6 Counting Methods and Pigeonhole Principle

6.1 The Basics of Counting

Product and Sum Rules

Suppose two task,

m number of ways to do task 1

n number of ways to do task 2

Do ***both*** *task 1* ***and*** *task 2:* ***Multiplication Principle***

m·n ways $|A \times B| = |A| \times |B|$.

Do ***either*** *task 1* ***or*** *task 2, but not both:* ***Addition Principle***

m+n ways $|A \cup B| = |A| + |B|$, *if* $A \cap B = \varnothing$.

Thm 6.1.12 Principle of Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$.

proof *trivial.*

6.2 Permutations and Combinations

Def. 6.2.1 *A **permutation** of n distinct elements $x_1, x_2, \ldots, x_k$ is **an ordering** of n elements $x_1, x_2, \ldots, x_k$.*

Thm. 6.2.3 *There are n! permutation of n elemetns.*

***Thm. 6.2.10 Permutation**(순열) of r-elements of set of n objects.*

$0 \le r < n, P(n, r) = n(n\text{-}1) \ldots (n\text{-}r\text{-}1)$

$0 \le r \le n, P(n, r) = n! / (n\text{-}r)!$

***Thm. 6.2.17 Combination**(조합) of r-elements of set of n objects.*

$0 \le r \le n, C(n, r) = P(n, r) / r! = n! / r!(n\text{-}r)!$

$C(n, r) = C(n, n\text{-}r)$

6.3 Generalized Permutations and Combinations

Thm 6.3.1 *r*-**permutation** *of set of n objects with* **repetition** *allowed.*

n^r. 중복순열

n-ary string of length r.

Thm 6.3.5 *r*-**combination** *of set of n objects with* **repetition** *allowed.*

$H(n, r) = C(n+r-1, r)$ 중복조합

n-ary string of length r.

		중복
순열	$P(n, r) = n! / (n-r)!$	$\Pi(n, r) = n^r$
조합	$C(n, r) = P(n, r) / r! = n! / r!(n-r)!$	$H(n, r)=C(n+r-1, r)$

6.7 Binomial Coefficient and Combinatorial Identities

Thm. 6.7.1 $(a+b)^n$

$= C(n, 0)a^n + C(n, 1)a^{n-1}b + \ldots + C(n, n\text{-}1)ab^{n-1} + C(n, n)ab^n.$

$= C(n, 0)a^n b^0 + C(n, 1)a^{n-1}b^1 + \ldots + C(n, n\text{-}1)a^1 b^{n-1} + C(n, n)a^0 b^n.$

$= \sum_{k=0}^{n} C(n, n-k)a^{n-k}b^k.$

Corollary *1* $\forall n \geq 0: \sum_{k=0}^{n} C(n, k) = (1+1)^n = 2^n.$

Corollary *2* $\sum_{k=0}^{n} (-1)^k C(n, k) = (1+(-1))^n = 0^n = 0.$

Thm. 6.7.6 ***<u>Pascal's Identity and Trialgle</u>***

$1 \leq \forall k \leq n: C(n+1, k) = C(n, k\text{-}1) + C(n, k).$

6.8 The Pigeonhole Principle

***First Form** 1 **The pigeonhole principle**(**First Form**)*

If n objects(pigeons) are placed into k boxes(holes) and $n > k$,

there is at least one or more box containing two or more objects.

Dirichlet drawer principle

***Second Form** If $f: X \to Y$ and $|X| > |Y|$, then f is **not** 1-1.*

Third Form**: **The generalized pigeonhole principle

If $f: X \to Y$ with $|X| = n$ and $|Y| = m$. Then

there is at least one or more boxes containing $\lceil n/m \rceil$ objects.

***proof** by **contradiction**, no box contains $\lceil n/m \rceil$ objects.*

$$n(\lceil n/m \rceil - 1) < m((n/m+1) - 1) = n$$

***Example** $n = 59$ students, $m = 12$ months,*

$$\lceil 59/12 \rceil = \lceil 4.91... \rceil = 5 \text{ students/month}$$