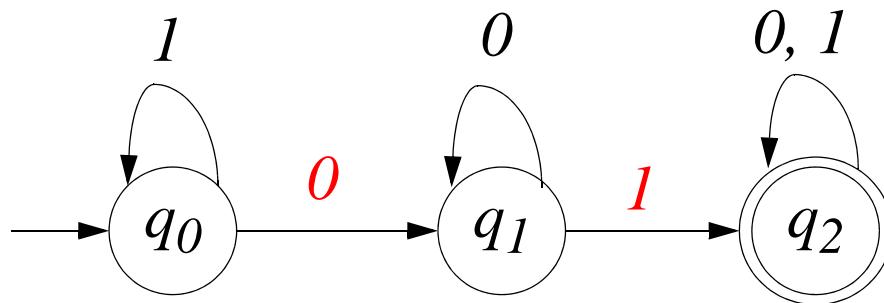
**if**  $q \in F \rightarrow$  **return**  $\underline{D}$  accept  $\underline{x}$  |  $q \notin F \rightarrow$  **return**  $\underline{D}$   $\neg$ accept  $\underline{x}$  **fi**

**end function** DFA.

### 2.2.3 Simple notations for DFA's

**Example 2.2** Let  $T = \{0, 1\}$  and  $L = \{x01y \in T^* \mid x, y \in T^*\}$ .  
 substring **01**.  $(0 \mid 1)^*01(0 \mid 1)^*$ .

1. **Transition diagram** Figure 2.4 in p. 48.



**State  $q_0$ :** *initial state*

If it sees **0**, go to a **new state  $q_1$** . If it sees **1** **stay** in the state  $q_0$ .  
 It has **never** seen 0.

**State  $q_1$ :** *The last symbol it has seen is 0.*

If it sees **1**, go to a **new state  $q_2$** . If it sees **0**, **stay** in the state  $q_1$ .

**State  $q_2$ :** *It has **ever** seen is 01.* **final** state

Whether it sees **0** or **1**, **stay** in the state  $q_2$ .

## 2. Transition table Figure 2.5 in p. 49.

		0	1
→	$q_0$	$q_1$	$q_0$
	$q_1$	$q_1$	$q_2$
*	$q_2$	$q_2$	$q_2$

### Formal definition of DFA $D$

$D = (Q, T, \delta, q_0, F)$  where

1.  $Q = \{q_0, q_1, q_2\}$
2.  $T = \{0, 1\}$
3.  $\delta = \{\delta(q_0, 0)=q_1, \delta(q_0, 1)=q_0, \delta(q_1, 0)=q_1, \delta(q_1, 1)=q_2, \delta(q_2, 0)=q_2, \delta(q_2, 1)=q_2\}$
4.  $q_0 \in Q$
5.  $F = \{q_2\}$

## 2.2.4 Extend the domain of transition function from symbols to **strings**

$$\delta: Q \times T \rightarrow Q$$

$$\hat{\delta}: Q \times T^* \rightarrow Q$$

$$\hat{\delta}(q, \varepsilon) =_B q \quad \text{basis}$$

$$\hat{\delta}(q, xa) =_R \delta(\hat{\delta}(q, x), a) \quad \text{recursion} \quad (2.1)$$

$$q \in Q, x \in T^*, a \in \Sigma, xa \in T^+.$$

$$\hat{\delta}(q_0, a_1 a_2 \dots a_n) =_R \delta(\hat{\delta}(q_0, a_1 a_2 \dots a_{n-1}), a_n) \quad \text{1st recursion}$$

$$=_R \delta(\delta(\hat{\delta}(q_0, a_1 a_2 \dots a_{n-2}), a_{n-1}), a_n) \quad \text{2nd recursion}$$

$$=_R \dots$$

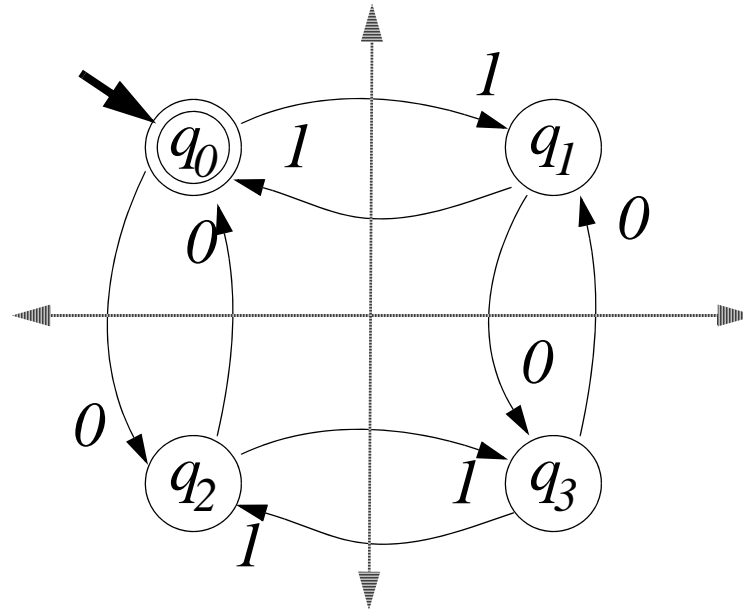
$$=_R \delta(\delta(\dots \delta(\delta(\hat{\delta}(q_0, \varepsilon), a_1), a_2), \dots), a_{n-1}), a_n) \quad \text{n-th recursion}$$

$$=_B \delta(\delta(\dots \delta(\delta(q_0, a_1), a_2), \dots), a_{n-1}), a_n) \quad \text{basis}$$

We may use  $\delta^*$  instead of  $\hat{\delta}$  to show that the 2nd domain of  $\delta$  is  $T^*$ .



