

1 Sets and Logic

1.1 Sets

An **unordered** (distinguished) collection of objects

An **object** in a set is called an **element** or a **member** of the set.

(An **ordered** (undistinguished) collection of objects
orderd pair, triple, quaduruple, ..., n -tuple)

Two ways to define sets

i) To **enumerate** the elements(원소나열법) (1.1.1)

$$A = \{a_1, a_2, \dots, a_n\} \quad \text{finite sets}$$

$$A = \{a_1, a_2, \dots\} \quad \text{infinite sets}$$

ii) To **specify conditions**(조건제시법) (1.1.2)

$$A = \{x \mid p(x)\} \quad p(x): \text{predicate (see Sec. 1.5 Quantifiers)}$$

$$A = \{x \in U \mid p(x)\} \quad U: \text{universe (domain) of discourse}$$

iii) to write a program(?; CS322)

Some important sets in discrete mathematics(p 2)

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

set of integers.

$$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z} \text{ and } q \neq 0\}$$

set of rational numbers.

R

set of real numbers.

$$\mathbf{Z}^+ = \{1, 2, 3, \dots\}$$

set of positive natural numbers.

$$\mathbf{N} = \{0, 1, 2, 3, \dots\}$$

set of natural numbers.

$$\mathbf{N}_n = \{n, n+1, \dots\}$$

$$n \in \mathbf{Z}$$

Def. Cardinality of a set(p 3)

$|A|$ *the number of elements in the set A*

A set contain its elements.(p 3)

$x \in X$ “*an object x is an element of the set X*”

$x \notin X$ “*an object x is not an element of the set X*”

Def. Empty set(p 3)

The set with no elements is **empty**(**null** or **void**) set denoted \emptyset or $\{\}$.

Note that $|\emptyset| = |\{\}| = 0$.

Def. Equality of two sets(p 3)

Two sets X and Y are **equal** and we write $X = Y$,

If X and Y have the **same** elements.

Otherwise

a set X is **not** equal to Y (written $X \neq Y$).

Note that $\{\} = \emptyset \neq \{\emptyset\}$.

Def. Subsets(p 4)

The set X is said to be a **subset** of Y (written $X \subseteq Y$),
if every elements of X is also an elements of Y .

Or

If for **every** x , if $x \in X$, then $y \in Y$.

Def. Equality of two sets, revisited

$X = Y$ if $X \subseteq Y \wedge Y \subseteq X$.

i) For **every** x , if $x \in X$, then $x \in Y$, and

ii) For **every** x , if $x \in Y$, then $x \in X$.

$$X \subseteq Y \wedge Y \subseteq X$$

Note that for **every** set X , $X \subseteq X$ and $\emptyset \subseteq X$.

Def. Proper subsets(p 5; 진부분집합)

If $X \subseteq Y$ and (but) $Y \not\subseteq X$, then we write $X \subset Y$ and

we say that X is a **proper subset** of Y , and we write $X \subset Y$.

Note that for **every** set X , $X \not\subset X$ and $\emptyset \not\subset X$.

Def. Power Set of a Set(p 5)

The set(**collection**) of all subsets of a set A , denoted $P(A)$ or 2^A , is the **power set** of A .

$$P(A) = 2^A = \{B \mid B \subseteq A\}$$

Note a **collection**(set) may have a set as an **element** of the **collection**.

$$X \subseteq A \quad \Leftrightarrow \quad X \in 2^A.$$

Exa. 1.1.13 Let $A = \{a, b, c\}$. Then

$$P(\{a, b, c\}) = 2^{\{a, b, c\}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$|P(\{a, b, c\})| = |2^{\{a, b, c\}}| = 8.$$

$$0_c 0_b 0_a \leftrightarrow \emptyset \quad 0_c 0_b 1_a \leftrightarrow \{a\} \quad 0_c 1_b 0_a \leftrightarrow \{b\} \quad 0_c 1_b 1_a \leftrightarrow \{a, b\}$$

$$1_c 0_b 0_a \leftrightarrow \{c\} \quad 1_c 0_b 1_a \leftrightarrow \{a, c\} \quad 1_c 1_b 0_a \leftrightarrow \{b, c\} \quad 1_c 1_b 1_a \leftrightarrow \{a, b, c\}$$

isomorphic(See **Sec. 3.1**) to the **binary number** of **length 3**

$$|\{a, b, c\}| = 3 \quad 8 = 2^3.$$

$$|2^A| = |P(A)| \stackrel{?}{=} 2^{|A|} \quad \text{See Sec 2.4 Mathematical Induction(Thm. 2.4.6)}$$

