7 Advanced Counting Techniques

7.1 Introduction

Definition *1 A* ***recurrence relation*** *for a sequence* $\{a_n\}$ *is*

an equation expresses a_n *in terms of* $a_0, a_1, \ldots, a_{n-1}$ $\forall n \geq n_0 \in \aleph$.

A sequence is said to be a **solution** *of a recurrence, if it consistent with the definition of the recurrence.*

Example *1* $a_n = a_{n-1} - a_{n-2}$ *for* $n \geq 2$ *with* $a_0 = 3$ *and* $a_1 = 5$.

Example *3* ***Compound Interests***

$$P_n = P_{n-1} + 0.11P_{n-1} = 1.11P_{n-1} \text{ for } n \geq 1 \text{ with } P_0 = 1.$$

$$P_n = 1.11P_{n-1} = 1.11(1.11P_{n-2}) = 1.11^3P_{n-3} = \ldots = 1.11^nP_0.$$

Example *4* ***Rabits and Fibonacci number***

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2 \text{ with } f_0 = 0 \text{ and } f_1 = 1.$$

Example 5 *Tower of Hanoi*

1) move (n-1) disks 2) move bottom 1 disk 3) move (n-1) disks.

$$H_n = 2H_{n-1} + 1 \; n \geq 2 \text{ with } H_1 = 1.$$

$H_n = 2H_{n-1} + 1$

$= 2(2H_{n-2} + 1) = 2^2H_{n-2} + 2 + 1$

$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1$

…

$= 2^{n-k-1}H_{k+1} + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

$= 2^{n-k-1}(2H_k + 1) + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

$= 2^{n-k}H_k + 2^{n-k-1} + 2^{n-k-2} + 2^{n-k-3} + \ldots + 2^2 + 2 + 1$

…

$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1 \text{(with } k = 1)$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1 = 2^n - 1.$

Tower of Hanoi with 64 disks

$2^{64} - 1 = 1844{,}6744{,}0737{,}0955{,}1615$
경 조 억 만

만 , 억 , 조 , 경 , 해 , 자 , 양 , 구 , 간 , 정 , 재 , 극
$10^4, 10^8, 10^{12}, 10^{16}, 10^{20}, 10^{24}, 10^{28}, 10^{32}, 10^{36}, 10^{40}, 10^{40}, 10^{44},$
항하사 , 아승기 , 나유타 , 불가사의 , 무량대수 , 대수 .
$10^{48}, 10^{52}, 10^{56}, 10^{60}, 10^{64}, 10^{68}.$

18,446,744,073,709,551,615
? ? T B M T

$= 1.844\ldots1615 \times 10^{19}.$

7.2 Solving Recurrence Relations

Definition *1 A* ***linear homogeneous recurrence relation*** *of* ***degree*** *k with* ***constant coefficient*** *is the recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k \in \mathbf{R}$, $c_k \neq 0$.

The ***solution*** *is uniquely determined if k* ***initial conditions***

$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$ *are provided.*

Characteristic equation

Assume $a_n = r^n$.

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$, *i.e.,*

$r^k - c_1 r^{k-1} - \ldots c_{k-1} r - c_k = 0$

k ***characteristic roots***

Consider a recurrence relation of degree 2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

Characteristic equation

$$r^2 - c_1 r - c_2 = 0$$

Theorem 2 *If C.E. of degree 2 has two* ***distict*** *roots $r_1 \neq r_2$, the* ***solution*** *is*

$$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

proof *Assume $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ is a solution.*

We know $r_1^{\,2} = c_1 r_1 + c_2$ and $r_2^{\,2} = c_1 r_2 + c_2$.

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{\,n-1} + \alpha_2 r_2^{\,n-1}) + c_2(\alpha_1 r_1^{\,n-2} + \alpha_2 r_2^{\,n-2})$$

$$= \alpha_1 r_1^{\,n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{\,n-2}(c_1 r_2 + c_2)$$

$$= \alpha_1 r_1^{\,n-2} r_1^{\,2} + \alpha_2 r_2^{\,n-2} r_2^{\,2} = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$$

$$= a_n.$$

Example *3.5 Consider* $h_n = 2h_{n-1}$ *with* $h_1 = 1$*(degree 1).*

characteristic equation $r - 2 = 0, r = 2.$

$h_n = \alpha 2^n.$ $h_1 = 2\alpha = 1$ $\therefore \alpha = 2^{-1}.$

$\therefore h_n = 2^{n-1}.$ *Note that Tower of Hanoi:* $H_n = 2^n - 1.$

Example *4 Fibonacci number* $f_n = f_{n-1} + f_{n-2}$ *with* $f_0 = 0, f_1 = 1.$