**Example 2.4** in p. 50.



*Even numbers of 0's and even numbers of 1's.*

### 2.2.5 The Language of DFA

We *define* the language of a DFA  $D = (Q, T, \delta, q_0, F)$ ,  $L(D)$ ,

$$L(D) = \{w \in T^* \mid \delta^*(q_0, w) \in F\}.$$

$$L(D) \subseteq T^* \text{ or } L(D) \in 2^{T^*}.$$

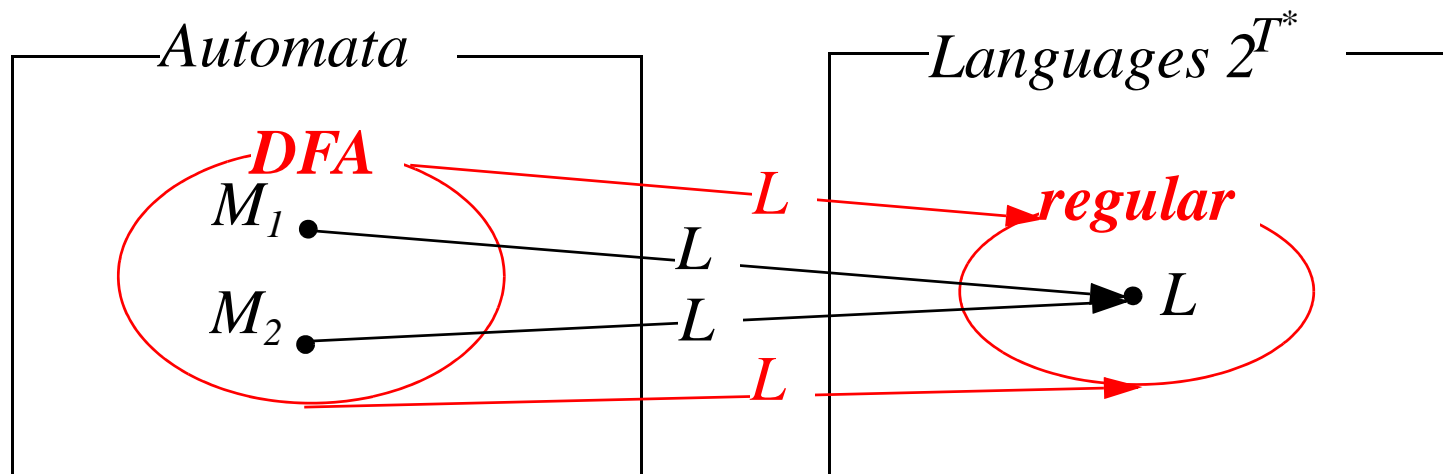
### **Def. A Class of Regular languages**

Let  $L$  be a language.

If  $L = L(A)$  for some **DFA**  $A$ , then we say  $L$  is a **regular language**.

A class of regular languages

type-3 languages in Chomsky's language hierarchy



### **Def. Equivalence of two Automata**

Let  $M_1$  and  $M_2$  be two automata. We say two automata  $M_1$  and  $M_2$  are **equivalent**, if  $L(M_1) = L(M_2) = L$ .

DFA:  $\delta: Q \times T \rightarrow Q$

$\forall q \in Q, \forall a \in T, \exists \delta(q, a) \in Q.$  (total) function

$\exists q \in Q, \exists a \in T. \nexists \delta(q, a).$  partial function

If  $A = (Q, T, \delta, q_0, F)$  be a FA with  $\delta: Q \times T \rightarrow Q$  partial function,

$L(A) = L(B)$  for DFA  $B = (Q \cup \{d\}, T, \delta', q_0, F)$  where  $d \notin Q$ ,

DFA but **partial** function:  $\delta: Q \times T \rightarrow Q \cup \{\emptyset\}.$

$\delta' = \delta \cup \{\delta(q, a) = d \mid \delta(q, a) \notin Q\} \cup \{\delta(d, a) = d \mid a \in T\}$

$d$ : **dead state**

$\delta'$  in  $B$  is the DFA (with **total** function) but?

$\delta': Q \times T \rightarrow Q.$

Dead state:

“ $m-cgw, mc-gw, mw-cg$ ” state in the 1st Example.

eliminate the state and relevant transition functions  $\rightarrow$  **partial** function.

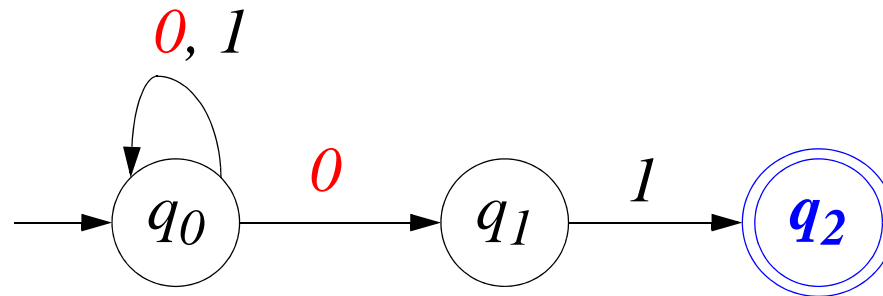
DFA with partial function can be considered as a DFA.

## 2.3 Nondeterministic Finite Automata (NFA)

### 2.3.1 An Informal View of NFA

**Example 2.9** An NFA accepting all strings that ends in 01 (postfix 01).

Fig. 2.9(p. 56)



Consider an input string  $00101 \in \{0, 1\}^*$ .

$$q_0 \xrightarrow{0} q_1 \xrightarrow{0} X(\text{stuck})$$

$$q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{0} X(\text{stuck})$$

$$q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_0 \notin F(\text{fail})$$

$$q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_0 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \in F(\text{success})$$

### 2.3.2 Definition of NFA

$A = (Q, T, \delta, q_0, F)$  is a *nondeterministic finite automaton*(NFA), if

1.  $Q$  is a *finite set of states*,
2.  $T$  is a *finite set of input symbols*,
3.  $\delta$  is a *transition function*,

$$\delta: Q \times T \rightarrow 2^Q.$$

If  $q \in Q$ ,  $a \in T$ , and  $\{p_1, p_2, \dots, p_n\} \subseteq Q$  ( $1 \leq \forall i \leq n, p_i \in Q$ )  
then we write  $\delta(q, a) = \{p_1, p_2, \dots, p_n\}$ , meaning that  
when current state is in  $q$  and input symbol is  $a$ ,  
the state may be changed to **any** one of  $p_1, p_2, \dots, p_n$ ,  
and input symbol is changed to the next(right) one.

