3. Functions, Sequences, and Relations

3.1 Functions

A ***function(mapping**, **transformation**) f from X to Y is*
an <u>assignment</u> of ***each(all)*** *element of X* ***exactly one*** *element of Y.*

Def. 3.1.1 *Let X and Y be sets. A* ***function*** $f \subseteq X \times Y$(***relation***) .∍.
$\forall x \in X, \exists_1 y \in Y$, .∍. $(x, y) \in f$.

X: **domain**(정의역) of the function f

Y: **codomain**(공역) of the function f

$\{y \in Y \mid \forall x \in X, (x, y) \in f\} \subseteq Y$ **range**(치역) of the function f

$f: X \to Y$ *f is a function from the* ***domain*** *X to* ***codomain*** *Y*
We write $f(x) = y$ *instead of* $(x, y) \in f$.

*A **function** is a relation(관계) satisfying two **conditions***

1) ***total**: for **all** elements of X(domain)*
2) ***unique**: **exactly one** elements of Y(codomain)*

Def. 3.1.10 *Let* $x \in \mathbf{Z}$ *and* $n \in \mathbf{Z}^+$. *Then*

x mod n = positive remainder when x is divided by n.

$$\text{mod}: \mathbf{Z} \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+.$$

We may write mod(x, n) = m instead of x mod n = m.

Def. 3.1.16 *The **floor** and **ceiling** function:* $\lfloor \cdot \rfloor \lceil \cdot \rceil : \mathbf{R} \rightarrow \mathbf{Z}$,

$\lfloor x \rfloor = n \in \mathbf{Z}$, *n is the **greatest integer** such that* $n \leq x$.*(floor)*

$\lceil x \rceil = n \in \mathbf{Z}$, *n is the **least integer** such that* $n \geq x$.*(ceiling)*

Let $x \in \mathbf{Z}$ *and* $y \in \mathbf{Z}^+$. *Then*

if $x \geq 0 \rightarrow x \div y = \lfloor x/y \rfloor + x \bmod y$

$\mid x \leq 0 \rightarrow x \div y = \lceil x/y \rceil + x \bmod y$

fi

Def. 3.1.21 *Let f: X → Y. f is* ***one-to-one****(1-1 or* ***injective;*** 단사 *), if*

$\forall x \in X \forall y \in Y: [(x \neq y) \Rightarrow (f(x) \neq f(y))]$, *or logically equivalent*

$\forall x \in X \forall y \in Y: [(f(x) = f(y)) \Rightarrow (x = y)]$. ***contrapositive****(* 대우 *)*

An ***injective function*** *is called* ***injection****.*

Thm. 3.1.21' *Let f: X → Y. f is* ***1-1*** $\Rightarrow |X| \leq |Y|$.

proof *Since f is* ***1-1****,* $|X| =_{1\text{-}1} |range(f)| \leq |codom(f)| = |Y| \therefore |X| \leq |Y|$.

Def. 3.1.28 *7 Let f: X → Y. f is* ***onto*** *(or* ***surjective;*** 전사 *), iff*

$\forall y \in Y \exists x \in X: (f(x) = y)$, *or* $f(X) = Y$*(****range*** *=* ***codomain****).*

A ***surjective function*** *is called* ***surjection*** *(or* ***correspondence****).*

Thm. 3.1.28' *Let f: X → Y. f is* ***onto*** $\Rightarrow |X| \geq |Y|$.

proof *Since f is* ***onto****,* $|X| \geq |range(f)| =_{onto} |codom(f)| = |Y| \therefore |X| \geq |Y|$.

Def. 3.1.34 *Let f: X → Y. If f is both* **1-1** *and* ***onto*** *then f is called* ***one-to-one onto*** *(or* ***bijective;*** 전단사 *).*
A ***bijective function*** *is called* ***bijection*** *or* ***one-to-one onto function*** *or* ***one-to-one correspondence;*** 짝짓기 *).*

Thm. 3.1.34' *Let f: X → Y and f be* ***one-to-one*** *and* ***onto(bijective)****.*
The ***inverse function*** *of f, denoted* $f^{-1}: Y \to X$*, is also is a* ***bijection****,*

$f^{-1} = \{(y, x) \mid x \in X, f(x) = y \in Y\}$. *Then*

$\forall y \in Y: \exists_1 f^{-1}(y) = x \in X$.϶. $f(x) = y$.