characteristic equation $r^2 - r - 1 = 0.$

$r_1 = (1+5^{1/2})/2, r_2 = (1-5^{1/2})/2.$

$f_n = \alpha_1((1+5^{1/2})/2)^n + \alpha_2((1-5^{1/2})/2)^n.$

$f_0 = \alpha_1 + \alpha_2 = 0.$

$f_1 = \alpha_1(1+5^{1/2})/2 + \alpha_2(1-5^{1/2})/2 = 1.$

$\alpha_1 = 5^{-1/2}, \alpha_2 = -5^{-1/2}.$

$\therefore f_n = 5^{-1/2}((1+5^{1/2})/2)^n - 5^{-1/2}((1-5^{1/2})/2).$

Theorem *2 If C.E. of degree 2 has one* ***double*** *roots* r_0*, the* ***solution*** *is*

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n, \text{ for } \forall \alpha_1, \alpha_2 \in \mathbf{R}.$$

proof *Since* r_0 *is a double root.* $(r - r_0)^2 = 0$.

Consider derivative, $2r(r - r_0) = 0$.

$\therefore$ $\alpha_2 n r_0^n$ *also is a homogeneous solution.*

Example *5* $a_n = 6a_{n-1} - 9a_{n-1}$ *with* $a_0 = 1$ *and* $a_1 = 6$.

C.E $r^2 - 6r + 9 = 0$. $(r-3)^2 = 0$

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$. $a_0 = \alpha_1 = 1$, $a_1 = 3\alpha_1 + 3\alpha_2$, $3\alpha_2 = 3$

$a_n = 3^n + n3^n$.

Theorem *3 If C.E. of degree k of* $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ *has k-distinct roots* $r_1, r_2, \ldots r_k$, *the* ***solution*** *is*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n, \text{ for } \forall \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbf{R}.$$

Example *6 C.E. of degree 3 has three roots 1, 2, 3.*

$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^n$.

Fix α_1, α_2, *and* α_3 *with 3 initial conditions.*

Theorem *4 If C.E. of degree k* $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ *has t-distinct roots* $r_1, r_2, \ldots r_t$ *with* ***muliplicities*** $m_1, m_2, \ldots m_t$, *where* $\Sigma_t\, m_t = k$, *the* ***solution*** *is*

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m_1-1}n^{m_1-1})r_1^{\,n} +$$
$$(\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m_2-1}n^{m_2-1})r_2^{\,n} + \ldots +$$
$$(\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,m_t-1}n^{m_t-1})r_t^{\,n}, \text{ for } \forall\alpha_{i,j} \in \mathbf{R}\ 1 \le i \le t,\ 0 \le j \le m_i-1.$$

Example *7 Suppose C.E. of degree 6 has roots 2, 2, 2, 5, 5, 9*

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)3^n + \alpha_{3,0}9^n.$$

Fix $\alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,0}, \alpha_{2,1}$, *and* $\alpha_{3,0}$ *with 6 initial conditions.*

*Linear **nonhomogeneous** recurrence relation of with constant coefficient*

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

Let $a_n^{(h)}$ *be a solution the **associated homogeneous** recurrence relation*

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}.$$ *Then*

Every solution is of the form $a_n = a_n^{(h)} + a_n^{(p)}$ *where* $a_n^{(p)}$ *is called **particular** solution associated with F(n).*

***Example** 10* $a_n = 3a_{n-1} + 2n$ *with* $a_1 = 3$. $a_n^{(h)} = \alpha 3^n$.

$F(n) = 2n$, *try* $a_n^{(p)} = cn + d$.

$cn + d \equiv 3(c(n-1) + d) + 2n$. *check recurrence*

$(2c+2)n + (2d - 3c) \equiv 0$ $\therefore c = -1, d -3/2.$

$a_n = \alpha 3^n - n - 3/2, a_1 = 3$ $\alpha = 11/6$ *check basis*

$a_n = (11/6)3^n - n - 3/2.$

Example *Tower of Hanoi* $H_n = 2H_{n-1} + 1$ *with* $H_1 = 1$*(degree 1).*

associated *characteristic equation* $r - 2 = 0, r = 2. \therefore H_n^{(h)} = \alpha 2^n$.

$H_n = H_n^{(p)} + \alpha 2^n$.

$F(n) = 1$, *try* $H_n^{(p)} = C$, $C = 2C + 1$(*recurrence*) $\therefore C = -1$.

$H_1 = -1 + 2\alpha = 1$(*basis*) $\therefore \alpha = 1$.

$\therefore H_n = 2^n - 1$.

Check base case, $H_1 = 2 - 1 = 1$. *O.K.*

Check the recurrence, $H_n = 2H_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$. *O.K.*

Example *11* $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$. $a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$.