4.  $q_0 \in Q$  is a *start state*, and
5.  $F \subseteq Q$  is a set of *final or accepting states*.

### 2.3.3 The Extended Transition Function

**DFA:**  $\delta: Q \times T \rightarrow Q$  Definition

$$\delta^*: Q \times T^* \rightarrow Q \quad \hat{\delta}: Q \times \Sigma^* \rightarrow Q \text{ in the text}$$

**NFA:**  $\delta: Q \times T \rightarrow 2^Q$  Definition

$$\delta': 2^Q \times T \rightarrow 2^Q \quad (1) \text{ set extension in this lecture}$$

$$\delta'^*: 2^Q \times T^* \rightarrow 2^Q \quad (2) \text{ string extension in this lecture}$$

$$\hat{\delta}: Q \times \Sigma^* \rightarrow 2^Q \text{ in the text}$$

#### Two step extensions

**1. Set extension:**  $\delta \Rightarrow \delta'$ : We extend the 1st domain of  $\delta'$  to set of states.

$$\delta': 2^Q \times T \rightarrow 2^Q.$$

Let  $P \subseteq Q$ . Then we define

$$\delta'(P, a) = \{q \in Q \mid p \in P, q \in \delta(p, a)\}$$

$$\text{or} \quad = \cup_{p \in P} \delta(p, a)$$

We may write  $\delta'(p, a)$  **instead** of  $\delta'(\{p\}, a)$  when needed.

**2. String extension:**  $\delta' \Rightarrow \delta'^*$  (same as  $\delta \Rightarrow \hat{\delta} = \delta^*$ )

$$\delta'^*: 2^Q \times T^* \rightarrow 2^Q.$$

$$\delta'^*(P, \varepsilon) =_B P \quad \text{basis}$$

$$\delta'^*(P, wa) =_R \delta'(\delta'^*(P, w), a) \quad \text{recursion}$$

$$P \subseteq Q (\text{or } P \in 2^Q), w \in T^*, a \in T, wa \in T^+.$$

We may **write**  $\delta$  and  $\delta^*$  instead of  $\delta'$  and  $\delta'^*$  since  $\delta \subseteq \delta'$ .

### 2.3.4 The Language of an NFA

Let  $N = (Q, \Sigma, \delta, q_0, F)$  be an NFA. Then the **language defined**

by the NFA  $N$ , denoted as  $L(N)$  is,

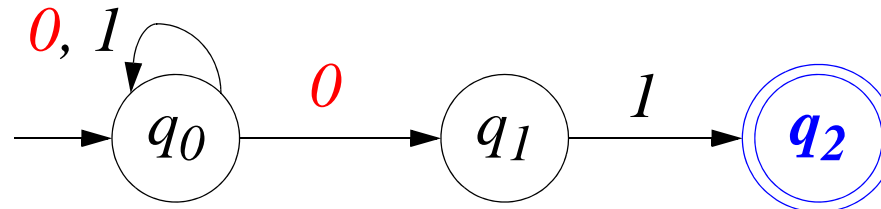
$$L(N) = \{w \in T^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}.$$

$$L(N) \subseteq T^* \text{ or } L(N) \in 2^{T^*}.$$

### 2.3.5 Equivalence of DFA and NFA

**Example 2.10** NFA of Fig. 2.9(p. 56)

postfix **01**.



**Transition table for NFA**

	0	1	
→ $q_0$	$\{q_0, q_1\}$	$\{q_0\}$	$\delta(q_0, 0) = \{q_0, q_1\}$ : NFA
$q_1$	$\emptyset$	$\{q_2\}$	
* $q_2$	$\emptyset$	$\emptyset$	

**Transition table for subsets of NFA**  $2^3 = 8$  subset-states (See Fig. 2.12)

	0	1		0	1	
	$\emptyset$	$\emptyset$		$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
→ $\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$	*	$\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$	*	$\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
* $\{q_2\}$	$\emptyset$	$\emptyset$	*	$\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



Starting from  $\{q_0\}$  See Fig. 2.14 in p. 63.

	$0$	$1$		$0$	$1$
$\rightarrow$	$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$	$\rightarrow$	$[\epsilon]$ $[0]$ $[\epsilon]$
	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$		$[0]$ $[0]$ $[01]$
$*$	$\{q_0, \mathbf{q_2}\}$	$\{q_0, q_1\}$	$\{q_0\}$	$*$	$\mathbf{[01]}$ $[0]$ $[\epsilon]$

Only **3 subset-states** are reachable from the initial state  $\{q_0\}$ .

**Renaming of states**  $\{q_0\} \leftrightarrow [\epsilon]$ ,  $\{q_0, q_1\} \leftrightarrow [0]$ , and  $\{q_0, \mathbf{q_2}\} \leftrightarrow \mathbf{[01]}$ .