Definitions of set operations Let X and Y be two sets. Then(p 6)

1. **Union** of X and Y $X \cup Y = \{x \mid x \in X \vee x \in Y\}$

2. **Intersection** of X and Y $X \cap Y = \{x \mid x \in X \wedge x \in Y\}$

3. **Difference** of X and Y $X - Y = \{x \mid x \in X \wedge x \notin Y\}$

Relative complement of Y from X

4. Let U be a **universe**(**universal set**) and $X \subseteq U$. Then(p 7)

Absolout complement of X $\bar{X} = U - X = \{x \in U \mid x \notin X\}$

Two sets X and Y are **disjoint**,

if their **intersection** is **empty**, i.e. $A \cap B = \emptyset$.

Let S be a **collection** of sets. The **collection** S is **pairwise disjoint**(p 6)

Every pair of set X and Y in S are **disjoint**, i.e. $X \cap Y = \emptyset$.

Venn diagram(p 7)

pictorial view of sets with **universe**

n sets

2^n regions

Thm. 1.1.21 Algebraic Rules of Set Equalities(p 8-9)

Let U be universal set and $A, B,$ and C be any subsets of U .

$$(\forall A, B, C \subseteq U \equiv \forall A, B, C \in 2^U)$$

1. **Associative laws** $(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$

2. **Commutative laws** $A \cup B = B \cup A$ $A \cap B = B \cap A$

3. **Distributive...** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **Identity laws** $A \cup \emptyset = A$ $A \cap U = A$

5. **Complement laws** $A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$

6. **Idempotent laws** $A \cup A = A$ $A \cap A = A$

7. **Domination(Bound) laws** $A \cup U = U$ $A \cap \emptyset = \emptyset$

8. **Absorption laws** $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

9. **Involution law** $\overline{\overline{A}} = A$

10. **0/1 laws** $\overline{\emptyset} = U$ $\overline{U} = \emptyset$

11. **De Morgan's laws** $\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof *insight for proof using Venn diagram*

Let S be a **collection** of sets. Then we can write

$$\cup S = \{a \mid \text{for **some** } A \in S \text{ } \exists. a \in A\} \quad \cap S = \{a \mid \text{for **all** } A \in S, a \in A\}$$

$$\cup S = \{a \mid \exists A \in S \text{ } \exists. a \in A\} \neq S \quad \cap S = \{a \mid \forall A \in S, a \in A\}$$

Let $S = \{A_1, A_2, \dots, A_n\}$ be a **collection** of n -sets. Then we can write

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n, \quad \bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n.$$

i : **index variable**

Let $I = \{1, 2, \dots, n\}$. Then

I : **index set**

$$\bigcup_{i \in I} A_i = \bigcup_{i=1}^n A_i,$$

$$\bigcap_{i \in I} A_i = \bigcap_{i=1}^n A_i.$$

Unions and Intersections are *associative* (semigroup)

If an operation is **associative**,

the **binary** operation extends to **n -ary** operation

Let $x_1, x_2, \dots, x_n \in \mathbf{R}$. Then

$$\sum_{i=1}^n x_i = x_1 + x_2 + \dots + x_n, \quad \prod_{i=1}^n x_i = x_1 \cdot x_2 \cdot \dots \cdot x_n.$$

Compare $\sum_{i=1}^n x_i$ with $\sum_{i \in I} x_i$ where $I = \{1, 2, \dots, n\}$.

for $i := 1$ *to* n *do* Σx_i *od*

for $i \in I$ *do* Σx_i *od*

Def. A *partion* of a set A is an *exhaustive pairwise disjoint collection* of *subsets* of A .