Thm. 3.1.34" *Let f: X → Y and f be* ***one-to-one*** *and* ***onto(bijective)****. Then*
$|X| = |Y|$.

Def. 3.1.40 *Let g: A → B and f: B → C. The* ***composition*** *of f and g, denoted by f ∘ g: A → C, is defined by*

$(f \circ g)(a) = f(g(a))$ or $f \circ g = \{(a, c) | f(a) = b, g(b) = c\}$.

Def. 3.1.40' Identity function(relation) *on A*

$\iota_A: A \to A$.∋. $\iota_A = \{(a, a) | a \in A\}$ or $\forall a \in A: \iota_A(a) = a$.

Thm. 3.1.40" *Let f: A → A. Then*

$f \circ \iota_A = \iota_A \circ f = f$.

ι_A *is a* ***identity*** *function for composition of functions on A.*

Def. 3.1.40"' *Let* $f_1, f_2: A \to \mathbf{R}$. $f_1+f_2, f_1 f_2: A \to \mathbf{R}$ *is defined by*

$(f_1+f_2)(x) = f_1(x) + f_2(x)$ and $f_1 f_2(x) = f_1(x)f_2(x)$.

Extension of Set Equivalence and Cardinality Revisited

We say that two sets A and B are ***isomorphic*** *with respect to f,* $A \cong_f B$.

If f is a ***bicection*** *from A to B,* $f: A \leftrightarrow B$.(짝짓기)

$^{\forall}a \in A: {}^{\exists}{}_1 f(a) \in B$ and $^{\forall}b \in B\ {}^{\exists}{}_1 f^{-1}(b) \in A$.

We can identify B with A and f, and identify A with B and f^{-1}*(vice versa)*

Set Isomorphim

Extended Set Equivalence

Def. Cardinality of Set, revisited

Let A and B are sets. We say the ***cardinalities*** *of A and B are same,*

$|A| = |B|$, *if there is a* ***bijection*** $f: A \leftrightarrow B$.

Operation *vs.* ***Function***

Operation *is a function on* ***same sets.***

Def. 3.1.45 *Let* $f: X \times X \rightarrow X$. *Then f is called* **binary operation** *on X.*

$\forall a, b, c \in X$, *we may write* $a\ f\ b = c$ *instead of* $f(a, b) = c$.

f is the ***infix binary operator.***

Def. 3.1.48 *Let* $f: X \rightarrow X$. *Then f is called* ***unary operation*** *on X.*

$\forall a, b \in X$, *we may write* $f\ a = b$ *instead of* $f(a) = b$.

f is the ***prefix unary operator.***

Def. 3.1.48' *Let $f: X^n \rightarrow X$. Then f is called n-**ary operation** on X.*

$^\forall a_1, a_2, \ldots, a_n, b \in X, f(a_1, a_2, \ldots, a_n) = b \in X,$

*f is the **prefix** n-**ary operator**.*

Def. 3.1.48" *Let $f: X^n \rightarrow X^{\mathbf{m}}$. Then f is called n-**ary operation** on X.*

$^\forall a_1, a_2, \ldots, a_n \in X, f(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m) \in X^{\mathbf{m}},$

*f is called **multiple value returning function***

Global variable considered harmful

Choe, Kwang-Moo

GOTO statemenet considered harmful

E. W. Dijkstra

3.2 Sequences and Strings

Sequence *a* ***function*** *whose* ***domain*** *is a set of* ***numbers***.

$a: \mathbf{N} \to A$. $\mathbf{N}$ *is a set of numbers* A *is a set.*

We write a_n *instead of* $a(n)$.

n is called ***index*** *of the sequence* a_n

$\mathbf{N}$ *is called* ***index*** *set of the sequence* a_n.

$\{a_n\} = \{a_n | n \in \mathbf{N}\} = \{a_n\}_{n \in \mathbf{N}}$.

