1 Sets and Logic

1.1 Sets

*An **unordered**(distinguished) **collection** of **objects***

*An **object** in a set is called an **element** or a **member** of the set.*

*(An **ordered**(undistinguished) **collection** of **objects***

orderd pair, triple, quadurple, …, n-tuple)

Two ways to define sets

*i) To **enumerate** the elements(원소나열법)* **(1.1.1)**

$A = \{a_1, a_2, \ldots, a_n\}$ ***finite* sets**

$A = \{a_1, a_2, \ldots\}$ ***infinite* sets**

*ii) To specify **conditions**(조건제시법)* **(1.1.2)**

$A = \{x | p(x)\}$ *p(x): **predicate**(see **Sec. 1.5 Quantifiers**)*

$A = \{x \in U | p(x)\}$ *U: **universe**(domain) of discourse*

iii) to write a program(?; CS322)

Some important sets in discrete mathematics(p 2)

$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ *set of* ***integers***.

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z} \text{ and } q \neq 0\}$ *set of* ***rational numbers***.

$\mathbf{R}$ *set of* ***real numbers***.

$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ *set of* ***positive natural numbers***.

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ *set of* ***natural numbers***.

$\mathbf{N}_n = \{n, n+1, \ldots\}$ $n \in \mathbf{Z}$

Def. Cardinality of a set(p 3)

$|A|$ *the* ***number*** *of elements in the set A*

A set ***contain*** *its* ***elements***.*(p 3)*

$x \in X$ "*an* ***object*** *x is an* ***element*** *of the* ***set*** *X*"

$x \notin X$ "*an* ***object*** *x is* ***not*** *an* ***element*** *of the* ***set*** *X*"

Def. Empty set(p 3)

The set with no elements is ***empty****(****null*** *or* ***void****) set denoted* $\varnothing$ *or* $\{\}$.

Note that $|\varnothing| = |\{\}| = 0$.

Def. Equality of two sets(p 3)

Two sets X and Y are ***equal*** *and we write* $X = Y$,

If X and Y have the ***same*** *elements.*

Otherwise

a set X is ***not*** *equal to Y(written* $X \neq Y$*).*

Note that $\{\} = \varnothing \neq \{\varnothing\}$.

Def. Subsets(p 4)

The set X is said to be a ***subset*** *of Y(written* $X \subseteq Y$*),*

if every elements of X is also an elements of Y.

Or

*If for **every** x, if $x \in X$, then $y \in Y$.*

Def. Equality of two sets**, **revisited

$X = Y$ if $X \subseteq Y \wedge Y \subseteq X$.

*i) For **every** x, if $x \in X$, then $x \in Y$, and* $X \subseteq Y \wedge$

*ii) For **every** x, if $x \in Y$, then $x \in X$.* $Y \subseteq X$

*Note that for **every** set X,* $X \subseteq X$ *and* $\emptyset \subseteq X$.

***Def. Proper subsets**(p 5; 진부분집합)*

If $X \subseteq Y$ *and(but)* $Y \not\subseteq X$*, then we write* $X \subset Y$ *and*

*we say that X is a **proper subset** of Y, and we write* $X \subset Y$.

*Note that for **every** set X,* $X \not\subset X$.

***Def. Power Set of a Set**(p 5)*

*The **set**(**collection**) of **all subsets** of a set A, denoted P(A) or 2^A, is the **power set** of A.* $P(A) = 2^A = \{B \mid B \subseteq A\}$

***Note** a **collection**(set) may have a set as an **elemement** of the **collection**.*

$$X \subseteq A \quad \Leftrightarrow \quad X \in 2^A.$$

***Exa. 1.1.13** Let A = {a, b, c}. Then*

$$P(\{a, b, c\}) = 2^{\{a, b, c\}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$|P(\{a, b, c\})| = |2^{\{a, b, c\}}| = 8.$$

$0_c0_b0_a \leftrightarrow \emptyset$ $0_c0_b1_a \leftrightarrow \{a\}$ $0_c1_b0_a \leftrightarrow \{b\}$ $0_c1_b1_a \leftrightarrow \{a, b\}$

$1_c0_b0_a \leftrightarrow \{c\}$ $1_c0_b1_a \leftrightarrow \{a, c\}$ $1_c1_b0_a \leftrightarrow \{b, c\}$ $1_c1_b1_a \leftrightarrow \{a, b, c\}$

***isomorphic**(See **Sec. 3.1**) to the binary number of length 3*

$$|\{a, b, c\}| = 3 \qquad 8 = 2^3.$$

$|2^A| = |P(A)| \overset{?}{=} 2^{|A|}$ *See **Sec 2.4 Mathematical Induction(Thm. 2.4.6)***

Definitions of set operations *Let X and Y be two sets. Then(p 6)*

1. ***Union*** *of X and Y* $X \cup Y = \{x \mid x \in X \vee x \in Y\}$
2. ***Intersection*** *of X and Y* $X \cap Y = \{x \mid x \in X \wedge x \in Y\}$
3. ***Difference*** *of X and Y* $X - Y = \{x \mid x \in X \wedge x \notin Y\}$

   Relative complement *of Y from X*
4. *Let U be a* ***universe(universal set****) and* $X \subseteq U$*. Then(p 7)*

   Absolout complement *of X* $\overline{X} = U - X = \{x \in U \mid x \notin X\}$

Two sets X and Y are ***disjoint****,*

if their ***intersection*** *is* ***empty****, i.e.* $A \cap B = \varnothing$.