$Par(A) = \{A_1, A_2, \dots, A_n\}$ such that $(\exists.)$

i) $\cup_{i \in N_n} A_i = A$ **exhaustive** $(1 \leq \forall i \leq n: A_i \subseteq A)$

ii) $\cap_{i \neq j \in N_n} (A_i \cap A_j) = \emptyset$ **pairwise disjoint**

Note that $\cup_{A \in Par(A)} Par(A) = A \wedge \cap_{A \in Par(A)} Par(A) = \emptyset$. BUT

$\cup_S S = S \wedge \cap_S S = \emptyset$ does not **gurantee** the collection S to be a partition.

Cartesian product of two sets(p10)

Def. An **ordered pair** (순서쌍) of elements.

$$(a, b) \neq (b, a) \quad \text{but } \{a, b\} = \{b, a\}.$$

Def. Let A and B be two sets. We define **Cartesian product** of A and B as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

(a, b) an **ordered pair**

$?X \times Y$ X - Y 평면 (중)

(Example) Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

$$|A \times B| = 2 \times 3 = |A| \times |B|$$

$(A \times B) \times C \neq A \times (B \times C)$, since $((a, 1), \alpha) \neq (a, (1, \alpha))$.

BUT

Def. The **Cartesian product** of the n sets A_1, A_2, \dots, A_n , denoted by

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

Def. The **ordered n -tuple** (a_1, a_2, \dots, a_n) is the **ordered collection**, that has a_1 as its first element, a_2 as its second element, \dots , a_n as its n -th element.

$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$, iff, $a_i = b_i$ for $i = 1, 2, \dots, n$.

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \equiv (a_1 = b_1) \wedge (a_2 = b_2) \wedge \dots \wedge (a_n = b_n).$$

Note that $(A \times B) \times C \neq A \times (B \times C) \neq A \times B \times C$.

$$\text{BUT } |(A \times B) \times C| = |A \times (B \times C)| \leftrightarrow |A \times B \times C|$$

$$\text{BUT } (A \times B) \times C \leftrightarrow A \times (B \times C) \leftrightarrow A \times B \times C \quad \text{See [Sec. 3.1](#)}$$

Four cases (binary relations) between two sets A and B.

1. equal $A = B$
2. subset $A \subseteq B$
 - 2.1. proper subset $A \subset B$
3. disjoint $A \cap B = \emptyset$
4. incomparable Otherwise

Four regions in the Venn diagram of two sets A and B.

$$i) A \cap B \quad ii) \bar{A} \cap B \quad iii) A \cap \bar{B} \quad iv) \bar{A} \cap \bar{B}$$

Truth table in predicate logic with two variables

Four cases for between two set and for regions in the Venn diagram

- i) $(A \cap \bar{B} = \emptyset) \wedge (\bar{A} \cap B = \emptyset) \equiv A = B,$ **equal**
- ii) $(A \cap \bar{B} = \emptyset) \text{ or } (\bar{A} \cap B = \emptyset) \equiv A \subseteq B \text{ or } B \subseteq A,$ **subset**
- iii) $(A \cap B = \emptyset) \equiv A \cap B = \emptyset,$ **disjoint**
- iv) otherwise **incomparable**

1.2 Propositions

Propositions(명제)

*a declarative sentence that is either true or false,
but not both nor neither*

letters denoting propositions p, q, r, s, \dots

T: true value

F: false value

propositional calculus or propositional logic

compounded propositions

*propositions that are formed from existing propositions
using logical operators(connectives)*

Def. 1.2.1; 3 Conjunction(And)

Let p and q be a propositions, $p \wedge q$ is a (compounded) proposition, called **conjunction** of p and q , or “ p and q ”.

truth table

Def. 1.2.1; 6 Disjunction(or; Inclusive or)

Let p and q be a propositions, $p \vee q$ is a proposition, called **disjunction** of p and q , or “ p or q ”.

Def. 1.2.9 Negation(not)

Let p be a proposition, $\neg p$ is a (new) **compounded** proposition, called **negation** of p , or “**not** p ”.

Compare the truth tables for \wedge , \vee , and \neg with the Venn diagrams for \cap , \cup , and $\bar{}$ in set operations (Sec. 1.1).

