Chapter 12 Boolean Algebras

12.1 Lattices and Algebraic Systems

Let (A, ≤) be a poset. Then

ub(a, b) = {c| a ≤ c, b ≤ c} **upper** *bound*

lub(a, b) = {c| c∈ub(a,b), ∄d .∍. d∈ub(a,b), d≤c,c≠d}

least *upper bound (d < c)*

lb(a, b) = {c| c≤a, c≤b} **lower** *bound*

glb(a, b) = {c| c∈lb(a,b), ∄d .∍. d∈lb(a,b), c≤d,c≠d}

greatest *lower bound (c < d)*

A poset (A, ≤) is called a **lattice***,*

*if ∀a, b ∈ A, ∃***unique** *lub(a, b) and glb(a, b).*

(A, ∨) and (A, ∧) are algebraic systems.

Let (A, ≤) be a **lattice***, an* **algebraic system** *(A, ∨, ∧),*

is defined by the **lattice** *(A, ≤) where*

$\vee: A \times A \rightarrow A$ *least upper bound(***join***)*

$\wedge: A \times A \rightarrow A$ *greatest lower bound(***meet***)*

(P(S), ∪, ∩) is an algebraic system

defined by the lattice (P(S), ⊆).

(N, max, min): *(N, ≤).*

(N^+, lcm, gcd): *(N^+, |). a|b, if a divides b.*

Theorem 12.1*(ub, lb)*

$a \wedge b \leq a, b \qquad a, b \leq a \vee b$

Proof

*$\wedge$(**glb**) is lower bound, and $\vee$(**lub**)is upper bound.*

Theorem 12.2

If $a \leq b$, $c \leq d$,

$a \vee c \leq b \vee d$, *and* $\quad a \wedge c \leq b \wedge d$.

Proof

$b \leq b \vee d, a \leq b \Rightarrow a \leq b \vee d.$ *ub, tran.*

$d \leq b \vee d, c \leq d \Rightarrow c \leq b \vee d.$ *ub, tran.*

$\therefore a \vee c \leq b \vee d.$ *lub ≤ ub*

$a \wedge c \leq a, a \leq b \Rightarrow a \wedge c \leq b.$ *lb, tran.*

$a \wedge c \leq c, c \leq d \Rightarrow a \wedge c \leq d.$ *lb, tran.*

$\therefore a \wedge c \leq b \wedge d.$ *lb ≤ glb*

12.2 Principle of Duality

left vs. right

freedom vs. equality

Capitalism vs. Socialism

Flowers in the mirror

$(A, \leq)$ *vs.* $(A, \geq)$

$(A, \vee, \wedge)$ *vs* $(A, \wedge, \vee)$.

12.3 Basic Properties of Algebraic systems defined by Lattices

Let $(A, \vee, \wedge)$ be a algebraic system defined by the lattice $(A, \leq)$.

Theorem 12.3 *Both $\vee$ and $\wedge$ are* ***commutative****.*

$a \vee b = b \vee a. \qquad a \wedge b = b \wedge a.$

Proof

by the definition lub and glb.

Theorem 12.4 *Both $\vee$ and $\wedge$ are* ***associative****.*

$a \vee (b \vee c) = (a \vee b) \vee c.$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$

Proof

Let $a \vee (b \vee c) = g$ and $(a \vee b) \vee c = h$.

