4/23 (Fri). Divide and Conquer

Thm 6. Assume $F(n) = p(n) \cdot s^n$

polynomial with order t and coeff. b_i ($i \in \{1, \dots, t\}$) — exponential

$\times$ polynomial (다항식) of order n.
$f(x) = ax + b$ order 1
$f(x) = ax^2 + bx + c$ " 2
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ order n

$a_n^{(h)}$ … same as in Thm 4.

i) s is not the root of C.E. (or basis of Homo. solution) See Thm 6

ii) s is ~~X~~ the root " (") See Thm 6

ex 12. $\underline{a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n}$ root $=3$ mult. $=2$

i) $F(n) = n^2 \cdot 2^n$. $a_n^{(p)} = P_2(n) \cdot 2^n$ polynomial exp. with order 2

Review

① Homogeneous RoR of ~~order~~ degree k

↳ Char. Eq: Polynomial of order k --- k roots

$P_i(n)$: polynomial of degree i

Thm ~~X~~-4 (in page 8)

$$a_n^{(h)} = P_{m_1 - 1}(n) \cdot (r_1)^n + P_{m_2 - 1}(n) \cdot (r_2)^n + \dots + P_{m_t - 1}(n) \cdot (r_t)^n$$

② $F(n) \neq 0$

i) Polynomial of order m $\quad a_n^{(p)} = \underline{P_m(n)} = d_m n^m + d_{m-1} n^{m-1} + \dots + d_0 n^0$.

① $\therefore a_n = a_n^{(h)} + a_n^{(p)} = $ ___ fix d_i with recurrence rel.

② finally fix k constants in homogeneous solution with k basis condition

ii) poly. exponential

↳ Thm 6

(Typo in 11 of TP.

$F(n) = (n^2 + 1) 3^n$ ---- $n^2 (\dots \; n^m 3^n$

2) Order of functions

i) logarithmic $O(\log n)$ → logarithmic order

ii) polynomial $O(n^1) \ldots O(n^k)$ … polynomial order

iii) exponential $O(k^n)$

$O(2^n) \subset O(3^n) \ldots$ exp.

$\cap$ … $\cap$

$O(1) \subset O(\log n) \subset O(n) \subset O(n \log n) \subset O(n^2) \ldots \subset O(n^3)$

$(O(n^4) \subset O(n^5) \subset) \ldots \subset O(2^n) \ldots \subset O(n!)$

Binary search :

$$B_n = B\left(\frac{n}{2}\right) + 2 \longrightarrow B\left(\frac{n}{2^2}\right) + 2$$

$$B_n = \cancel{B\left(\frac{n}{2}\right)+} \; \underbrace{B\left(\frac{n}{2^2}\right)} + 2 + 2$$

$$= B\left(\frac{n}{2^3}\right) + 2 \cdot 3$$

$$\vdots$$

$$= \underbrace{B\left(\frac{n}{2^k}\right)}_{\Downarrow\; 0} + k \cdot 2$$

$$f(n) = af(n/b) + g(n) \qquad af(n/b) = af(n/b^2) + g(n/b)$$

$$= a^2 f\left(\frac{n}{b^2}\right) + ag(n/b) + g(n)$$

$$= a^3 f\left(\frac{n}{b^3}\right) + a^2 g\left(\frac{n}{b^2}\right) + ag\left(\frac{n}{b}\right) + g(n)$$

$$\vdots$$

$$= a^k f\left(\frac{n}{b^k}\right) + a^{k-1} g\left(\frac{n}{b^{k-1}}\right) + \cdots + a^1 g\left(\frac{n}{b}\right) + g(n)$$

$$= \;\; '' \;\; + \sum_{j=0}^{k-1} g\left(\frac{n}{b^j}\right)$$

3) when $\boxed{n = b^k}$

$$f(n) = a^k \cdot f(1) + \sum_{j=0}^{k-1} a^j \underbrace{g(b^{k-j})}_{C}$$

case $g(n) = C$

Thm 1 i) $f(n) \in O(n^{\log_b a})$ if $a > 0$

ii) $f(n) \in O(\log_b n)$ if $a = 1$.

Prof $f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j \cdot c = a^k f(1) + c\left(\sum_{j=0}^{k-1} a^j\right)$

i) $a = 1$ $\quad f(n) = f(1) + C \cdot k = \underline{f(1)} + \underline{C} \cdot \log_b n \quad (\because n = b^k)$

$\therefore f(n) \in \underline{O(\log_b n)}$

ii) $\underline{a > 1}$. $\quad f(n) = a^k f(1) + c \dfrac{a^k - 1}{a - 1}$

$= O(a^k) = \cancel{O(\log n^{\log_b a})}$

$= O(a^{\log_b n}) \quad (\because n = b^k)$

$= O(n^{\log_b a})$

~~$H_n = 2H_{n-1} + 1$~~

~~$f(n) = 2f\left(\frac{n}{2}\right) + 1$~~

Example) Binary search

$B(n) = B\left(\frac{n}{2}\right) + 2$

$\therefore a = 1, b = 2$

$\therefore B(n) \in O(\log_2 n)$