1.2.13' Algebraic Rules of Logical Equivalences

1. **Associative laws** $(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
2. **Commutative laws** $p \vee q = q \vee p$ $p \wedge q = q \wedge p$
3. **Distributive laws** $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
4. **Identity laws** $p \vee \mathbf{F} = p$ $p \wedge \mathbf{T} = p$
5. **Complement laws** $p \vee \neg p = \mathbf{T}$ $p \wedge \neg p = \mathbf{F}$
6. **Idempotent laws** $p \vee p = p$ $p \wedge p = p$
7. **Domination laws** $p \vee \mathbf{T} = \mathbf{T}$ $p \wedge \mathbf{F} = \mathbf{F}$
8. **Absorption laws** $p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$
9. **Double negation law** $\neg(\neg p) = p$
10. **T/F laws** $\neg \mathbf{T} = \mathbf{F}$ $\neg \mathbf{F} = \mathbf{T}$
11. **De Morgan's laws** $\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$

Compare with the algebraic rules of set equalities in p. 6(Thm 1.1.21)

$$(\{\mathbf{T}, \mathbf{F}\}, \vee, \wedge, \neg, \equiv) \leftrightarrow (\{U, \emptyset\}, \cup, \cap, \bar{}, =)$$

isomorphism between positional logic and set algebra

Syntax (문법) of *composite propositions*

\neg	<i>negation</i>	<i>unary prefix operator</i>
\wedge, \vee	<i>conjunctive, disjunctive</i>	<i>binary infix operator</i>
$(,)$		<i>precedence of operator</i>
T, F		<i>constant</i>
p, q, r, s		<i>variables</i>

Syntactic grammars for propositional logic, set operation and 식

p	$::=$	T F v	/	$p \vee p$	/	$p \wedge p$	/	$\neg p$ (p)
S	$::=$	\emptyset U S	/	$S \cup S$	/	$S \cap S$	/	\overline{S} (S)
식	$::=$	수 v		식 + 식		식 * 식		- 식 $(식)$
				/ $\sin(식)$		식 ^식	/	$\log(식)$

1.3 Conditional Propositions and Logical Equivalence

Def. 1.3.1 Let p and q be propositions, the **proposition**

“if p then q ” (1.3.2) “ p , only if p ”

conditional proposition(or **implication**) written $p \rightarrow q$,

p : **hypothesis**(or **antecedent**; **premise**; 가정) of $p \rightarrow q$.

q : **conclusion**(or **consequence**; 결론) of $p \rightarrow q$.

Def. 1.3.3 Truth table for implication $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If $p = \mathbf{F}$, then $p \rightarrow q = \mathbf{T}$ regardless of whether q is **T** or **F**.

true by default or **vacuously true**.

An **implication** $p \rightarrow q$ says **nothing** when the **hypothesis** p is **false**

Definition 1.3.8 biconditional(equivalence)

Let p and q be a propositions, $p \leftrightarrow q$ is a proposition,
called **biconditional** of p and q , or “ p , if and only if, q ”.

Def. 1.3.8’ Truth table for implication $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$	
T	T	T	if
T	F	F	$p = q \rightarrow p \leftrightarrow q = \mathbf{T}$
F	T	F	$ p \neq q \rightarrow p \leftrightarrow q = \mathbf{F}$
F	F	T	fi

Def. 1.3.10 Let two propositions P and Q are made up of the n propositions p_1, p_2, \dots, p_n . We say P and Q are **logically equivalent** and write $P \equiv Q$, if given any truth value of p_1, p_2, \dots, p_n , either P and Q are both **true** or P and Q are both **false**.

Def. 1.3.16 The *contrapositive* (transposition; 대우) of the *conditional proposition* $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

$q \rightarrow p$ (converse; 역)

$\neg p \rightarrow \neg q$ (inverse; 이)

Thm. 1.3.18 The *conditional proposition* $p \rightarrow q$ is and its *contrapositive* $\neg q \rightarrow \neg p$ are *logically equivalent*.

proof The *truth table*

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Def. A (compound) proposition that is always true: **tautology**
 A proposition that is always false: **contradiction**
 Otherwise: **contingency**.