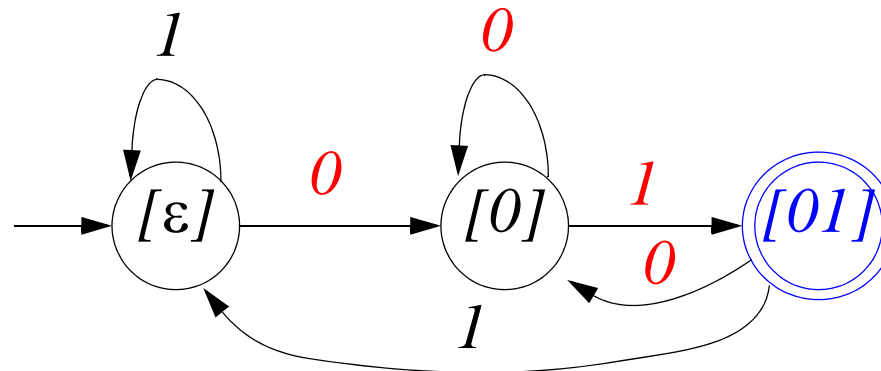


Fig. 2.14 DFA from NFA Fig. 2.9.(**postfix 01**)

**Subset Construction (NFA  $\Rightarrow$  DFA)**

$$Q_D = 2^{Q_N}.$$

Given an NFA  $N = (Q_N, T, \delta_N, q_{0N}, F_N)$ ,

**construct an equivalent DFA  $D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$**

*such that  $L(D) = L(N)$ .*

1.  $Q_D = \{P \mid P \subseteq Q_N\} = 2^{Q_N}$ .

3.  $\delta_D(P, a) = \{q \in Q_N \mid p \in P \subseteq Q_N, q \in \delta_N(p, a)\} \subseteq Q_N$  or  $\in Q_D$ .

or  $= \cup_{p \in P} \delta_N(p, a)$

or  $= \delta_N'(P, a)$  **set extension of  $\delta_N$ .**

$\delta_D(P, a) = \delta_N'(P, a)$  or  $\delta_D = \delta_N'$  for short,

where  $\delta_D: Q_D \times T \rightarrow Q_D$  and  $\delta_N': 2^{Q_N} \times T \rightarrow 2^{Q_N}$  ( $Q_D = 2^{Q_N}$ ).

4.  $q_{0D} (\in Q_D) = \{q_{0N}\} (\subseteq Q_N)$ .

5.  $F_D = \{F \in Q_D \mid F \cap F_N \neq \emptyset\}$

**Thm. 2.11**  $L(D) = L(N)$  in the above subset construction

**Proof**  $\delta_D^*(q_{0D}, w) = \delta_N'^*(\{q_{0N}\}, w)$  for all  $w \in \Sigma^*$ .

$$\delta_D^*: Q_D \times T^* \rightarrow Q_D (= 2^{Q_N} \times \Sigma^* \rightarrow 2^{Q_N}).$$

$$\delta_N'^*: 2^{Q_N} \times T^* \rightarrow 2^{Q_N}. \text{ We may use } q_{0N} \text{ instead of } \{q_{0N}\}.$$

**Induction on  $|w|$ .**

**basis:** Let  $|w| = 0$ ,  $\delta_D^*(q_{0D}, \epsilon) = \delta_D^*(\{q_{0N}\}, \epsilon) = \{q_{0N}\} = \delta_N'^*(\{q_{0N}\}, \epsilon)$ .

**induction:** Let  $w = xa$  where  $a \in T$ ,  $x \in T^*$ , and  $w \in T^+$ . ( $|w| = |x| + 1$ )

$$\delta_D^*(q_{0D}, xa) = \delta_D(\delta_D^*(\{q_{0N}\}, x), a) \quad \text{by string extension of } \delta_D^*.$$

$$= \delta_N'(\delta_D^*(\{q_{0N}\}, x), a) \quad \text{by subset construction}$$

$$= \delta_N'(\delta_N'^*(\{q_{0N}\}, x), a) \quad \text{by induction hypothesis}$$

$$= \delta_N'^*(\{q_{0N}\}, xa) \quad \text{by string extension of } \delta_N'^*.$$

$\delta_D^*(q_{0D}, x) \in F_D$ , if and only if,  $\delta_N'^*(q_0, x) \cap F_N \neq \emptyset$ .  $\therefore L(D) = L(N)$ .

**Thm. 2.12** A language  $L$  is defined by some DFA, if and only if  $L$  is defined by some NFA. ( $NFA \Leftrightarrow DFA$ )

**Proof:** (If) subset construction ( $NFA \Rightarrow DFA$ ) **Thm. 2.11**

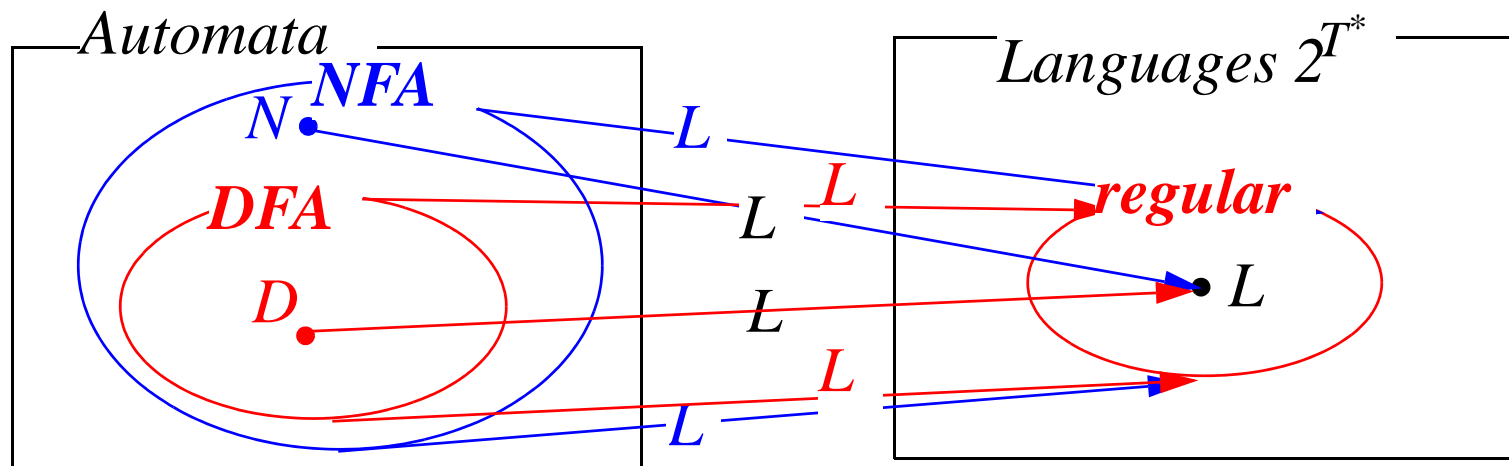
(Only if) NFA can easily simulate DFA ( $DFA \Rightarrow NFA$ ).