Let S be a ***collection*** *of sets. The* ***collection*** *S is* ***pairwise disjoint****(p 6)*

Every pair of set X and Y in S are ***disjoint****, i.e.* $X \cap Y = \varnothing$.

Venn diagram*(p 7)*

pictorial *view of sets with* ***universe***

n ***sets*** 2^n ***regions***

Thm. 1.1.21 Algebraic Rules of Set Equalites(p 8-9)

Let U be universal set and A, B, and C be **any** subsets of U.

$({}^{\forall}A, B, C \subseteq U \;\equiv\; {}^{\forall}A, B, C \in 2^U)$

1. **Associative laws** $(A \cup B) \cup C = A \cup (B \cup C)$ $\quad (A \cap B) \cap C = A \cap (B \cap C)$
2. **Commutative laws** $A \cup B = B \cup A$ $\quad A \cap B = B \cap A$
3. **Distributive...** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **Identity laws** $A \cup \varnothing = A$ $\quad A \cap U = A$
5. **Complement laws** $A \cup \overline{A} = U$ $\quad A \cap \overline{A} = \varnothing$
6. **Idempotent laws** $A \cup A = A$ $\quad A \cap A = A$
7. **Domination(Bound) laws** $A \cup U = U$ $\quad A \cap \varnothing = \varnothing$
8. **Absorption laws** $A \cup (A \cap B) = A$ $\quad A \cap (A \cup B) = A$
9. **Involution law** $\neg\overline{(A)} = A$
10. **0/1 laws** $\overline{\varnothing} = U$ $\quad \overline{U} = \varnothing$
11. **De Morgan's laws** $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\quad \overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof **insight for proof** using **Venn diagram**

Let S be a ***collection*** *of sets. Then we can write*

$\cup S = \{a | a \in A$, *for* **some** $A \in S\}$ $\cap S = \{a | a \in A$, *for* **all** $A \in S\}$

$\cup S = \{a | \exists A \in S$.϶. $a \in A\} \neq S$ $\cap S = \{a | \forall A \in S, a \in A\}$

Let $S = \{A_1, A_2, \ldots, A_n\}$ *be a* ***collection*** *of n-sets. Then we can write*

$\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$, $\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$.

i: ***index variable***

Let $I = \{1, 2, \ldots, n\}$. *Then* *I*: ***index set***

$\cup_{i \in I} A_i = \cup_{i=1}^{n} A_i$, $\cap_{i \in I} A_i = \cap_{i=1}^{n} A_i$.

Unions *and* ***Intersections*** *are* ***assosiative***(***semigroup***)

If an operation is ***associative***,

the ***binary*** *operation* ***extends*** *to* ***n-ary*** *operation*

Let $x_1, x_2, \ldots, x_n \in \mathbf{R}$. *Then*

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n, \qquad \prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n.$$

Compare $\sum_{i=1}^{n} x_i$ *with* $\sum_{i\in I} x_i$ *where* $I = \{1, 2, \ldots, n\}$.

for $i := 1$ **to** n **do** Σx_i **od**   **for** $i \in I$ **do** Σx_i **od**

Def. *A* ***partion*** *of a set A is an* ***exhaustive pairwise disjoint collection*** *of* ***subsets*** *of A.*

$$Par(A) = \{A_1, A_2, \ldots, A_n\} \text{ such that}(.\ni.)$$

i) $\cup_{i\in N_n} A_i = A$ ***exhaustive*** $(1 \le {}^{\forall} i \le n: A_i \subseteq A)$

ii) $\cap_{i\neq j\in N_n} (A_i \cap A_j) = \varnothing\Sigma)$ ***pairwise disjoint***

Note that $\cup_{A\in Par(A)} Par(A) = A \wedge \cap_{A\in Par(A)} Par(A) = \varnothing$. *BUT* $\cup_S S = S \wedge \cap_S S = \varnothing$ *does not* ***gurantee*** *the collection S to be a partition.*

Cartesian product of two sets(p10)

Def. An **ordered pair** (순서쌍) of elements.

$(a, b) \neq (b, a)$ but $\{a, b\} = \{b, a\}$.