If an **implication** $p \rightarrow q$ is **always true (tautology)**, we write $p \Rightarrow q$.

$p \Rightarrow q$ p is a **sufficient condition** to be q .
 $\neg q \Rightarrow \neg p$ q is a **necessary condition** to be p .

If a **biconditional** $p \leftrightarrow q$ is a **tautology**, we write $p \Leftrightarrow q$ (or $p \equiv q$)

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
 $p \rightarrow q \equiv \neg p \vee q$ *disjunctive normal form*
 $p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$ *disjunctive normal form*
 $\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$ *distribution law*
 $\equiv (\neg p \wedge \neg q) \vee \mathbf{F} \vee \mathbf{F} \vee (q \wedge p) \equiv (\neg p \wedge \neg q) \vee (p \wedge q)$
 $\equiv (p \equiv q) \equiv (p = q).$ *We will use = instead of \equiv or \Leftrightarrow .*

Two ways of proofs for *logical equivalences*

1. algebraic **rules** of logical equivalences
2. **truth tables** n propositional variables, 2^n rows in the table

Precedence of Logical Operators \neg high \wedge \vee \rightarrow \leftrightarrow $(= (\equiv, \Leftrightarrow))$ low

*Disjunction(\vee) and conjunction(\wedge) are **associative**,*

Let p_1, p_2, \dots, p_n be n propositions. Then

$$\bigvee_{i=1}^n p_i = p_1 \vee p_2 \vee \dots \vee p_n \qquad \bigwedge_{i=1}^n p_i = p_1 \wedge p_2 \wedge \dots \wedge p_n.$$

*Let I be a set(**index set**). Then we can define*

$$\bigvee_{i \in I} p_i \qquad \text{disjunctive normal form}$$

$$\bigwedge_{i \in I} p_i \qquad \text{conjunctive normal form}$$

See p8 of set equalities

Extended De Morgan's law

$$\neg \bigvee_{i \in I} p_i = \bigwedge_{i \in I} \neg p_i \qquad \neg \bigwedge_{i \in I} p_i = \bigvee_{i \in I} \neg p_i.$$

$$\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i} \qquad \overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

Logical equivalences involving conditional statements

$$p \rightarrow q = \neg p \vee q$$

disjunctive normal form

$$\neg(p \rightarrow q) = p \wedge \neg q$$

conjunctive normal form

$$p \rightarrow q = \neg q \rightarrow \neg p$$

contrapositive

$$p \vee q = \neg p \rightarrow q$$

$$p \wedge q = \neg(p \rightarrow \neg q)$$

$$(p \rightarrow q) \wedge (p \rightarrow r) = p \rightarrow (q \wedge r) \quad (p \rightarrow q) \wedge (r \rightarrow q) = (p \vee r) \rightarrow q$$

$$(p \rightarrow q) \vee (p \rightarrow r) = p \rightarrow (q \vee r) \quad (p \rightarrow q) \vee (r \rightarrow q) = (p \wedge r) \rightarrow q$$

Set identities involving subset relations

$$A \subseteq B = (\overline{A} \cup B = U)$$

disjunctive normal form

$$\overline{(A \subseteq B)} = (A \cap \overline{B} = U)$$

conjunctive normal form

$$A \subseteq B = \overline{B} \subseteq \overline{A}$$

contrapositive

$$A \cup B = \overline{A} \subseteq B$$

$$A \cap B = \neg(A \subseteq \overline{B})$$

$$(A \subseteq B) \cap (A \subseteq C) = A \subseteq (B \cap C) \quad (A \subseteq B) \cap (C \subseteq B) = (A \cup C) \subseteq B$$

$$(A \subseteq B) \cup (A \subseteq C) = A \subseteq (B \cup C) \quad (A \subseteq B) \cup (C \subseteq B) = (A \cap C) \subseteq B$$