Some Useful Sequences

polynomial *sequences* $n^2, n^3, n^4, \ldots n^k, \ldots$

exponential *sequences* $2^n, 3^n, \ldots k^n, \ldots, n!, \ldots, n^n, \ldots$

Definition *6 Let* $\{s_n\}_{n\in\mathbf{N}}$*(*s*:* $\mathbf{N} \to A$*) be a sequence over A and*

$\leq$ *is a* ***partial order*** *on A(See 3.3). Then*

If $s_n < s_{n+1}$, *s is* ***increasing*** *or (****strictly increasing****),*

If $s_n > s_{n+1}$, *s is* ***decreasing*** *or (****strictly decreasing****),*

If $s_n \leq s_{n+1}$, *s is* ***nondecreasing*** *or (****increasing****), and*

If $s_n \geq s_{n+1}$, *s is* ***nonincreasing*** *or (****decreasing****).*

Def. 3.2.11 *Let* $\{s_n\}_{n\in\mathbf{N}}$ *be a sequence.*

If $\mathbf{I} \subseteq \mathbf{N}$, *then* $\{s_i\}_{i\in\mathbf{I}}$ *is the* ***subsequence*** *of* $\{s_n\}_{n\in\mathbf{N}}$.

Def. 3.2.14

$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n = \sum_{i \in \mathbf{N}_{m,n}} a_i$, ***sum(sigma)*** *notation*

$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot \ldots \cdot a_n. = \prod_{i \in \mathbf{N}_{m,n}} a_i.$ ***product(pi)*** *notation*

where $\mathbf{N}_{m,n} = \{m, m+1, \ldots, n\}$ $(m \le n)$.

m: lower limit *n: upper limit* *i: index variable*

$\mathbf{N}_{m,n}$*: index set*

Def. 3.2.10 *A* ***String*** *over V.*

Let $n \geq 0$, $s: \{1, 2, \ldots, n\} \rightarrow V = \{a, b, \ldots, z\}$.

We write s = (b, o, y) or s = boy for short

instead of s(1) = b, s(2) = o, s(3) = y.

s is called the finite ***string*** *over V of length n.*

V is called the ***vocabulary***(***alphabet***) *of string s.*

length *of a string* $|\alpha|$

number of elements in α.

null(***empty***) *string* λ *or* ε

Recursive definition of V^n.

basis $V^0 = \{\lambda\}$

recursion $V^n = V^{n-1} \cdot V$.

universe *of string over V*

$V^* = V^0 \cup V^1 \cup V^2 \cup \ldots$

where $V^0 = \{\lambda\}, V^1 = V, V^2 = V \cdot V, \ldots, V^n = V^{n-1} \cdot V, , \ldots$

$\forall \alpha \in V^*., \alpha \cdot \varepsilon = \varepsilon \cdot \alpha = \alpha.$

ε *is a* ***identity*** *element on* ***concatenation***.

concatenation *of two string* α *and* $\beta(\alpha, \beta \in X^*)$

$\cdot : X^* \times X^* \rightarrow X^*$

$\alpha \cdot \beta = \alpha\beta$ $|\alpha \cdot \beta| = |\alpha| \cdot |\beta|$

school·boy = schoolboy

Def 3.2.26 *Let* $\alpha = \gamma\beta\delta$ *where* $\alpha, \beta, \gamma, \delta \in V^*$.
Then β *is a* ***substring*** *of* α, γ *is a* ***prefix*** *of* α, *and* δ *is a* ***postfix*** *of* α.
Ex. α = *boy. substring*(α) = {λ, *b, o, y, bo, oy, boy*}
prefix(α) = {λ, *b, bo, boy*}
posfix(α) = {λ, *y, oy, boy*}

<u>한글</u>

A set of ***strings*** *over* Σ_{24} = { ㄱ , ㄴ , …, ㅎ } ∪ { ㅏ , ㅑ , …, ㅣ }.
My MS dissertation $\Sigma_{29} = \Sigma_{24} \cup$ { ㄲ , ㄸ , ㅃ , ㅆ , ㅉ }.
computer keyboard $\Sigma_{33} = \Sigma_{29} \cup$ { ㅐ , ㅒ , ㅔ , ㅖ }.
삼성 hp Σ_{11} = { ㅣ , ·, ㅡ, ㄱ , ㄴ , ㄷ , ㅂ , ㅅ , ㅈ , ㅇ , *blank*}.
LG hp Σ_{12} = { ㅏ , ㅗ , ㅡ , ㅣ , ㄱ , ㄴ , ㄹ , ㅁ , ㅅ , *, +}.
한글음절 (unicode) = $19 \times 21 \times 28 = 1{,}1172$

3.3 Relations

Def. 3.3.1 *If $R \subseteq X \times Y$, then R is called a (**binary**) **relation** from X to Y.*

We may write $a\ R\ b$, if $(a, b) \in R$.

