3-1 Cardinality of Sets and Infinite Sets

Def. *Let X and Y be two sets. We define a set of functions from X to Y as*

$$Y^X = \{f \mid f: X \rightarrow Y\}.$$

Thm. *Let X and Y be two sets. The* $|Y^X| = |Y|^{|X|}$.

proof *Let* $|X| = m$, $X = \{1, 2, \ldots, m\}$ *and* $|Y| = n$.

$$Y^X = \{(y_1, y_2, \ldots, y_m) \in Y^m \mid 1 \leq {}^{\forall}i \leq m: f(i) = y_i\}$$

m-bit n-ary string. $\therefore\ |Y^X| = n^m = |Y|^{|X|}$.

Lemma *Let f be a function. Then* $|dom(f)| \leq |codom(f)|$.

proof *f is not* ***onto***.

Fact *Let f be a function. Then* $|range(f)| \leq |codom(f)|$.

proof $range(f) \subseteq codom(f)$.

Question *Let f be a function. Then* $|dom(f)|\ <^{?}=^{?}>\ |range(f)|$. 정의역:치역

Lemma *If there is a* **1-1** *function* $f: X \rightarrow^{1\text{-}1} Y$. *Then* $|X| \leq |Y|$.
proof *Since f is* **1-1**, $|X| =_{\mathbf{1\text{-}1}} |range(f)| \leq |codom(f)| = |Y| \therefore |X| \leq |Y|$.

Lemma *If there is a* **onto** *function* $f: X \rightarrow^{onto} Y$. *Then* $|X| \geq |Y|$.
proof *Since f is* **onto**, $|X| \geq_{\mathbf{onto}} |codom(f)| = |Y| \therefore |X| \geq |Y|$.

Theorem *If there is a* **1-1** *and* **onto** *function* $f: X \rightarrow^{1\text{-}1,\ onto} Y$
or $f: X \leftrightarrow Y$, *for short. Then* $|X| = |Y|$.

Cardinality of Infinite Sets

Def. *A set that has* ***same cardinality*** *with a* ***subset*** *of* $\aleph$*(****natural number****) is called* ***countable****. A set that is* ***not countable*** *is called* ***uncountable****.*

When an ***infinite*** *set A is* ***countable****, we denote cardinality of A as* $\aleph_0$*(****aleph null****).* $|A| = \aleph_0$ *(or* $\aleph$ *for short).*

Three ***disjoint*** *cases (or* ***partitions****) of the* ***cardinalities*** *of* ***Sets***

1) ***finite****(or* ***countable****)*

   $\{x_1, x_2, \ldots, x_n\}$

2) ***countably infinite****(or* ***countable****)*

   $\{x_1, x_2, \ldots, x_n, \ldots\}$

3) ***uncountable****(or* ***uncountably infinite****)See supplement of Ch. 12*

   $\{x_1, x_2, \ldots\}$???

Extension of Set Equivalence and Cardinality Revisited

Definition *5.1 Let A and B are sets. We say the* ***cardinalities*** *of A and B are same,* $|A| = |B|$*, if there is a* ***bijection*** $f: A \leftrightarrow B$.

We say that two sets A and B are ***isomorphic*** *with respect to f,* $A \cong_f B$.

If f is a ***bicection*** *from A to B and vice versa:* $f: A \leftrightarrow B$.

$$^{\forall}a \in A \ ^{\exists 1}f(a) \in B \text{ and } ^{\forall}b \in B \ ^{\exists 1}f^{-1}(b) \in A.$$

We can identify B with A and f, and identify A with B and f^{-1}*(vice versa)*

Set Isomorphim

Extended Set Equivalence

If $A \subset B$, then $|A| < |B|$ is ***true****, only if A and B are finite.*

<u>Counter example</u>

Let $\aleph_0 = \{0, 1, \ldots\}$ and $\aleph_1 = \{1, 2, \ldots\} = \mathbf{Z}^+$ be two ***infinite*** *sets.*

f: $\aleph_0 \rightarrow \aleph_1$ *where $f(n) = n+1$*

Is f a ***function****?*

Is f ***1-1****?* *Is f* ***onto****?* *What is f^{-1}?*

$\therefore \aleph_0 \cong_f \aleph_1$ $\therefore |\aleph_0| = |\aleph_1|$

$\aleph_0 \supset \aleph_1$ *but* $|\aleph_0| = |\aleph_1|$.

Can you ***beleive*** *it?*

But it is ***true*** *in the world of* ***infinite*** *sets!*

$\therefore$ *So we add* ***0*** *to the* ***natural numbers****.*

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$

But let A and B be ***infinite*** *sets. If $A \subset B$, then $|A| \leq |B|$.*

$\mathbf{E} = \{n \mid {}^{\exists} n \in \aleph,\ n = 2k\}$ $\aleph \supset \mathbf{E}$ *but* $|\aleph| = |\mathbf{E}|$. *What is a* ***bijection****?*

$|\mathbf{Z}|$ ¿= $2|\aleph| + 1$ *and* $\aleph \subset \mathbf{Z}$ *but* $|\aleph| = |\mathbf{Z}|$. *What is a* ***bijection****?*

Consider a ***pair*** *of natural numbers* $\aleph_0 \times \aleph_0 = \{(i, j) \mid i, j \in \aleph_0\}$.

$f: \aleph_0 \times \aleph_0 \rightarrow \aleph_0$ $f(i, j) = (i+j)(i+j+1)/2 + j$. *What is* f^{-1}*?*

f is a bijection.

$\therefore |\aleph| = |\aleph \times \aleph| = |\aleph^2| = |\mathbf{Q}| = \aleph$.

$|\aleph^k| = \aleph$ ${}^{\forall} k \geq 1$ *What is a* ***bijection****?*

Even(Odd) numbers, positive integers, natural numbers, and even rational numbers are all ***countably infinite****.*

Are there any ***infinite*** *sets whose cardinality is larger that* $\aleph$*?*