$F(n) = 7^n$, *try* $a_n^{(p)} = C7^n$ $C7^n = 5C7^{n-1} - 6C7^{n-2} + 7^n$.

$7^2C = 5 \cdot 7 \cdot C - 6C + 49$ $\therefore C = 49/20$.

Theorem *6* $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$,

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$.

*When s is **not** the root of the associated characteristics equation*

$a_n^{(p)} = (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$.

*When s is the **root** of the associated C. E. of **mutiplicity** m*

$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$.

Example *12* $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$.

characteristic root 3 of multiplicity 2

$F(n) = 3^n$	$a_n^{(p)} = p_0 n^2 3^n$.
$F(n) = n3^n$	$a_n^{(p)} = n^2 (p_1 n + p_0) 3^n$.
$F(n) = n^2 2^n$	$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$.
$F(n) = (n^2 + 1) 3^n$	$a_n^{(p)} = n^2 (p_2 n^2 + p_1 n + p_0) 3^n$.

Example *13* $a_n = \sum_{k=1}^{n} k$.

Recurrence relation $a_n = a_{n-1} + n$ *with* $a_1 = 1$.

Characteristic root $r = 1$ *with mumtiplicity 1. and* $F(n) = n(1)^n$.

$\therefore a_n^{(h)} = \alpha \cdot 1^n = \alpha$ *and*

$\therefore a_n^{(p)} = n(p_1 n + p_0) = p_1 n^2 + p_0 n$.

$p_1 n^2 + p_0 n \equiv p_1(n-1)^2 + p_0(n-1)$ *check recurrence*

$\ldots$ $p_0 = p_1 = 1/2$.

$\therefore a_n^{(p)} = 1/2n^2 + 1/2n = n(n+1)/2$

$\therefore a_n = a_n^{(p)} + a_n^{(h)} = n(n+1)/2 + \alpha$ *with* $a_1 = 1$. *check basis*

$a_1 = 1 \cdot 2/2 + \alpha = 1$ $\therefore \alpha = 0$.

$\therefore a_n = n(n+1)/2$.

Basis and recurrence are already checked!

7.3 Application to Analysis of Algorothms(Divide-and-Conquer)

Example 1 Binary search

$$f(n) = f(n/2) + 2$$

Example 3 Merge Sort.

$$M(n) = 2M(n/2) + n$$

Divide-and-Conquer Recurrence Relation

$$f(n) = af(n/b) + g(n) \qquad a \geq 1, \text{ integer } b \geq 2.$$

$$f(n) = af(n/b) + g(n)$$
$$= a^2f(n/b^2) + ag(n/b) + g(n)$$
$$= a^3f(n/b^3) + a^2g(n/b^2) + ag(n/b) + g(n)$$
$$\ldots$$
$$= a^kf(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j)$$

when $n = b^k$, $f(n) = a^kf(1) + \sum_{j=0}^{k-1} a^j g(b^{k-j})$

Theorem *1* $f(n) = af(n/b) + c$ *with* $a \geq 1$, *integer* $b \geq 2$, $c \in \mathbf{R}+$.

$f(n) \in O(\log_b n)$, *if* $a = 1$,

$\in O(n^{\log_b a})$, *if* $a > 1$.

When $n = b^k$, *where k is a positive integer,*

$f(n) = C_1 n^{\log_b a} + C_2$ *where* $C_1 = f(1) + c/(a-1)$ *and* $C_2 = -c/(a-1)$

proof *Let* $n = b^k$ *and* $g(n) = c$.

$f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c = a^k f(1) + c\sum_{j=0}^{k-1} a^j$.

If $a = 1$, $f(n) = f(1) + ck = f(1) + c\log_b n$ $\therefore O(\log_b n)$.

If $a > 1$, $f(n) = a^k f(1) + c(a^k-1)/(a-1) = a^k[f(1) + c/(a-1)] - c/(a-1)$

$= C_1 a^{\log_b n} + C_2 = C_1 n^{\log_b a} + C_2$. $\therefore O(n^{\log_b a})$

Theorem 2 Master Theorem

Let $f(n) = af(n/b) + cn^d$ *with* $a \geq 1$, *integer* $b \geq 2$, $c > 0$, $d \geq 0$.

$$
\begin{aligned}
f(n) &\in O(n^d) && \text{if } a < b^d, \\
&\in O(n^d \log_b n) && \text{if } a = b^d, \\
&\in O(n^{\log_b a}) && \text{if } a > b^d.
\end{aligned}
$$

When $n = b^k$, $f(n) = a^k f(1) + c\sum_{j=0}^{k-1} a^j (b^{k-j})^d$.

$$= a^k f(1) + a^{k-1}b^d + a^{k-2}b^{2d} \ldots + a^1 b^{(k-1)d} + a^1 b^d.$$