Logical equivalences involving biconditional statements

$p \leftrightarrow q = q \leftrightarrow p$	<i>commutative(symetric)</i>
$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$	<i>definition of biconditional</i>
$p \leftrightarrow q = \neg p \leftrightarrow \neg q$	<i>symetricity of biconditional</i>
$p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)$	<i>disjuntive normal form(truth table)</i>
$\quad = (p \vee \neg q) \wedge (\neg p \vee q)$	<i>conjuntive normal form</i>
$\neg(p \leftrightarrow q) = (p \leftrightarrow \neg q) = (\neg p \leftrightarrow q)$	<i>De Mogan's law for bicondi.</i>

Set identities involving set equalities

$(A = B) = (B = A)$	<i>commutative(symetric)</i>
$(A = B) = (A \subseteq B) \wedge (B \subseteq A)$	<i>definition of set equality</i>
$(A = B) = (\bar{A} = \bar{B})$	<i>symetricity of equality</i>
$(A = B) = (A \wedge B) \vee (\bar{A} \wedge \bar{B})$	<i>disjuntive normal form(Venn diagram)</i>
$\quad = (A \vee \bar{B}) \wedge (\bar{A} \vee B)$	<i>conjuntive normal form</i>
$\neg(A = B) = (A = \bar{B}) = (\bar{A} = B)$	<i>De Mogan's law for set equa.</i>

1.4 Arguments and Rules of Inference

Deductive(연역) reasoning

Hypotheses(Premises) \Rightarrow Conclusion

Proof: *valid arguments that*

establish the truth of logical statements

arguments *a sequence of statement that ends with a **conclusion***

valid *the **conclusion** must follow from the truth of*

*the **preceding statements** or **premises** of the argument*

*An **argument** is **valid**, if and only if,*

*it is impossible for **all premises** to be **true** and **conclusion** to be **false***

*or If **all premises** are **true**, then the **conclusion** is true.*

Rules of inference

***deducing** new statements from statements we already have.*

Valid Arguments in Propositional Logic

argument a sequence of propositions with
preceding *premises* and finally a *conclusion*.

argument form

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$$

valid argument

$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is *tautology*.

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Rightarrow q.$$

Def. 1.4.1

p_1

p_2

...

p_n

$q.$

Fig. 1.4.1 Rules of Inference**Modus ponens**

$$p \rightarrow q$$

$$\underline{p}$$

$$\therefore q$$

Modus tollens

$$p \rightarrow q$$

$$\underline{\neg q}$$

$$\therefore \neg p$$

Addition

$$\underline{p}$$

$$\therefore p \vee q$$

Simplification

$$\underline{p \wedge q}$$

$$\therefore p$$

Conjunction

$$p$$

$$\underline{q}$$

$$\therefore p \wedge q$$

Hypothetical syllogism

$$p \rightarrow q$$

$$\underline{q \rightarrow r}$$

$$\therefore p \rightarrow r$$

Disjunction syllogism

$$p \vee q$$

$$\underline{\neg p}$$

$$\therefore q$$

Resolution

$$p \vee q$$

$$\underline{\neg p \vee r}$$

$$\therefore q \vee r$$

Resolution*Automatic theorem proving tools**Useful rule in reducing the size of propositions*

$$\frac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$$

$$\frac{p \vee q \quad \neg p \vee q}{\therefore q}$$

$$\frac{p \vee q \quad \neg p}{\therefore q}$$

disjunction syllogism

Fallacies in Implication

$((p \rightarrow q) \wedge q) \not\Rightarrow p$ *fallacy of affirming the conclusion*

*Implication says **nothing** even though the conclusion is true!*

$((p \rightarrow q) \wedge \neg p) \not\Rightarrow \neg q$ *fallacy of denying the hypothesis*

*Implication says **nothing** when the hypotheses are false!*

1.5 Predicates and Quantifiers

Def. 1.5.1 *predicate (propositional function; 조건명제)* $P(x)$ is a boolean function with variable x in the domain of discourse D .
A predicate $P(x)$ is either **T** or **F** for each $d \in D$.

Exa. 1.5.2 Let $P(n)$ denotes “ n is an odd integer”. Then

$P(3)$ denotes “3 is an odd integer” is T .	proposition
$P(4)$ denotes “4 is an odd integer” is F .	proposition
$P(n)$ denotes “ n is an odd integer” is ?	not a proposition

Def. 1.5.1' *predicate* $P: D \rightarrow \{\mathbf{T}, \mathbf{F}\} = \mathbf{B}$. **B: boolean**
for each variable $x \in D: P(x) \in \mathbf{B}$.