Def. Let A and B be two sets. We define **Cartesian product** of A and B as

$A \times B = \{(a, b) | a \in A, b \in B\}$

(a, b) an **ordered pair**

$^{?}X \times Y$ X-Y 평면 (중)

(Example) Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

$|A \times B| = 2 \times 3 = |A| \times |B|$

$(A \times B) \times C \neq A \times (B \times C)$, since $((a, 1), \alpha) \neq (a, (1, \alpha)) \neq (a, 1, \alpha)$

BUT

Def. *The* ***Cartesian product*** *of the n sets* $A_1, A_2, \ldots, A_n$, *denoted by*

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Def. *The* ***ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ *is the* ***ordered collection***, *that has* a_1 *as its first element,* a_2 *as its second element, …,* a_n *as its n-th element.*

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n), \text{ iff, } a_i = b_i \text{ for } i = 1, 2, \ldots, n.$$

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \equiv (a_1 = b_1) \wedge (a_2 = b_2) \wedge \ldots \wedge (a_n = b_n).$$

Note that $(A \times B) \times C \neq A \times (B \times C) \neq A \times B \times C$.

BUT $|(A \times B) \times C| = |A \times (B \times C)| \leftrightarrow |A \times B \times C|$

BUT $(A \times B) \times C \leftrightarrow A \times (B \times C) \leftrightarrow A \times B \times C$ *See* ***Sec. 3.1***

***Four cases**(binary **relations**) between two sets A and B.*

1. ***equal*** $A = B$
2. ***subset*** $A \subseteq B$
    - 2.1. ***proper subset*** $A \subset B$
3. ***disjoint*** $A \cap B = \emptyset$
4. ***incomparable*** *Otherwise*

***Four regions** in the **Venn diagram** of **two** sets A and B.*

i) $A \cap B$ ii) $\overline{A} \cap B$ iii) $A \cap \overline{B}$ iv) $\overline{A} \cap \overline{B}$

***Truth table** in **predicate logic** with **two** variables*

***Four cases** for between two set and for **regions** in theVenn diagram*

i) $(A \cap \overline{B} = \emptyset) \wedge (\overline{A} \cap B = \emptyset)$ $\equiv A = B$, ***equal***

ii) $(A \cap \overline{B} = \emptyset)$ or $(\overline{A} \cap B = \emptyset)$ $\equiv A \subseteq B$ or $B \subseteq A$, ***subset***

iii) $(A \cap B = \emptyset)$ $\equiv A \cap B = \emptyset$, ***disjoint***

iv) *otherwise* ***incomparable***

1.2 Propositions

Propositions(명제)

a declarative ***sentence*** *that is either* **true** *or* ***false****,*
but **not both nor neither**

letters denoting propositions $p, q, r, s, \ldots$

T*: true value*

F*: false value*

propositional calculus *or* ***propositional logic***

compounded propositions

propositions *that are* ***formed*** *from* ***existing*** *propositions*
using ***logical operators(connectives)***

Def. 1.2.1; 3 Conjunction(And)

Let p and q be a propositions, $p \wedge q$ is a (compounded) proposition, called ***conjunction*** *of p and q, or "p* ***and*** *q".*

truth table

Def. 1.2.1; 6 Disjunction(or; Inclusive or)

Let p and q be a propositions, $p \vee q$ is a proposition, called ***disjunction*** *of p and q, or "p or q".*

Def. 1.2.9 Negation(not)

Let p be a proposition, $\neg p$ is a (new) ***compounded*** *proposition, called* ***negation*** *of p, or "****not*** *p".*

Compare *the* ***truth tables*** *for $\wedge$, $\vee$, and $\neg$ with the* ***Venn diagrams*** *for $\cap$, $\cup$, and $\overline{}$ in* ***set operations(Sec. 1.1)****.*

1.2.13' Algebraic Rules of Logical Equivalences

1. **Associative laws** $(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
2. **Commutative laws** $p \vee q = q \vee p$ $p \wedge q = q \wedge p$
3. **Distributive laws** $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
4. **Identity laws** $p \vee \mathbf{F} = p$ $p \wedge \mathbf{T} = p$
5. **Complement laws** $p \vee \neg p = \mathbf{T}$ $p \wedge \neg p = \mathbf{F}$
6. **Idempotent laws** $p \vee p = p$ $p \wedge p = p$
7. **Domination laws** $p \vee \mathbf{T} = \mathbf{T}$ $p \wedge \mathbf{F} = \mathbf{F}$
8. **Absorption laws** $p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$
9. **Double negation law** $\neg(\neg p) = p$
10. **T/F laws** $\neg \mathbf{T} = \mathbf{F}$ $\neg \mathbf{F} = \mathbf{T}$
11. **De Morgan's laws** $\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$

Compare with the ***algebraic rules of set equalities*** *in p. 7(**Thm 1.1.21**)*