Let a DFA  $D = (Q, T, \delta_D, q_0, F)$ . We define an NFA  $(Q, T, \delta_N, q_0, F)$

$$\delta_N(q, a) = \{p \mid \delta_D(q, a) = p\}.$$

**Proof for**  $\forall w \in \Sigma^*$ , if and only if,  $\hat{\delta}_D(q, w) = p$ , then  $\hat{\delta}_N(q, w) = \{p\}$ .

$\therefore L$  is regular, iff  $L = L(N)$  for some NFA  $N$ .



### 2.3.6 A Bad Case for the Subset Construction

See Ex. 2.13 (p64-65)

$$(0+1)^*1(0+1)^{n-1}.$$

The state of DFA

**summarizes** the information concerning **past inputs** that is needed to **determine** the **behaviour** of the automata on **subsequent inputs**.

Aho and Ullman in the 2nd Edition.

$$\begin{array}{ll}
 [(0+1)^*] \xrightarrow{1} [(0+1)^*1] \xrightarrow{0} [(0+1)^*10] \xrightarrow{0} [(0+1)^*100] & \Rightarrow^* [(0+1)^*10^{n-2}0] \\
 & \Rightarrow^* [(0+1)^*10^{n-2}1] \\
 & \rightarrow^1 [(0+1)^*101] \quad \dots \\
 & \dots \\
 & \rightarrow^1 [(0+1)^*11] \xrightarrow{0} [(0+1)^*110] \quad \dots \\
 & \dots \\
 & \rightarrow^1 [(0+1)^*111] \quad \dots \\
 & \Rightarrow^* [(0+1)^*11^{n-2}0] \\
 & \Rightarrow^* [(0+1)^*11^{n-2}1]
 \end{array}$$

## 2.4 An Application: Text Search Read text

## 2.5 Finite Automata with Epsilon-Transitions

$A = (Q, T, \delta, q_0, F)$  is a fa with epsilon transition ( $\epsilon$ -NFA), if

$Q, T, q_0,$  and  $F$  are same as NFA, but

3.  $\delta$  is a transition function,

$$\delta: Q \times (T \cup \{\epsilon\}) \rightarrow 2^Q.$$

If  $q \in Q$  and  $\{p_1, p_2, \dots, p_n\} \subseteq Q,$

then we write  $\delta(q, \epsilon) = \{p_1, p_2, \dots, p_n\},$  meaning that

when current state is in  $q$  and **regardless** of input symbol

the state may be changed to **any** one of  $p_1, p_2, \dots, p_n,$

and next input symbol is **not changed**.

Ex. 2.16 in Fig. 2.18 for an  $\epsilon$ -NFA accepting decimal numbers in p. 73.

Ex. 2.17 in Fig. 2.19 for an  $\epsilon$ -NFA

accepting keywords “web” and “ebay” in p. 73.

### 2.5.3 Epsilon-closure

**basis:**  $q \in \varepsilon^*(q)$  (same as  $ECLOSE(q)$  in the text)

**recursion:** If  $p \in \varepsilon^*(q)$ , and  $r \in \delta(p, \varepsilon)$ , then  $r \in \varepsilon^*(q)$ .

*Example 2.19 in p. 74.*

Why  $\varepsilon^*(q)$  instead of  $ECLOSE(q)$

Let  $\varepsilon = \{(p, q) \in Q \times Q \mid q \in \delta(p, \varepsilon)\}$ .

Then  $\varepsilon \subseteq Q \times Q$  (a binary relation on  $Q$ )

and  $ECLOSE(q) = \varepsilon^*(q)$  reflexive-transitive closure of  $\varepsilon$ .

### 2.5.4 The Extended Transition and Language for $\varepsilon$ -NFA's

NFA:  $\delta: Q \times T \rightarrow 2^Q$ .

$\varepsilon$ -NFA:  $\delta: Q \times (T \cup \{\varepsilon\}) \rightarrow 2^Q$ .

Let's *divide*  $\delta$  of  $\varepsilon$ -NFA into  $\delta^1$  and  $\delta^0 (= \varepsilon)$ .

$\delta^1: Q \times T \rightarrow 2^Q$ .

$\delta^0: Q \times \{\varepsilon\} \rightarrow 2^Q$ .

$\delta = \delta^1 \cup \delta^0$ .

$\delta^1$ : is *same* as  $\delta$  in NFA, we use  $\delta$  *instead* of  $\delta^1$ , since  $\delta^1 = \delta$ .

$\delta^0 \leftrightarrow_f \varepsilon$  what is the *bijection*  $f$ ? We use  $\varepsilon$  *instead* of  $\delta^0$ .

$\varepsilon = \{(p, q) \in Q \times Q \mid q \in \delta(p, \varepsilon)\}$  binary *relation* on  $Q$ .

$\varepsilon^* = \bigcup_{i \in \mathbb{N}_0} \varepsilon^i \subseteq Q \times Q$  binary *relation* on  $Q$ .

$\varepsilon, \varepsilon^*: Q \rightarrow 2^Q$ . *function* from  $Q$  to  $2^Q$ .



Let's **extend** the domain of  $\delta(=\delta^1)$ ,  $\varepsilon$ , and  $\varepsilon^*$  from **state** to **set of states**.

Let  $P \subseteq Q$ , we define

$$\delta': 2^Q \times T \rightarrow 2^Q.$$

$$\delta'(P, a) = \{q \in Q \mid p \in P, \delta(p, a) = q\}$$

$$\text{or} \quad = \cup_{p \in P} \delta(p, a)$$

We may write  $\delta'(p, a)$  **instead** of  $\delta'(\{p\}, a)$  when needed.