Quantifiers

*A predicate is not a proposition only if, variables are not fixed.
If all the variables are fixed, the predicate becomes a propositions.
How can we fix variables?*

*Let **universe(domain) of discourse** is D for each predicates.*

Let D be a set and $P: D \rightarrow \mathbf{B}$.

*If $P(x)$ is true for **all** values of x in the **universe of discourse**,*

$\forall x \in D: P(x)$ is true

otherwise $\forall x P(x)$ is false.

*$\therefore \forall x \in D: P(x)$ becomes a **proposition**.*

Def. 1.5.4 Universal quantifier

“ $\forall x \in D: P(x)$ ” (or “ $\forall x P(x)$ ” for short) is a **proposition** such that
 “ $P(x)$ (is **T**) for **all** values in the **domain** D .”

\forall is called **universal** quantifier.

We read “ $\forall x \in D: P(x)$ ” as “for **all** x in D , $P(x)$ (is **T**).”

Def. 1.5.9 Existential quantifier

“ $\exists x \in D: P(x)$ ” (or “ $\exists x P(x)$ ” for short) is a **proposition** such that
 “There **exists** an element x in D such that $P(x)$ (is **T**).”

\exists is called **existential** quantifier.

We read “ $\exists x \in D: P(x)$ ” as “there **exists** x in D , $P(x)$ ”.

Let D be the universe of domains. Then

$$\forall x \in D: P(x) = \bigwedge_{d \in D} P(d). \quad \exists x \in D: P(x) = \bigvee_{d \in D} P(d).$$

function $\forall x \in D \in B$ (Exa. 1.5.7)
for $d \in D$ **do** **if** $P(d) \rightarrow \text{skip} \mid \neg P(d) \rightarrow \text{return } F$ **fi** **od**; **return** T .

function $\exists x \in D \in B$ (Exa. 1.5.12)
for $d \in D$ **do** **if** $P(d) \rightarrow \text{return } T \mid \neg P(d) \rightarrow \text{skip}$ **fi** **od**; **return** F .

pseudo code of E. W. Dijkstra

do-od and if-fi bracket $()$, $\{\}$, $[]$

dangling-else problem in if-then-else statement

if B_1 then if B_2 then S_A else S_B .

for $x \in X$ do ... od structure

if $B_1 \rightarrow \dots \mid B_2 \rightarrow \dots \mid \dots \mid B_n \rightarrow \dots$ fi

Difficult to prove $\forall x \in D: P(x)$, **BUT** An element

$d \in D \ni \neg P(d)$ is enough to prove $\neg \forall x \in D: P(x)$

counterexample (반례) of $\forall x \in D: P(x)$.

Binding variables

A variable is said to be **bound**, if the variable binds to

(1) quantifiers (\forall , \exists) or

(2) specific value (in the domain), and

it is said to be **free**, otherwise.

A predicate with **bound variables only** is a proposition but

A predicate with **free variable** is **not** a proposition

Extension 1.5.1'' predicate $P: D_1 \times D_2 \times \dots \times D_n \rightarrow \mathbf{B}$.

for a n -tuple variable $(x_1, x_2, \dots, x_n) \in D_1 \times D_2 \times \dots \times D_n$:

$$P(x_1, x_2, \dots, x_n) \in \mathbf{B}.$$

scope of quantifier

the **part of logical expression** to which the quantifier is applied

Example for scope of variables

$$\exists x(P(x) \wedge R(x)) \vee \forall xR(x) \equiv \exists x(P(x) \wedge R(x)) \vee \forall yR(y)$$

Negation of Quantifiers

Let $P(x)$ be “Every student in CS204 has taken CS101.”

$$\neg(\forall x P(x))$$

“It is **not** true that **every** student in CS204 has taken CS101.”
is logically equivalent to

“**There is** a student in CS204 who has **not** taken CS101”

$$\exists x (\neg P(x))$$

$$\therefore \neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \stackrel{?}{\equiv} \forall x \neg P(x)$$