$\therefore a \leq g, b \vee c \leq g, \therefore b \leq g, c \leq g.$ *g is ub.*

$\therefore a \vee b \leq g, c \leq g, (a \vee b) \vee c \leq g.$ *lub ≤ ub*

$\therefore h \leq g.$

$a \leq h, b \leq h, c \leq h.$ *h is ub.*

$\therefore a \leq h, b \vee c \leq h, a \vee (b \vee c) \leq h,$ *lub ≤ ub*

$\therefore g \leq h.$

Theorem 12.5 *Both $\vee$ and $\wedge$ are* ***idempotent****.*

$a \vee a = a \qquad a \wedge a = a$

Proof

$a \leq a \vee a$, *and since* $a \leq a$, $a \vee a \leq a$ *lub≤ub***.**

$\therefore a \vee a = a$

***Theorem 12.6 abortion property**.*

$a \vee (a \wedge b) = a \qquad a \wedge (a \vee b) = a.$

Proof

$a \le a \vee (a \wedge b).$ *$a \vee (a \wedge b)$ is ub of a*

Since, $a \wedge b \le a$ *$a \wedge b$ is lb of a*

$a \vee (a \wedge b) \le a \vee a = a.$ *T12.2, T12.5*

$\therefore\ a \vee (a \wedge b) = a.$

*$(A, \vee)$ and $(A, \wedge)$ are (**commutative**) **semigroups**.*

Example (N, max, min) defined by (N, ≤).

min(a, b) ≤ a, b ≤ max(a, b)

a ≤ b, and c ≤ d ⇒

min(a,c) ≤ min(b,d), and max(a,c) ≤ max(b,d).

max(a, b) = max(b, a),

min(a, b) = min(b, a). ***commutative***

min(a, min(b, c)) = min(min(a, b), c),

max(a, max(b, c)) = max(max(a, b), c). ***assoc**.*

max(a, a) = min(a, a) = a. ***idempotent***

max(a, min(a, b)) = a,

min(a, max(a, b)) = a. ***absortion***

(N, min) is a semigroup.

(N, max) is a momoid with identity 0.

Example (N^+, lcm, gcd) *defined by* $(N^+, |)$.

$gcd(a, b) \mid a, b \mid lcm(a, b)$.

$a \mid b$ *and* $c \mid d \Rightarrow$

$gcd(a, c) \mid gcd(b, d)$, *and* $lcm(a, c) \mid lcm(b, d)$.

$lcm(a, b) = lcm(b, a)$

$gcd(a, b) = gcd(b, a)$. ***commutative***

$lcm(a, lcm(b, c)) = lcm(lcm(a, b), c)$,

$lcm(a, lcm(b, c)) = lcm(lcm(a, b), c)$. ***assoc***.

$lcm(a, a) = gcd(a, a) = a$. ***idempotent***

$lcm(a, gcd(a, b)) = a$,

$gcd(a, lcm(a, b)) = a$. ***absortion***

(N^+, lcm) *is a monoid with identity with 1*

(N^+,gcd) *is a semigroup.*

Example $(P(S), \cup, \cap)$ *defined by* $(P(S), \subseteq)$.

$A \cap B \subseteq A, B \subseteq A \cup B$.

$A \subseteq B$ *and* $C \subseteq D$

$A \cup C \subseteq B \cup D$, *and* $A \cap C \subseteq B \cap D$.

$A \cup B = B \cup A$

$A \cap B = B \cap A$ ***commutative***

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$ ***associative***

$A \cup A = A \cap A = A$ ***idempotent***

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$ ***absortion***

$(P(S), \cup)$ *is a monoid with identity* $\emptyset$.

$(P(S), \cap)$ *is a monoid with identity S.*

12.4 Distributive and Complemented Lattices

A lattice is a **distributive** *lattice, if*

the meet(∧) operation distributes over the join(∨) operation and

the join(∨) operation distributes over the meet(∧) operation.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Since ∨ and ∧ are comutative

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$
$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

Theorem 12.7 *If the meet operation distributes over the join operation, the join operation distributes over the meet operation, and vice versa.*

Proof

Assume ∧ distributes over ∨ and consider

$(a \vee b) \wedge (a \vee c)$	
$= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$	*∧ distributes over ∨*
$= a \vee ((a \vee b) \wedge c)$	*abortion*
$= a \vee ((a \wedge c) \vee (b \wedge c))$	*∧ distributes over ∨*
$= ((a \vee (a \wedge c)) \vee (b \wedge c)$	*∨ is associative*
$= a \vee (b \wedge c)$	*abortion*

Let $(A, \leq)$ be a lattice.