We may write  $\delta(P, a)$  **instead** of  $\delta'(P, a)$  when needed.

$$\varepsilon': 2^Q \rightarrow 2^Q.$$

$$\varepsilon'(P) = \{q \in Q \mid p \in P, \varepsilon(p) = q\}$$

$$\text{or} \quad = \cup_{p \in P} \varepsilon(p)$$

$$\varepsilon'^*: 2^Q \rightarrow 2^Q.$$

$$\varepsilon'^*(P) = \{q \in Q \mid p \in P, \varepsilon^*(p) = q\} = \cup_{p \in P} \varepsilon^*(p)$$

We write  $\varepsilon(P)$  **instead** of  $\varepsilon'(P)$  and  $\varepsilon^*(P)$  **instead** of  $\varepsilon'^*(P)$ .

Let's **extend** the 2nd domain of  $\delta$  from **symbol** to **strings**.

$$\delta^*: 2^Q \times T^* \rightarrow 2^Q.$$

$$\delta^*(P, \varepsilon) =_B \varepsilon^*(P) \quad \text{basis}$$

$$\delta^*(P, wa) =_R \varepsilon^*(\delta(\delta^*(P, w), a)) \quad \text{recursion}$$

$$P \subseteq Q (\text{or } P \in 2^Q), w \in \Sigma^*, a \in T, wa \in T^+.$$

$$\delta^n(q_0, a_1 a_2 \dots a_n) =_R \varepsilon^*(\delta(\delta^{n-1}(q_0, a_1 a_2 \dots a_{n-1}), a_n)) \quad \text{1st recursion}$$

$$=_R \varepsilon^*(\delta(\varepsilon^*(\delta(\delta^{n-2}(q_0, a_1 a_2 \dots a_{n-2}), a_{n-1})), a_n)) \quad \text{2nd recursion}$$

$$=_R \dots$$

$$=_R \varepsilon^*(\delta(\varepsilon^*(\delta(\dots \varepsilon^*(\delta(\varepsilon^*(\delta(\delta^0(q_0, \varepsilon), a_1)), a_2)), \dots)), a_{n-1})), a_n)) \quad \text{n-th}$$

$$=_B \varepsilon^*(\delta(\varepsilon^*(\delta(\dots \varepsilon^*(\delta(\varepsilon^*(\delta(\varepsilon^*(q_0), a_1)), a_2)), \dots)), a_{n-1})), a_n)) \quad \text{basis}$$

$$\delta^n = \varepsilon^{* \circ} \delta^{\circ} \varepsilon^{* \circ} \dots \circ \delta^{\circ} \varepsilon^{* \circ} = \varepsilon^{* \circ} (\delta^{\circ} \varepsilon^{* \circ})^n = \varepsilon^{* \circ} (\delta^1)^n. \quad \text{where } \delta^1 = \delta^{\circ} \varepsilon^{* \circ}.$$

$$\therefore \delta^* = \cup_{i \in \mathbb{N}_0} \delta^i = \delta^*.$$

### 2.5.5 Eliminating $\varepsilon$ -Transitions

Given an  $\varepsilon$ -NFA  $E = (Q_E, T, \delta_E, q_{0E}, F_E)$ , construct the equivalent DFA  $D = (Q_D, T, \delta_D, q_{0D}, F_D)$  such that  $L(D) = L(N)$ .

1.  $Q_D = \{P \mid P \subseteq Q_E\} = 2^{Q_E}$ .
3.  $q_{0D} = \varepsilon^*({q_{0E}})$
4.  $\delta_D(P, a) = \varepsilon^*(\delta_E(P, a))$  set extension of  $\delta_E$  and  $\varepsilon$ -closure ( $\delta^1 = \delta \circ \varepsilon^*$ )  
where  $P \subseteq Q_E$  ( $P \in Q_D$ ),  $a \in T$ , and  
 $\delta_D(P, a) \subseteq Q_E$  (or  $\delta_D(P, a) \in Q_D$ )
5.  $F_D = \{F \in Q_D \mid F \cap F_E \neq \emptyset\}$

**Theorem 2.22** A language  $L$  is accepted by some  $\varepsilon$ -NFA if and only if  $L$  is accepted by some DFA.

**Proof:** (If)  $\varepsilon$ -NFA can easily simulate DFA (DFA  $\Rightarrow$   $\varepsilon$ -NFA).

(Only if) Eliminating  $\varepsilon$ -transitions ( $\varepsilon$ -NFA  $\Rightarrow$  DFA)

Let  $E = (Q_E, T, \delta_E, q_{0E}, F_E)$  be a  $\varepsilon$ -NFA and

DFA  $D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$  is the one in 2.5.5.

We show  $\delta_E^*(q_{0E}, w) = \delta_D^*(q_{0D}, w)$  by induction on  $|w|$ .

$$\begin{aligned}
 \text{basis: } \delta_E^*(q_{0E}, \varepsilon) &= \varepsilon^*(q_{0E}) && \text{by basis definition of } \delta_E^*. \\
 &= q_{0D} && \text{by construction of } \delta_D. \\
 &= \delta_D^*(q_{0D}, \varepsilon) && \text{by basis definition of } \delta_D^*.
 \end{aligned}$$

**Induction:** Let  $w = xa \in T^+$ ,  $a \in T$ , and  $x \in T^*$ .

$$\begin{aligned}
 \delta_E^*(q_0, xa) &= \delta_E(\varepsilon^*(\delta_E^*(q_0, x), a)) && \text{by recursive definition of } \delta_E^*. \\
 &= \delta_E(\varepsilon^*(\delta_D^*(q_{0D}, x), a)) && \text{by induction hypothesis.} \\
 &= \delta_D(\delta_D^*(q_{0D}, x), a) && \text{by construction of } \delta_D. \\
 &= \delta_D^*(q_{0D}, xa) && \text{by recursive definition of } \delta_D^*.
 \end{aligned}$$

$\varepsilon$ -NFA Transition Table for Fig. 2.18 in p. 73.