*Let $R \subseteq X \times X$, R is called a **relation on** X.*

*A **digraph**(directed graph) $G = (V, E)$ where $E \subseteq V \times V$.*

1) *V: a set of vertices*
2) *$E \subseteq V \times V$: a set of edges*

Let R be a relation on $X(R \subseteq X \times X)$. Then

Def. 3.3.6 *R is **reflexive**, if $\forall x \in X: (x, x) \in R$(or $x\ R\ x$). or $\iota_X \subseteq R$.*

Def. 3.3.6' *R is **irreflexive**, if $\forall x \in X: (x, x) \notin R$(or $x \not R\ x$). or $R \cap \iota_X = \emptyset$.*

Def. 3.3.9 *R is **symmetric**, if $\forall x, y \in X: (x, y) \in R \Rightarrow (y, x) \in R$.*

(or $x\ R\ y \Rightarrow y\ R\ x$) or $R = R^{-1}$.

Def. 3.3.12 *R is* ***antisymmetric****, if* $\forall x, y \in X$: $(x,y) \in R \wedge (y,x) \in R \Rightarrow x=y$.

(or $((x\ R\ y) \wedge (x \neq y)) \Rightarrow (y \not R\ x)$ or $R \cap R^{-1} \subseteq \iota_X$.

Thm. 3.3.12’ *If R is* ***antisymmetric****,*

then R is not (always) ***reflexive*** *nor* ***irreflexive****.*

Def. 3.3.12” *R is* ***asymmetric****,()* $\forall x, y \in X$, $(x, y) \in R \Rightarrow (y, x) \notin R$.

(or $x\ R\ y \Rightarrow y \not R\ x$) or $R \cap R^{-1} = \emptyset$.

Thm. 3.3.12”’ *If R is* ***asymmetric****,*

then R is ***irreflexive*** *and* ***antisymmetric****.*

Def. 3.3.13 *R is* ***transitive****, if* $\forall x, y, z \in X$: $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$.

(or $x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z$ or $R \cdot R \subseteq R$.

Def. 3.3.20 *Let* $R \subseteq X \times X$. *Then R is a* ***partial order****,*
if R is ***reflexive****,* ***antisymmetrux****, and* ***transitive****.*

Def. 3.3.20' *Let* $\leq \subseteq X \times X$. *Then If* $\leq$ *is a* ***partial order****. Then*
$(X, \leq)$ *is called a* ***poset****(or* ***partially ordered sets****).*
If $(x \leq y) \vee (x \leq y)$, *then x and y are* ***comparable****.*
Otherwise$((x \neg\leq y) \wedge (y \neg\leq x))$, ***incomparable****.*
If $^\forall x, y \in X$, *x and y are compareble* $\leq$ *is* ***total****(or* ***linear****) order.*

Exa. 3.3.21 *Let* $R \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$. *and* $(x, y) \in$ ***Divides****, if x* ***divides*** *y. Then*
$(\mathbf{Z}^+$, ***Divides****) is a* ***poset****.*
$(\mathbf{Z}^+, \leq)$ *is a* ***total*** *order.*
$(2^S, \subseteq)$ *is a* ***poset****.*

Hasse Diagram

Let $(A, \leq)$ be a poset. We say a **covers** b, if $a < b \land \neg\exists c \in A, a \leq c \land c \leq b$.
(A, covers) is a Hasse Diagram.

Maximal and Minimal Elements

Let $(A, \leq)$ be a poset.
We say a is a **maximal**, if $\neg\exists b \in A, a < b$.
We say a is a **minimal**, if $\neg\exists b \in A, b < a$.
We say a is a **greatest element**, if $\forall b \in A, b \leq a$.
We say a is a **least elements**, if $\forall b \in A, a \leq b$.
Let $S \subseteq A$. $u \in A$ is called **upper bound** of S, if $\forall a \in S, a \leq u$.
$l \in A$ is called **lower bound** of S, if $\forall a \in S, l \leq a$.
$x \in A$ is called the **least upper bound** of S, if $\forall a \in S, a \leq x$,
$x \leq z$, if $\forall z \in$ upper bound of S.
$y \in A$ is called the **greatest lower bound** of S, if $\forall a \in S, y \leq a$,
$z \leq y$, if $\forall z \in$ lower bound of S.