*$0 \in A$ is called **universal lower bound(bottom)**, if $^{\forall}a \in A, 0 \leq a$.*

*$1 \in A$ is called **universal upper bound(top)**, if $^{\forall}a \in A, a \leq 1$.*

***Theorem 12. 7** If there exists a **universal lower(upper)bound**, it is **unique**.*

***Proof** Assume $^{\exists}$**two** universal lower bound a and b*

$a \leq b, b \leq a. \therefore a = b.$

Example

$(2^A, \subseteq)$ universal lower bound(0) is $\emptyset$,

universal upper bound(1) is A.

***Theorem 12.8** Let $(A, \leq)$ ba a lattice, and 0 and 1 be **universal lower** and **upper bounds**. Then $^{\forall}a \in A$,*

$0 \vee a = a,$	*0 is **identity** for* $\vee$
$0 \wedge a = 0,$	*0 is **zero** for* $\wedge$
$1 \vee a = 1,$	*1 is **zero** for* $\vee$
$1 \wedge a = a.$	*1 is **identity** for* $\wedge$

Proof

$1 \leq 1 \vee a$,(ub) and $1 \vee a \leq 1$(uub)

$\therefore 1 \vee a = 1.$

$1 \wedge a \leq a$(lb) and $1 \wedge a \geq a \wedge a$(uub, Thm12.2) $= a$

$\therefore 1 \wedge a = a.$

Let (A, ≤) ba a lattice, and 0 and 1 be a ***universal lower*** *and* ***upper bounds****, respectively. Then*

a is said to be a ***complement*** *of b, if*

$a \vee b = 1$ *and* $a \wedge b = 0$.

(Example) (P(A), $\subseteq$*)* $A \cup \overline{A} = S$, $A \cap \overline{A} = \emptyset$.

If a is a ***complement*** *of b, b also is a* ***complement*** *of a. (commutative)*

An element may have ***many*** *complements, if any.*

Example: Fig 12.1

But 0 and 1 are ***unique complements*** *of each other.*

A lattice is said to be a ***complemented lattice****, if every elements has* ***complement(s)****.*

Theorem 12.9 *In a* ***distributive*** *lattice, if an element has a* ***complement*** *then this* ***complement*** *is* ***unique****.*

Proof *Suppose a has two complements* a_1 *and* a_2

$a \vee a_1 = 1, a \wedge a_1 = 0.$

$a \vee a_2 = 1, a \wedge a_2 = 0.$

$a_1 = a_1 \wedge 1$	*1 is the* ***identity*** *for* $\wedge$
$= a_1 \wedge (a \vee a_2)$	a_2 *is the* ***complement*** *of a*
$= (a_1 \wedge a) \vee (a_1 \wedge a_2)$	$\wedge$ ***distributes*** *over* $\vee$

$= \underline{0} \vee (a_1 \wedge a_2)$	*a_1 is the **complement** of a*
$= (\underline{a \wedge a_2}) \vee (a_1 \wedge a_2)$	*a_2 is the **complement** of a*
$= (a \vee a_1) \wedge a_2$	*$\wedge$ **distributes** over $\vee$*
$= 1 \wedge a_2$	*a_1 is the **complement** of a*
$= a_2$	*1 is an **identity** for $\wedge$.*

12.5 Boolean Lattice and Boolean algebra

*A **complemented** and **distributive** lattice is called a **boolean lattice**.*

*Since every element has a unique complement, **complement** is an **algebraic operator**.*

***complement**: **unary** operator on A.*

$\neg: A \to A$

*A boolean lattice $(A, \leq)$ defines an **boolean algebra** $(A, \vee, \wedge, \neg)$*