	$\delta^I$	+, -	.	$d$		$\varepsilon$	$\varepsilon^*$
$\rightarrow$	$q_0$	$\{q_1\}$	$\emptyset$	$\emptyset$	<b><math>q_0</math></b>	$\{q_1\}$	$\{q_0, q_1\}$
	$q_1$	$\emptyset$	$\{q_2\}$	$\{q_1, q_4\}$	$q_1$	$\emptyset$	$\{q_1\}$
	$q_2$	$\emptyset$	$\emptyset$	$\{q_3\}$	$q_2$	$\emptyset$	$\{q_2\}$
*	$q_3$	$\emptyset$	$\emptyset$	$\{q_3\}$	<b><math>q_3</math></b>	$\{q_5\}$	$\{q_3, q_5\}$
	$q_4$	$\emptyset$	$\{q_3\}$	$\emptyset$	$q_4$	$\emptyset$	$\{q_4\}$
	$q_5$	$\emptyset$	$\emptyset$	$\emptyset$	$q_5$	$\emptyset$	$\{q_5\}$

DFA Transition Table for Fig. 2.22 in p. 79.

	$\varepsilon^* \circ \delta^I \circ \varepsilon^*$	+, -	.	$d$
$\rightarrow$	$\{q_0, q_1\}$	$\{q_1\}$	$\{q_2\}$	$\{q_1, q_4\}$
	$\{q_1\}$	$\emptyset$	$\{q_2\}$	$\{q_1, q_4\}$
	$\{q_2\}$	$\emptyset$	$\emptyset$	$\{q_3, q_5\}$
	$\{q_1, q_4\}$	$\emptyset$	$\{q_2, q_3, q_5\}$	$\{q_1, q_4\}$
*	$\{q_3, q_5\}$	$\emptyset$	$\emptyset$	$\{q_3, q_5\}$
*	$\{q_2, q_3, q_5\}$	$\emptyset$	$\emptyset$	$\{q_3, q_5\}$

## 2.A Extended Finite Automata(XFA)

$X = (Q, T, q_0, \delta, F)$  is an **eXtended Finite state Automata(XFA)**, if

$$\delta: Q \times T^* \rightarrow 2^Q.$$

Given an XFA  $X = (Q_X, T, q_0, \delta_X, F)$ , **construct**

an **equivalent  $\varepsilon$ -NFA**  $E = (Q_E, T, q_0, \delta_E, F)$  such that  $L(X) = L(E)$ .

**Construction algorithm**

$Q_E := Q_X; \delta_E := \emptyset;$

**for**  $\forall (\delta_X(q, x) = P) \in \delta_X$  where  $x = a_1a_2\dots a_n$  **do**

**if**  $0 \leq n \leq 1 \rightarrow \delta_E := \delta_E \cup \{\delta_E(q, x) = P\}$  where  $x \in \Sigma \cup \{\varepsilon\}$

$| n \geq 2 \rightarrow Q_E := Q_E \cup \{p_1\}; \delta_E := \delta_E \cup \{\delta_E(q, a_1) = \{p_1\}\};$

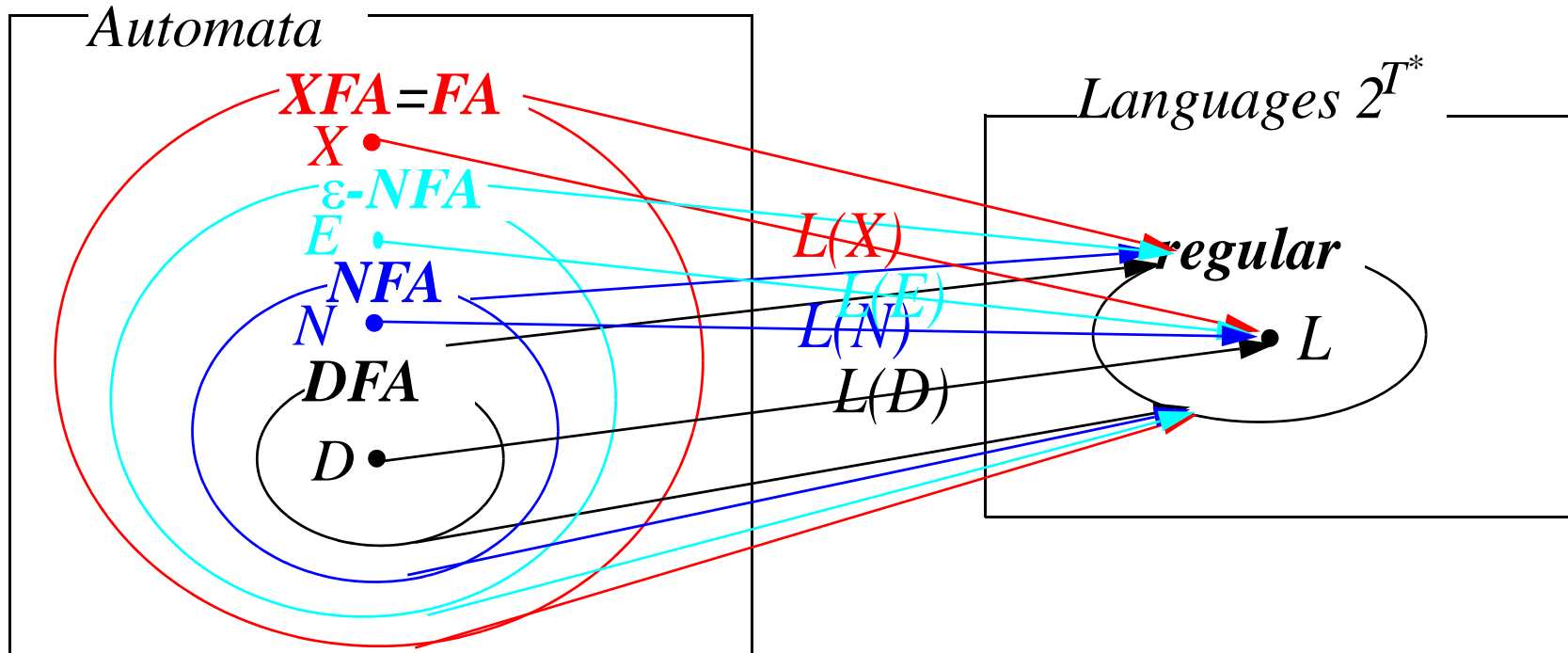
**for**  $2 \leq \forall i < n$  **do**  $Q_E := Q_E \cup \{p_i\}; \delta_E := \delta_E \cup \{\delta_E(p_{i-1}, a_i) = \{p_i\}\}$  **od;**