$(\{\mathbf{T}, \mathbf{F}\}, \vee, \wedge, \neg, \equiv) \leftrightarrow (\{U, \emptyset\}, \cup, \cap, \bar{\ }, =)$

isomorphism *between positional logic and set algebra*

Syntax *(문법) of* ***composite*** *propositions*

$\neg$	***negation***	***unary prefix*** *operator*
$\wedge$, $\vee$	***conjunctive***, ***disjunctive***	***binary infix*** *operator*
(,)		***precedence*** *of operator*
T, **F**		***constant***
p, *q*, *r*, *s*		***variables***

Syntactic ***grammars*** *for propositional logic, set operation and 식*

$p \;::=\; \mathbf{T} \mid \mathbf{F} \mid v \mid p \vee p \mid p \wedge p \mid \neg p \mid (p)$

$S \;::=\; \varnothing \mid U \mid b \mid S \cup S \mid S \cap S \mid \overline{S} \mid (S)$

식 ::= 수 | 변수 | 식 + 식 | 식 * 식 | − 식 | (식)
| 식^식 | *sin*(식) | *log*(식)

1.3 Conditional Propositions and Logical Equivalence

Def. 1.3.1 *Let p and q be prpositions, the* ***proposition***

"if p then q" *(1.3.2)* *"q, only if p"*

conditional proposition*(or* ***implication****) written* $p \to q$,

p: ***hypothesis****(or* ***antecedent****;* ***premise****; 가정) of* $p \to q$.

*q:****conclusion****(or* ***consequence****; 결론) of* $p \to q$.

Def. 1.3.3 Truth table *for* ***implication*** $p \to q$.

p	q	$p \to q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***F***	***T***

If p = ***F****, then* $p \to q$ *=* ***T*** *regardless of whether q is* ***T*** *or* ***F****.*

true by default *or* ***vacuously true****.*

An ***implication*** $p \to q$ *says* ***nothing*** *when the* ***hypothesis*** *p is* ***false***

Definition 1.3.8 biconditional(equivalence)

Let p and q be a propositions, $p \leftrightarrow q$ is a proposition, called **biconditional** *of p and q, or "p, if and only if, q".*

Def. 1.3.8' Truth table *for* **biconditional** $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***T***

if

$p = q \rightarrow p \leftrightarrow q = \mathbf{T}$

$| \; p \neq q \rightarrow p \leftrightarrow q = \mathbf{F}$

fi

Def. 1.3.10 *Let two propositions P and Q are made up of the n propositions $p_1, p_2, \ldots, p_n$. We say P and Q are* **logically equivalent** *and write $P \equiv Q$, if given any truth value of $p_1, p_2, \ldots, p_n$, either P and Q are both* ***true*** *or P and Q are both* ***false***.

Def. 1.3.16 *The* ***contrapositive****(****transposision****; 대 우) of the* ***conditional proposition*** $p \to q$ *is* $\neg q \to \neg p$.

$q \to p$(***converse****; 역)*

$\neg p \to \neg q$(***inverse****; 이)*

Thm. 1.3.18 *The* ***conditional proposition*** $p \to q$ *is and its* ***contrapositive*** $\neg q \to \neg p$ *are* ***logically equivalent****.*

proof *The* ***truth table***

p	q	$p \to q$	$\neg q \to \neg p$
T	***T***	***T***	***T***
T	***F***	***F***	***F***
F	***T***	***T***	***T***
F	***F***	***T***	***T***

Def. *A (compound) proposition that is always true:* ***tautology***
A proposition that is always false: ***contradiction***
Otherwise: ***contingency***.

If an ***conditional*** $p \to q$ *is* ***always true(tautology)****, we write* $p \Rightarrow q$.

$p \Rightarrow q$ — *p is a* ***sufficient condition*** *to be q.*

$\neg q \Rightarrow \neg p$ — *q is a* ***necessary condition*** *to be p.*

If a ***biconditional*** $p \leftrightarrow q$ *is* ***a tautology****, we write* $p \Leftrightarrow q$ *(or* $p \equiv q$*)*

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \to q \equiv \neg p \vee q$ — *disjunctive form*

$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$ — *def. of* $\leftrightarrow$ *and disjunctive form*

$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$ *distribution law*

$\equiv (\neg p \wedge \neg q) \vee \mathbf{F} \vee \mathbf{F} \vee (q \wedge p) \equiv (\neg p \wedge \neg q) \vee (p \wedge q)$

$\equiv (p \equiv q) \equiv (p = q)$. — *We will use* $=$ *instead of* $\equiv$ *or* $\Leftrightarrow$.