$\vee$ *join*

$\wedge$ *meet*

$\neg$ *complement*

$a \vee \neg a = 1,$ $\qquad a \wedge \neg a = 0.$

$\neg(\neg a) = a.$

Example

$(2^A, \cup, \cap, \overline{\ })$ *is a **boolean algebra** defined by the **boolean lattice** $(2^A, \subseteq)$*

Theorem 12.10 DeMorgan's laws

$\neg(a \vee b) = \neg a \wedge \neg b,$

$\neg(a \wedge b) = \neg a \vee \neg b.$

Proof

$(a \vee b) \vee (\neg a \wedge \neg b)$	
$= ((a \vee b) \vee \neg a) \wedge ((a \vee b) \vee \neg b)$	$\vee$ dist. over $\wedge$
$= ((a \vee \neg a) \vee b) \wedge (a \vee (b \vee \neg b))$	$\vee$ comm, assoc.
$= (1 \vee b) \wedge (a \vee 1)$	complement
$= 1 \wedge 1 = 1$	uub
$(a \vee b) \wedge (\neg a \wedge \neg b)$	
$= (a \wedge(\neg a \wedge\neg b)) \vee (b \wedge(\neg a \wedge\neg b)$	$\wedge$ dist. over $\vee$
$= 0 \vee 0 = 0$	..., ulb

$\therefore$ $a \vee b$ is a complement of $\neg a \wedge \neg b$

$\neg(a \vee b) = \neg a \wedge \neg b,$

$\neg(a \wedge b) = \neg a \vee \neg b.$

$(\{0, 1\}, \vee, \wedge, \neg)$ is a ***boolean algebra*** *defined by a lattice $(\{0, 1\}, \leq)$ with $(0 \leq 1)$*

$0 \vee 0 = 0,\ \ 0 \vee 1 = 1,\ \ 0 \vee 0 = 1,\ \ 1 \vee 1 = 1.$

$0 \wedge 0 = 0,\ \ 0 \wedge 1 = 0,\ \ 0 \wedge 1 = 0,\ \ 1 \wedge 1 = 1.$

$\neg 0 = 1, \neg 1 = 0.$

$(\{0, 1\}, \wedge, \vee, \neg)$ is a ***boolean algebra*** *defined by a lattice $(\{0, 1\}, \leq)$ with $(1 \leq 0)$*

$0 \wedge 0 = 0,\ \ 0 \wedge 1 = 0,\ \ 0 \wedge 1 = 0,\ \ 1 \wedge 1 = 1.$

$0 \vee 0 = 0,\ \ 0 \vee 1 = 1,\ \ 0 \vee 0 = 1,\ \ 1 \vee 1 = 1.$

Principle of Duality

12.6 Uniqueness of Finite Boolean Algebra

Boolean algebra
freedom?

Every *finite* ***boolean algebra*** *has* 2^n *elements*
for ***some*** $n > 0$.

There is a ***unique boolean algebra*** *of* 2^n *element*
for ***every*** $n > 0$.

$a \in A$ *is called to* ***cover*** $b \in A$,
if $b \le a$, $\not\exists c \in A$.э. $b < c < a$.*(Hasse diagram)*
$b \le c \le a,\ b \ne c,\ c \ne a$

Let $(A, \le)$ *be a lattice with* ***bottom*** *0.*
An element is called ***atom****, if it covers 0.*

Lemma 12.0
Let $(A, \le)$ *be a* ***finite*** *lattice with* ***bottom*** *0.*
$\forall b \ne 0 \in A$, $\exists$*an* ***atom*** $a \in A$.э. $a \le b$.