Lattice

*A poset $(A, \leq)$ is called **lattice**. If every pair of elements has both a **least upper bound**, and a **greatest lower bound**.*

$(\mathbf{Z}, \leq)$ is a lattice. $\quad$ min, max: $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$

$(\mathbf{Z}^+, |)$ is a lattice. $\quad$ gcd, lcm: $\mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$

$(2^S, \subseteq)$ is a lattice. $\quad$ $\cap, \cup: 2^S \times 2^S \to 2^S$

Def. 3.3.23 *Let $R \subseteq X \times Y$, The **inverse** of R, denoted $R^{-1} \subseteq Y \times X$,*

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$$

Def. 3.3.25 *Let $R_1 \subseteq X \times Y$, and $R_2 \subseteq Y \times Z$. The **composition** of R_1 and R_2, denoted $R_2 \circ R_1 \subseteq X \times Z$,*

$$R_2 \circ R_1 = \{(x, z) \in X \times Z \mid (x, y) \in R_1, (y, z) \in R_2\}$$

Three faces *of the relation R form A to B:* $R \subseteq A \times B$.

i) R is a ***set of pairs****(subset of Cartesian product)*

$R \subseteq A \times B$, $(a, b) \in R$ where $a \in A$ and $b \in B$.

ii) R is a ***infix binary boolean(relational) function***

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$, $a\ R\ b$ where $a \in A$ and $b \in B$.

*R is r****ectangular boolean marix****. (See Sec. 3.5)*

iii) R is a ***function*** *from A to* 2^B.

$R: A \rightarrow 2^B$.

$R(a) = \{b_1, b_2, \ldots, b_n\}$ where $a \in A$ and $\{b_1, b_2, \ldots, b_n\} \in 2^B$.

if $1 \leq {}^\forall i \leq n$, $(a, b_i) \in R$ or $a\ R\ b_i$ for $n \geq 0$,

if $n=0$, $R(a) = \{b_1, b_2, \ldots, b_n\} = \emptyset$.

Note that ${}^\forall a \in A$, ${}^\exists{}_1\{b_1, b_2, \ldots, b_n\} \in 2^B$. ***(unique)***

set valued *function.*

3.4 Equivalence Relations

Thm 3.4.1 *Let S be a **partition** of a set X. We define a relation on X as*

$R = \{(x, y) \in X \times X \mid x, y \in s, s \in S\}$.

*Then R is **reflexive**, **symmetric**, and **transitive**.*

proof $\forall x, y, z \in X$: $\exists_1 s \in S$.э. $x, y, z \in s$.

$\therefore (x, x) \in R$. $\therefore$ *R is **reflexive**.*

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$. $\therefore$ *R is **symmetric**.*

$\therefore (x, y), (y, x) \in R \Rightarrow (x, z) \in R$. $\therefore$ *R is **transitive**.*

Thm 3.4.3 *A relation that is **reflexive**, **symmetric**, and **transitive** is called an **equivalent** relation.*

Exa. $= (\equiv \iota_{\mathbf{N}}) \subseteq \mathbf{N} \times \mathbf{N}$ *is an **equivalent** relation*

$< \subseteq \mathbf{N} \times \mathbf{N}$ *is a **irreflexive total** order*

$\leq (\equiv < \cup =) \subseteq \mathbf{N} \times \mathbf{N}$ *is a **reflexive total** order*

Thm 3.4.8 *Let* $R \subseteq X \times X$ *be an* ***equivalent*** *relation on* X*, let*

$[a]_R = \{x \in X \mid a\ R\ x\}$ ***equivalent class*** *of* X *given by the relation* R.

Then $S = \{[a]_R \mid a \in X\}$ *is a* ***partion*** *of* X.

proof *Since* R *is* ***reflexive***, $^{\forall}a \in X: a\ R\ a.\ \therefore\ a \in [a]_R$.

If $a\ R\ b$*, then* $[a]_R \subseteq [b]_R$*, since* R *is* ***transitive*** *and*

$[b]_R \subseteq [a]_R$*, since* R *is* ***symmetric*** *and* ***transitive***. $\therefore$ *If* $a\ R\ b$ $[a]_R = [b]_R$.

If $a \not R\ b$, $[a]_R \cap \subseteq [b]_R = \emptyset$.

Realtion $R \subseteq X \times X$ $|R| \le n^2$ $O(n^2)$ *where* $|X| = n$.

Equivalent *relation* $S = \{[a]_R \mid a \in X\}$ $|S| \le n$ $O(n)$.

3.5 Matrices of Relations
3.6 Relational Database