Three ways of proofs *for* ***logical equivalences***

1. *Algebraic* ***rules*** *of logical equivalences*
2. ***Truth tables*** *n propositional variables,* 2^n *raws in the table*
3. ***Disjunctive(Conjubctive) normal*** *form.*

Precedence of Logical Operators

$\neg$ *high*

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

(=(≡, ⇔)) *low*

Disjunction($\vee$) and conjunction($\wedge$) are ***associative****,*
Let $p_1, p_2, \ldots, p_n$ be n propositions. Then

$$\vee_{i=1}^{n} p_i = p_1 \vee p_2 \vee \ldots \vee p_n, \qquad \wedge_{i=1}^{n} p_i = p_1 \wedge p_2 \wedge \ldots \wedge p_n.$$

*Let I be a set(***index set***). Then we can define*

$\vee_{i\in I}\, p_i$ ***disjunctive($\vee$) normal form*** *(***Truth table*** for 2^n rows)*

where p_i's are ***conjunction***$(\wedge)$ *of p_i and $\neg p_i$.*

$\wedge_{i\in I}\, p_i$ ***conjunctive($\wedge$) normal form***

where p_i's are ***disjunction***$(\vee)$ *of p_i and $\neg p_i$.*

Extened De Morgan's law

$$\neg\vee_{i\in I}\, p_i = \wedge_{i\in I}\, \neg p_i, \qquad \neg\wedge_{i\in I}\, p_i = \vee_{i\in I}\, \neg p_i.$$

$$\overline{\cup}_{i\in I}\, A_i = \overline{(\cup_{i\in I}\, \overline{A_i})} = \cap_{i\in I}\, \overline{A_i}, \qquad \overline{\cap}_{i\in I}\, A_i = \cup_{i\in I}\, \overline{A_i}.$$

Logical equivalences involving conditional statements

$p \to q = \neg p \vee q$ — disjunctive form($\vee$)

$\neg(p \to q) = p \wedge \neg q$ — conjunctive form($\wedge$)

$p \to q = \neg q \to \neg p$ — contrapositive

$p \vee q = \neg p \to q$ $\qquad$ $p \wedge q = \neg(p \to \neg q)$

$(p \to q) \wedge (p \to r) = p \to (q \wedge r)$ $\qquad$ $(p \to q) \wedge (r \to q) = (p \vee r) \to q$

$(p \to q) \vee (p \to r) = p \to (q \vee r)$ $\qquad$ $(p \to q) \vee (r \to q) = (p \wedge r) \to q$

Set identities involving subset relations

$A \subseteq B = (\overline{A} \cup B = U)$ — disjunctive form($\cup$)

$\overline{(A \subseteq B)} = (A \cap \overline{B} = U)$ — conjunctive form($\cap$)

$A \subseteq B = \overline{B} \subseteq \overline{A}$ — contrapositive

$A \cup B = \overline{A} \subseteq B$ $\qquad$ $A \cap B = \neg(A \subseteq \overline{B})$

$(A \subseteq B) \cap (A \subseteq C) = A \subseteq (B \cap C)$ $\qquad$ $(A \subseteq B) \cap (C \subseteq B) = (A \cup C) \subseteq B$

$(A \subseteq B) \cup (A \subseteq C) = A \subseteq (B \cup C)$ $\qquad$ $(A \subseteq B) \cup (C \subseteq B) = (A \cap C) \subseteq B$

Logical equivalences involving biconditional statements

$p \leftrightarrow q = q \leftrightarrow p$ *commutative(symetric)*

$p \leftrightarrow q = (p \to q) \wedge (q \to p)$ *definition of biconditional*

$p \leftrightarrow q = \neg p \leftrightarrow \neg q$ *symetricity of biconditional*

$p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)$ *disjunctive normal form(truth table)*

$= (p \vee \neg q) \wedge (\neg p \vee q)$ *conjunctive normal form*

$\neg(p \leftrightarrow q) = (p \leftrightarrow \neg q) = (\neg p \leftrightarrow q)$ *De Mogran's law for bicondi.*

Set identities involving set equilities

$(A = B) = (B = A)$ *commutative(symetric)*

$(A = B) = (A \subseteq B) \wedge (B \subseteq A)$ *definition of set equality*

$(A = B) = (\overline{A} = \overline{B})$ *symetricity of equality*

$(A = B) = (A \wedge B) \vee (\overline{A} \wedge \overline{B})$ *disjunctive normal form(Venn diagram)*

$= (A \vee \overline{B}) \wedge (\overline{A} \vee B)$ *conjunctive normal form*

$\neg(A = B) = (A = \overline{B}) = (\overline{A} = B)$ *De Mogran's law for set equa.*

1.4 Arguments and Rules of Inference

Deductive(연역) reasoning

$\wedge$ ***Hypotheses(Premises)*** $\Rightarrow$ ***Conclusion***

Proof: ***valid arguments*** *that*

establish the truth of logical statements

arguments *a sequence of statement that ends with a* ***conclusion***

valid *the* ***conclusion*** *must follow from the truth of*

the ***preceding statements*** *or* ***premises of the argument***

An ***argument*** *is* ***valid****, if and only if,*

it is impossible for ***all premises*** *to be* ***true*** *and* ***conclusion*** *to be* ***false***

or If ***all premises*** *are* ***true****, then the conclusion is true.*

Rules of inference

deducing *new statements from statements we already have.*

Valid Arguments in Propositional Logic

argument *a sequence of propositions with preceding **hypotheses(premises)** and finally a **conclusion**.*

argument form

$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q.$

Usually, p_i's does not have conjunctions($\wedge$))