Proof
If b is an atom, $b \le b$.
If b is not an atom, since $(A, \le)$ *is a* ***finite*** *lattice,*
$\exists$*a chain* $0 < b_i < \ldots < b_2 < b_1 < b$
b_i *is an atom*

Lemma 12.1 *In a* ***distributed*** *lattice,*
if $b \wedge \neg c = 0$*, then* $b \leq c$*.*

Proof

$(b \wedge \neg c) \vee c = c$	$b \wedge \neg c = 0$
$(b \vee c) \wedge (\neg c \vee c) = c$	*distributive*
$b \vee c = c$	*1 is a identity for* $\wedge$*.*
$\therefore\ b \leq c$	*c is lub.*

Lemma 12.2 *Let* $(A, \vee, \wedge, \neg)$ *be a* ***finite boolean*** *algebra. Then*

$^{\forall}b \in A\text{-}\{0\}$, $1 \leq {}^{\exists}k \leq |A|$, $1 \leq {}^{\forall}i \leq k$, a_i*'s are* ***atoms*** *of A,*

$$b = a_1 \vee a_2 \vee \ldots \vee a_k.$$

Proof

Since $^{\forall}b \in A\text{-}\{0\}$, $1 \leq {}^{\exists}k \leq |A|$, $1 \leq {}^{\forall}i \leq k$,
a_i*'s are* ***atoms*** *of A and* $a_i \leq b$*.* — *L12.0*
$c = a_1 \vee a_2 \vee \ldots \vee a_k \leq b$*.* — *T12.2*
Suppose $b \wedge \neg c \neq 0$*.*
$^{\exists}$*an atom* $a \in A$ *.∋.* $a \leq b \wedge \neg c$*.* — *L12.0*
$\therefore\ a \leq b$ *and* $a \leq \neg c$*.* — *T12.2*
Since a is an atom, $a \leq c$
$\therefore\ a \leq c \wedge \neg c = 0$ — *T12.2*
But contradiction!
$\therefore\ b \wedge \neg c = 0$
$\therefore\ b \leq c$ — *L12.1*
$b = c = a_1 \vee a_2 \vee \ldots \vee a_k$*.*

Lemma 12.3 *Let $(A, \vee, \wedge, \neg)$ be a **finite** boolean algebra. Then*
$^{\forall}b \in A\text{-}\{0\}, 1 \le {}^{\exists}k \le |A|, 1 \le {}^{\forall}i \le k$, *$a_i$'s are **atoms** of A,*

$$b = a_1 \vee a_2 \vee \ldots \vee a_k.$$

*is the **unique** way to represent b as **join** of **atoms**.*

Proof

Suppose an alternative representation

$b = c_1 \vee c_2 \vee \ldots \vee c_t$.   *join of atoms*

$\therefore$ *For* $1 \le {}^{\forall}j \le t$,

$c_j \le b$.   *c_j's are atoms*

$\therefore c_j \wedge b = c_j$.

$c_j \wedge (a_1 \vee a_2 \vee \ldots \vee a_k) = c_j$.   *repalce b*

$(c_j \wedge a_1) \vee \ldots \vee (c_j \wedge a_k) = c_j$.   *$\wedge$ dist. over $\vee$.*

$\therefore 1 \le {}^{\exists}i \le k$.э. $c_j \wedge a_i \ne 0$.

Since both c_j and a_i are atoms, $c_j = a_i$.

$\therefore 1 \le {}^{\forall}j \le t, 1 \le {}^{\exists}i \le k$.э. $c_j = a_i$.

$\therefore t = k$.

Q.E.D.

Theorem 12.11 *Let (A, $\vee$, $\wedge$, $\neg$) be a* ***finite*** *boolean algebra and S be a set of* ***atoms****. Then (A, $\vee$, $\wedge$, $\neg$) is* ***isomorphic*** *to the boolean algebra (2^S, $\cup$, $\cap$, $^-$).*