$\delta_E := \delta_E \cup \{\delta_E(p_{n-1}, a_n) = P\}$  where  $Q_X \cap \{p_1, p_2, \dots, p_{n-1}\} \neq \emptyset$

**fi**

**od**

**$XFA's = \epsilon-NFA's = NFA's = DFA's = Class\ of\ regular\ languages = FA's$**



The following **six** statements are **equivalent**,

1. A language  $L$  is **regular**.

2.  $L = L(D)$  for some **DFA**  $D$  where  $\delta: Q \times T \rightarrow Q$ .

3.  $L = L(P)$  for some DFA  $P$  where  $\delta: Q \times T \rightarrow Q \cup \{\emptyset\}$ .  
with **partial function**  $\delta$ .

4.  $L = L(N)$  for some **NFA**  $N$  where  $\delta: Q \times T \rightarrow 2^Q$ .

5.  $L = L(E)$  for some  **$\epsilon$ -NFA**  $E$  where  $\delta: Q \times (\{\epsilon\} \cup T) \rightarrow 2^Q$ .

6.  $L = L(X)$  for some **XFA**  $X$  where  $\delta: Q \times T^* \rightarrow 2^Q$ .

The following **two** statements are **equivalent**,

1. A language  $L$  is **regular**.

2.  $L = L(A)$  for some FA  $A$  where  $\delta: Q \times T^* \rightarrow 2^Q$ .

*We've changed the definition of FA from DFA to XFA.*



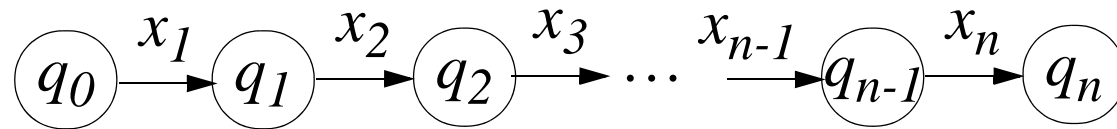
## 2.B Configuration rewriting system for FA

Let  $A = (Q, T, \delta, q_0, F)$  be a Finite automata where  $\delta: Q \times T^* \rightarrow 2^Q$ . Then

A **configuration rewriting system**  $R_A = (Q \times T^*, \rightarrow)$  on  $Q \times T^*$  is

$\rightarrow = \{(q, x) \rightarrow (p, \varepsilon) \mid p \in \delta(q, x)\}$ .  $(q, x), (p, \varepsilon) \in Q \times T^*$ : **configurations**

Assume  $x = x_1x_2\dots x_n \in T^*$  and  $1 \leq \forall i \leq n: q_i \in \delta(q_{i-1}, x_i)$ . Then



$$\begin{aligned} \delta(\dots\delta(\delta(q_0, x_1), x_2), \dots, x_{n-1}), x_n) &= q_1 \in \delta(q_0, x_1) \delta(\delta(\dots\delta(q_1, x_2), \dots, x_{n-1}), x_n) = \\ \dots &= \delta(q_{n-2}, x_{n-1}), x_n) = q_{n-1} \in \delta(q_{n-2}, x_{n-1}) \delta(q_{n-1}, x_n) = q_n \in \delta(q_{n-1}, x_n) q_n. \end{aligned}$$

$$\begin{aligned} (q_0, x_1x_2\dots x_{n-1}x_n) &\Rightarrow (q_0, x_1) \rightarrow (q_1, \varepsilon) (q_1, x_2\dots x_{n-1}x_n) \Rightarrow \\ \dots &\Rightarrow (q_{n-2}, x_{n-1}x_n) \Rightarrow (q_{n-2}, x_{n-1}) \rightarrow (q_{n-1}, \varepsilon) (q_{n-1}, x_n) \Rightarrow (q_{n-1}, x_n) \rightarrow (q_n, \varepsilon) (q_n, \varepsilon). \end{aligned}$$

$$L(R_A) = \{x \in T^* \mid (q_0, x) \Rightarrow^* (f, \varepsilon), f \in F\}.$$

Read this page for the *minimal state DFA TBD* in Chap. 4.

Consider a language  $L \subseteq T^*$  and a **binary relation**  $R_L$  on  $T^*$  where

$x, y \in \Sigma^* : x R_L y$ , if  $\forall z \in T^* : xz, yz \in L$  or  $xz, yz \notin L$ .

Then  $R_L$  is an **equivalent**(proof?) **binary relation** on  $T^*$ .

Consider a DFA  $M = (Q, T, \delta, q_0, F)$  and a **binary relation**  $R_M$  on  $T^*$

$x, y \in \Sigma^* : x R_M y$ , if  $\delta^*(q_0, x) = \delta^*(q_0, y)$ . Then  $Par_{R_M}(\Sigma^*) \cong Q$ .

**Compare two equivalent relation** on  $T^*$ ,  $R_L$  and  $R_M$  where  $L = L(M)$ .

$|Par_{R_L}(T^*)| \leq |Par_{R_M}(T^*)|$  in general.

If  $|Par_{R_L}(T^*)| = |Par_{R_{M'}}(T^*)|$  for some  $M' = (Q', T, \delta', q_0', F')$ .

$M'$  is a **minimal state DFA** that accepting  $L$ .