$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow q.$

valid argument

Def. 1.4.1

p_1

p_2

$\ldots$

$\underline{p_n}$

$q.$

Fig. 1.4.1 Rules of Inference

Modus ponen

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

Modus tollen

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore\ \neg p \end{array}$$

Addition

$$\begin{array}{l} \underline{p \qquad\qquad} \\ \therefore\ p \vee q \end{array}$$

Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore\ p \end{array}$$

Conjunction

$$\begin{array}{l} \quad p \\ \underline{\quad q \qquad} \\ \therefore\ p \wedge q \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore\ p \to r \end{array}$$

Disjunction syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p \quad} \\ \therefore\ q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore\ q \vee r \end{array}$$

Resolution

Automatic theorem proving tools

Useful *rule in reducing the size of propositions*

Z3 *propositional logic*

coq *high-order logic, Inria U. in France,* 류석영교수

$p \vee q$	$p \vee q$	$p \vee q$
$\underline{\neg p \vee r}$	$\underline{\neg p \vee q}$	$\underline{\neg p}$
$\therefore\ q \vee r$	$\therefore\ q$	$\therefore\ q$
		disjunction syllogsm

Fallacies in Implication

$((p \to q) \wedge q) \not\Rightarrow p$ ***fallacy*** *of* ***affirming*** *the* ***conclusion***

Implication *says* ***nothing*** *even though the* ***conclusion*** *is* ***true****!*

$((p \to q) \wedge \neg p) \not\Rightarrow \neg q$ ***fallacy*** *of* ***denying*** *the* ***hypothesis***

Implication *says* ***nothing*** *when the* ***hypotheses*** *are* ***false****!*

1.5 Predicates and Quantifiers

***Def. 1.5.1 predicate(propositional function;** 조건명제)* *P(x) is a **boolean function** with **variable** x in the **domain of discourse** D.*
*A **predicate** P(x) is either **T** or **F** for each d* $\in$ *D.*

Exa. 1.5.2 *Let P(n) denotes "n is an odd integer". Then*

*P(3) denotes "3 is an odd integer" is **T**.*	*proposition*
*P(4) denotes "4 is an odd integer" is **F**.*	*proposition*
P(n) denotes "n is an odd integer" is ?	***not** a proposition*

Def. 1.5.1' predicate *P: D* $\rightarrow$ *{**T**, **F**} = **B**.* ***B**: **boolean***
*for each **variable** x* $\in$ *D: P(x)* $\in$ ***B**.*

Quantifiers

*A **predicate** is **not** a **proposition** only if, **variables** are **not fixed**.*
*If **all** the variables are fixed, the predicate **becomes** a **propositions**.*
How can we fix variables?

*Let **universe(domain) of discourse** is D for each **predicates**.*

Let D be a set and $P: D \rightarrow \mathbf{B}$.
*If P(x) is true for **all** values of x in the **universe of discourse**,*

$\forall x \in D$*: P(x) is true.*

otherwise $\forall x P(x)$ *is false.*

$\therefore\ \forall x \in D: P(x)$ *becomes a **proposition**.*

Def. 1.5.4 Universal quantifier

"$\forall x \in U: P(x)$" (or "$\forall x\, P(x)$" for short) is a ***proposition*** such that "$P(x)$ (is **T**) for **all** values in the **domain** U."

$\forall$ is called ***universal*** quantifier.

We read "$\forall x \in U: P(x)$" as "for **all** x in U, $P(x)$ (is **T**)."

Def. 1.5.9 Existential quantifier

"$\exists x \in U: P(x)$" (or "$\exists x\, P(x)$" for short) is a ***proposition*** such that "There **exists** an element x in U such that $P(x)$ (is **T**)."

$\exists$ is called ***existential*** quantifier.

We read "$\exists x \in U: P(x)$" as "there **exists** x in U, $P(x)$".

*Let U be the **universe of domains**. Then*

$\forall x \in U: P(x) = \wedge_{d \in U} P(d)$. $\exists x \in U: P(x) = \vee_{d \in U} P(d)$.