Proof

Let (A, $\leq$) be a ***boolean lattice*** *and S be set of* ***atoms****.*

$f: A \rightarrow 2^S$

$f(b) = \{a \in S \mid a \leq b\}$

f is ***one-to-one onto****.(Lemma 12.2. 12.3)*

Thers is a ***isomorphism*** *f from*

any ***finite boolean lattice*** *(A, $\leq$) to*

finite power set latice *(2^S, $\subseteq$)*

where S is a set of ***atoms*** *of A.*

Any ***finite boolean algebra*** *(A, $\vee$, $\wedge$, $\neg$)*

is ***isomorphic*** *to the*

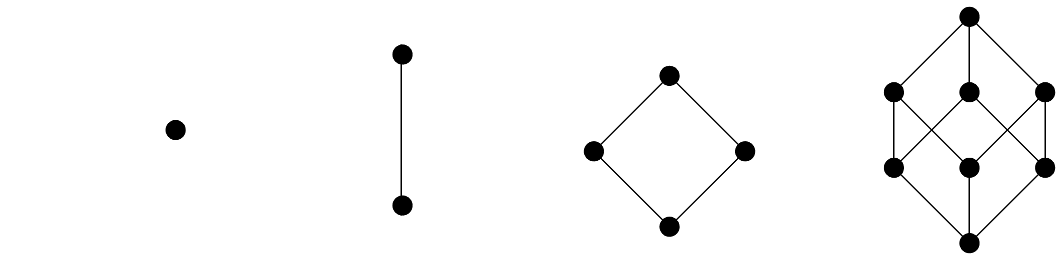
finite power set boolean algebra *(2^S, $\cup$, $\cap$, $^-$).*

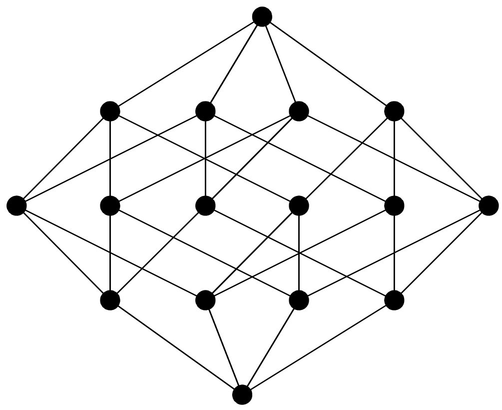
where S is a set of ***atoms*** *of A.*

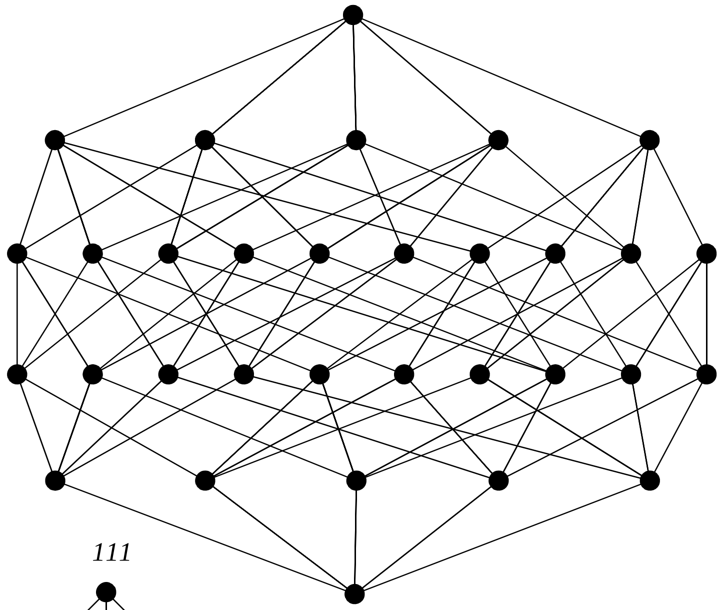
There exists a ***unique finite boolean algebra***