“It is **not** true that **there’s** a student in CS204 who has taken CS101.”
is logically equivalent to

“**Every** student in CS204 has **not** taken CS101”

Thm. 1.5.14 De Morgan's Laws for quantified predicates

$$\neg \forall x P(x) = \exists x \neg P(x), \quad \neg \exists x P(x) = \forall x \neg P(x).$$

proof Let D be a set of discourse. Then (different form text)

$$\neg \forall x P(x) = \neg \bigwedge_{d \in D} P(d) = \bigvee_{d \in D} \neg P(d) = \exists x \neg P(x).$$

$$\neg \exists x P(x) = \neg \bigvee_{d \in D} P(d) \equiv \bigwedge_{d \in D} \neg P(d) = \forall x \neg P(x).$$

Rules of Inferences**Universal instantiation**

$$\frac{\forall x \in D: P(x)}{\therefore P(d), \text{ if } d \in D}$$

Existential instantiation

$$\frac{\exists x \in D: P(x)}{\therefore P(d) \text{ for some } d \in D}$$

Universal generalization

$$\frac{P(d) \text{ for every } d \in D}{\therefore \forall x P(x)}$$

Existential generalization

$$\frac{P(d) \text{ for some } d \in D}{\therefore \exists x P(x)}$$

1.6 Nested Quantifier

$$\forall x \forall y P(x, y)$$

function $\forall x \in D \forall y \in E: P(x, y) \in \mathbf{B}$

$$\forall x \forall y P(x, y)$$

for $d \in D$ **do**

for $e \in E$ **do**

if $P(d, e) \rightarrow \text{skip} \mid \neg P(d, e) \rightarrow \text{return } F$ **fi**

od

od;

return T

function $\exists x \in D \exists y \in E: P(x, y) \in \mathbf{B}$

$$\exists x \exists y P(x, y)$$

for $(d, e) \in D \times E$ **do**

if $P(d, e) \rightarrow \text{return } T \mid \neg P(d, e) \rightarrow \text{skip}$ **fi**

od;

return F

function $\forall x \in D \exists y \in E: P(x, y) \in B$ $\forall x \exists y P(x, y)$
for $d \in D$ **do** $\forall x(d)$
 for $e \in E$ **do** $\exists y(e)$
 if $P(d, e) \rightarrow \text{exit } T \mid \neg P(d, e) \rightarrow \text{skip}$ **fi**
 od;
 if $P(d, e) \rightarrow \text{skip} \mid e \notin E \rightarrow \text{return } F$ **fi**
od;
return T

function $\exists x \in D \forall y \in E: P(x, y) \in B$ $\exists x \forall y P(x, y)$
for $d \in D$ **do** $\exists x(d)$
 for $e \in E$ **do** $\forall y(e)$
 if $P(d, e) \rightarrow \text{skip} \mid \neg P(d, e) \rightarrow \text{exit } F$ **fi**
 od;
 if $e \notin E \rightarrow \text{return } T \mid \neg P(d, e) \rightarrow \text{skip}$ **fi**
od;
return F

function $\forall x \in D \exists y \in E: P(x, y) \in B$ $\forall x \exists y P(x, y)$
do $d \in D \rightarrow$ $\forall x(d)$
 do $e \in E \rightarrow$ $\exists y(e)$
 if $P(d, e) \rightarrow \text{exit } T \mid \neg P(d, e) \rightarrow \text{skip } \text{fi}$
 od;
 if $P(d, e) \rightarrow \text{skip} \mid e \notin E \rightarrow \text{return } F \text{ fi}$
od;
return T

function $\exists x \in D \forall y \in E: P(x, y) \in B$ $\exists x \forall y P(x, y)$
do $d \in D \rightarrow$ $\exists x(d)$
 do $e \in E \rightarrow$ $\forall y(e)$
 if $P(d, e) \rightarrow \text{skip} \mid \neg P(d, e) \rightarrow \text{exit } F \text{ fi}$
 od;
 if $e \notin E \rightarrow \text{return } T \mid \neg P(d, e) \rightarrow \text{skip } \text{fi}$
od;
return F