Truth set

Let P(x) be a ***predicate*** *whose universe is U.*

We define the ***truth set*** *of the predicate P(x), denoted P, as follow*

$P = \{x \in U | P(x)\}.$

$(P = U) \Rightarrow \forall x \in U: P(x).$

$(P = \emptyset) \Rightarrow \exists x \in U: \neg P(x).$

$P \subseteq U$ ***membership problem.***

$x \in P$ *or* $x \notin P.$

function *“$^{\forall}x \in D\ P(x)$” $\in$* ***B*** ***(Exa. 1.5.7)***

for *$d \in D$* ***do if*** *$P(d) \rightarrow$* ***skip*** *$\mid \neg P(d) \rightarrow$* ***return F fi od; return T***.

function *“$^{\exists}x \in D\ P(x)$” $\in$* ***B*** ***(Exa. 1.5.12)***

for *$d \in D$* ***do if*** *$P(d) \rightarrow$* ***return T*** *$\mid \neg P(d) \rightarrow$* ***skip fi od; return F***.

pseudo code *of E. W. Dijkstra*

do-od *and* ***if-fi*** *bracket (), {}, []*

dangling-else probem in ***if-then-else*** *statement*

if B_1 **then** **if** B_2 **then** S_A **else** S_B.

for *$x \in X$* ***do*** *…* ***od*** *structure*

if *$B_1 \rightarrow \ldots \mid B_2 \rightarrow \ldots \mid \ldots \mid B_n \rightarrow \ldots$* ***fi***

Difficult *to proove $^{\forall}x \in U$: $P(x)$, BUT An element*

$d \in U$.∍. $\neg P(d)$ is enough to proove $\neg {}^{\forall}x \in U$: $P(x)$

counterexample *(반례) of $^{\forall}x \in U$: $P(x)$.*

Binding variables

A variable is said to be ***bound****, if the variable binds to*

(1) quantifiers($\forall$, $\exists$) or

(2) specific value(in the domain), and

it is said to be ***free****, otherwise.*

A predicate with ***bound variables only*** *is a proposition but*

A peedicate with ***free varaible*** *is* ***not*** *a proposition*

Extension 1.5.1" predicate $P: D_1 \times D_2 \times \ldots \times D_n \rightarrow \mathbf{B}$.

for a n-tuple bariable $(x_1, x_2, \ldots, x_n) \in D_1 \times D_2 \times \ldots \times D_n$:

$$P(x_1, x_2, \ldots, x_n) \in \mathbf{B}.$$

scope *of* ***quantifier***

the ***part of logical expression*** *to which the quantifier is applied*

Example for ***scope*** *of* ***variables***

$$\exists x(P(x) \wedge R(x)) \vee \forall x R(x) \equiv \exists x(P(x) \wedge R(x)) \vee \forall y R(y)$$

Negation of Quantifiers

Let P(x) be "Every student in CS204 has taken CS101."

$\neg\forall x\, P(x)$

*"It is **not** **true** that **every** student in CS204 who has taken CS101."*

*is **logically equivalent** to*

*"**There is** a student in CS204 who has **not** taken CS101"*

$\exists x \neg P(x)$

$\therefore \neg\forall x\, P(x)) \equiv \exists x\, \neg P(x)$

$\neg\exists x\, P(x) \stackrel{?}{\equiv} \forall x \neg P(x)$

*"It is **not** **true** that **there's** a student in CS204 who has taken CS101."*

*is **logically equivalent** to*

*"**Every** student in CS204 has **not** taken CS101"*

Thm. 1.5.14 De Morgan's Laws for quantified predicates

$\neg\forall x P(x) = \exists x \neg P(x)$, $\quad \neg\exists x P(x) = \forall x \neg P(x)$.

proof *Let D be a set of discourse. Then* *(diffrent form text)*

$\neg\forall x P(x) = \neg\wedge_{d\in D} P(d) = \vee_{d\in D} \neg P(d) = \exists x \neg P(x)$.

$\neg\exists x P(x) = \neg\vee_{d\in D} P(d) \equiv \wedge_{d\in D} \neg P(d) = \forall x \neg P(x)$.

<u>Rules of Inferences</u>

Universal instatiation

$\forall x \in D: P(x)$

$\therefore P(d)$, if $d \in D$

Universal generalization

$P(d)$ for every $d \in D$

$\therefore \forall x P(x)$

Exsistential instatiation

$\exists x \in D: P(x)$

$\therefore P(d)$ for some $d \in D$

Exsistential generalization

$P(d)$ for some $d \in D$

$\therefore \exists x P(x)$

1.6 Nested Quantifier

$\forall x \forall y\, P(x, y)$

function "$\forall x{\in}D\forall y{\in}E$: $P(x, y)$" $\in$ ***B*** $\forall x \forall y\, P(x, y)$

for $d \in D$ ***do***

for $e \in E$ ***do***

if $P(d, e) \rightarrow$ ***skip*** $\mid \neg P(d, e) \rightarrow$ ***return F fi***

od

od;

return T

function "$\exists x{\in}D\exists y \in E$: $P(x, y)$" $\in$ ***B*** $\exists x \exists y\, P(x, y)$

for $(d, e) \in D \times E$ ***do***

if $P(d, e) \rightarrow$ ***return T*** $\mid \neg P(d, e) \rightarrow$ ***skip fi***

od;

return F

function "$\forall x\in D\exists y\in E: P(x, y)$" $\in$ ***B*** $\forall x\exists y\ P(x, y)$