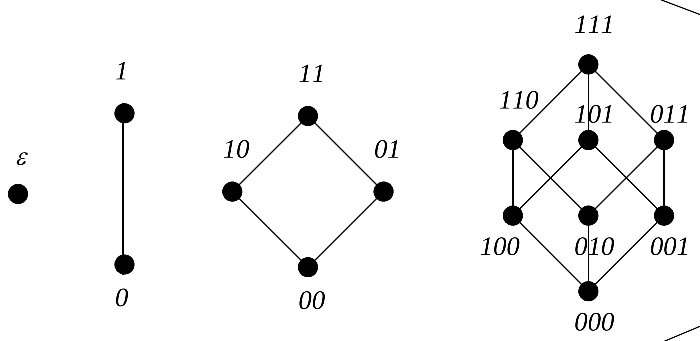
of $\mathbf{2^n}$ *elements for any n > 0, and*

there is ***no*** *other finite boolean algebra.*





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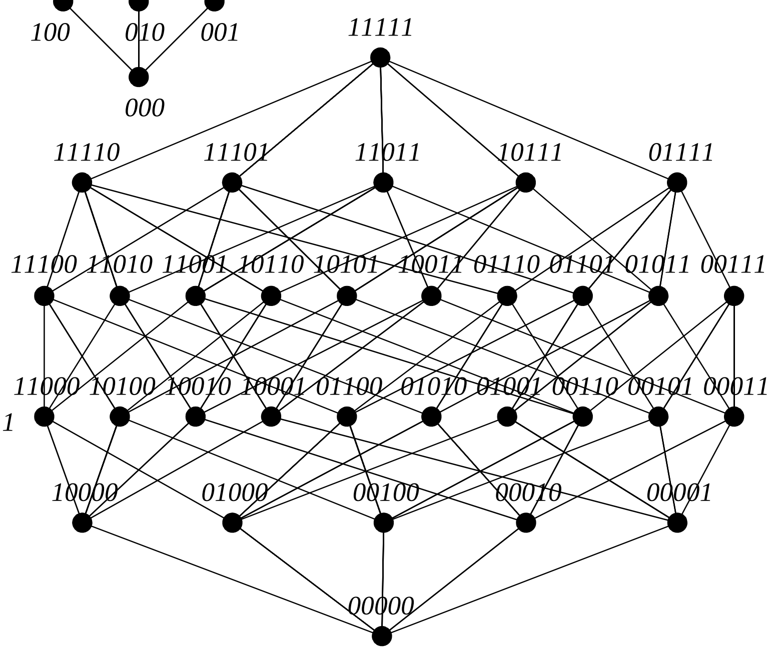
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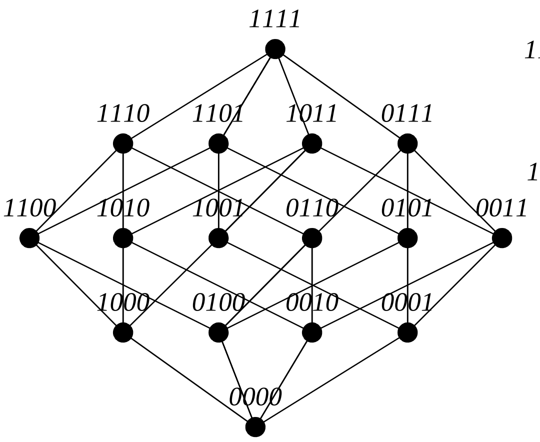
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12.7 Boolean Functions and Boolean Expressions

Let $(A, \vee, \wedge, \neg)$ be a boolean algebra.
Consider a function $f: A^n \rightarrow A$
    function
        $(|A|^n \times 1)$ table of $|A|$ values.
        ***closed form expression** with n variables.*

Let $(A, \vee, \wedge, \neg)$ be a boolean algebra.
*A **boolean expression** over $(A, \vee, \wedge, \neg)$ is defined*
1. *$a \in A$ is a **boolean expression**,*
2. *Any variable name is a **boolean expression**,*
3. *If e_1 and e_2 are **boolean expressions**, then*
   *$e_1 \vee e_2$, $e_1 \wedge e_2$, $\neg e_1$ are **boolean expressions**.*