 for d $\in$ ***D do*** $\forall x(\mathbf{d})$

  for e $\in$ ***E do*** $\exists y(\mathbf{e})$

   if $P(d, \mathbf{e}) \rightarrow$ ***break*** $\mid \neg P(d, \mathbf{e}) \rightarrow$ ***skip fi***

  od;

  if $P(\mathbf{d}, e) \rightarrow$ ***skip*** $\mid\ e \notin E \rightarrow$ ***return F fi***

 od;

 return T

function "$\exists x\in D\forall y\in E: P(x, y)$" $\in$ ***B*** $\exists x\forall y\ P(x, y)$

 for d $\in$ ***D do*** $\exists x(\mathbf{d})$

  for e $\in$ ***E do*** $\forall y(\mathbf{e})$

   if $P(d, \mathbf{e}) \rightarrow$ ***skip*** $\mid \neg P(d, \mathbf{e}) \rightarrow$ ***break fi***

  od;

  if $e \notin E \rightarrow$ ***return T*** $\mid \neg P(\mathbf{d}, e) \rightarrow$ ***skip fi***

 od;

 return F

function “$\forall x \in D \exists y \in E$: $P(x, y)$” $\in$ **B** $\qquad$ $\forall x \exists y\ P(x, y)$

$\quad (d, \forall)$ **:=** (first(**D**), **F**); **while** $\neg\forall$ **do** $\qquad$ $\forall x(\mathbf{d})$

$\qquad (e, \exists)$ **:=** (first(**E**), **F**); **while** $\neg\exists$ **do** $\qquad$ $\exists y(\mathbf{e})$

$\qquad\quad$ **if** $P(d, \mathbf{e}) \to \exists$ **:= T** $\mid \neg P(d, \mathbf{e}) \to (e, \exists)$ **:=** $next^2(\mathbf{E})$ **fi**

$\qquad$ **od;**

$\qquad$ **if** $\exists \to (d, \forall)$ **:=** $next^2(\mathbf{D}) \mid \neg\exists \to$ **return F fi**

$\quad$ **od;**

$\quad$ **return T**

function “$\exists x \in D \forall y \in E$: $P(x, y)$” $\in$ **B** $\qquad$ $\exists x \forall y\ P(x, y)$

$\quad (e, \exists)$ **:=** (first(**D**), **F**); **while** $\neg\exists$ **do** $\qquad$ $\exists x(\mathbf{d})$

$\qquad (d, \forall)$ **:=** (first(**E**), **F**); **while** $\neg\forall$ **do** $\qquad$ $\forall y(\mathbf{e})$

$\qquad\quad$ **if** $P(d, \mathbf{e}) \to (e, \exists)$ **:=** $next^2(\mathbf{E}) \mid \neg P(d, \mathbf{e}) \to \exists$ **:= F fi**

$\qquad$ **od;**

$\qquad$ **if** $\forall \to$ **return T** $\mid \neg\forall \to (d, \exists)$ **:=** $next^2(\mathbf{D})$ **fi**

$\quad$ **od;**

$\quad$ **return F**

function "$\forall x \in D \exists y \in E: P(x, y)$" $\in \mathbf{B}$ $\forall x \exists y\ P(x, y)$

$(d, \forall) := (\mathit{first}(\mathbf{D}), \mathbf{F})$; **do** $\neg\forall \rightarrow$ $\forall x(\mathbf{d})$

$(e, \exists) := (\mathit{first}(\mathbf{E}), \mathbf{F})$; **do** $\neg\exists \rightarrow$ $\exists y(\mathbf{e})$

if $P(d, \mathbf{e}) \rightarrow \exists := \mathbf{T} \mid \neg P(d, \mathbf{e}) \rightarrow (e, \exists) := \mathit{next}^2(\mathbf{E})$ ***fi***

od;

if $\exists \rightarrow (d, \forall) := \mathit{next}^2(\mathbf{D}) \mid \neg\exists \rightarrow$ ***return*** **F** ***fi***

od;

return T

function "$\exists x \in D \forall y \in E: P(x, y)$" $\in \mathbf{B}$ $\exists x \forall y\ P(x, y)$

$(e, \exists) := (\mathit{first}(\mathbf{D}), \mathbf{F})$; **do** $\neg\exists \rightarrow$ $\exists x(\mathbf{d})$

$(d, \forall) := (\mathit{first}(\mathbf{E}), \mathbf{F})$; **do** $\neg\forall \rightarrow$ $\forall y(\mathbf{e})$

if $P(d, \mathbf{e}) \rightarrow (e, \exists) := \mathit{next}^2(\mathbf{E}) \mid \neg P(d, \mathbf{e}) \rightarrow \exists := \mathbf{F}$ ***fi***

od;

if $\forall \rightarrow$ ***return*** **T** $\mid \neg\forall \rightarrow (d, \exists) := \mathit{next}^2(\mathbf{D})$ ***fi***

od;

return F