***Syntactic grammar** of boolean expressions*

$B \rightarrow a$	$a \in A$.	
$\quad \mid \; v$	$v \in V$	*variable name.*
$\quad \mid \; B \vee B \mid B \wedge B \mid \neg B$		

Boolean expression over $(A, \vee, \wedge, \neg)$ is
    *a **language** over $A \cup V \cup \{\vee, \wedge, \neg\}$*
        *where V is a set of **variable** names.*

*A boolean expression that contains n **distinct** variable, is called a boolean expression with n-variables.*

*Let $E(x_1, \ldots, x_n)$ be a **boolean expression** of n-variables over an boolean algebra $(A, \vee, \wedge, \neg)$.*

$E(a_1, \ldots, a_n) \in A$ where $a_1, \ldots, a_n \in A$

assignment of values $a_1, \ldots, a_n$

to the variable $x_1, \ldots, x_n$.

Example

$E(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge \neg(x_2 \vee x_3)$

$E(0, 1, 0) = (0 \vee 1) \wedge (1 \vee 0) \wedge \neg(1 \vee 0) = 0$

*Two boolean expressions of n variables are **equivalent**, if **every** assignment of values to n variables results same values.*

Let $E_1, E_2: A^n \to A$.

$E_1(x_1, \ldots, x_n) = E_2(x_1, \ldots, x_n)$, *if and only if,*

$\forall (a_1, \ldots, a_n) \in A^n,$

$E_1(a_1, \ldots, a_n) = E_2(a_1, \ldots, a_n).$

Example

$(x_1 \wedge x_2) \vee (x_1 \wedge \neg x_3) = x_1 \wedge (x_2 \vee \neg x_3)$

***Equivalence** of two boolean expressions*

1. *Check every $|A|^n$ cases.*
2. *Properties of boolean algebra.*

$f: A^n \to A$ vs $E(x_1, \ldots, x_n)$

Not every function f has

*the **equivalent** boolean expression $E(x_1, \ldots, x_n)$*

*$f: A^n \to A$ is called **boolean function**,*

*if it can be specified by a **boolean expression***

*But $\{0, 1\}^n \to \{0, 1\}$ is always a **boolean function***

***Proof** Let $f: \{0, 1\}^n \to \{0, 1\}$.*

Let $(x_1, \ldots, x_n) \in V^n$. Then

*$\tilde{x}_1 \wedge \ldots \wedge \tilde{x}_n$ is a **minterm**, if $\tilde{x}_i = x_i$ or $\tilde{x}_i = \neg x_i$.*

*$\tilde{x}_1 \vee \ldots \vee \tilde{x}_n$ is a **maxterm**, if $\tilde{x}_i = x_i$ or $\tilde{x}_i = \neg x_i$.*

There are 2^n minterms and maxterms, resp.

***Disjunctions**($\vee$) of minterms whose f value is 1 is*

***equivalent** to f.*

$$\vee_{f(x_1 x, \ldots, x_n)=1} \tilde{x}_1 \wedge \ldots \wedge \tilde{x}_n$$

$$= (\tilde{x}_1 \wedge \ldots \wedge \tilde{x}_n) \vee \ldots \vee (\tilde{z}_1 \wedge \ldots \wedge \tilde{z}_n)$$

disjunctive normal form

***Conjunctions**($\wedge$) of maxterms whose f value is 0 is*

***equivalent** to f.*

$$\wedge_{f(x_1 x, \ldots, x_n)=0} \tilde{x}_1 \vee \ldots \vee \tilde{x}_n$$

$$= (\tilde{x}_1 \vee \ldots \vee \tilde{x}_n) \wedge \ldots \wedge (\tilde{z}_1 \vee \ldots \vee \tilde{z}_n)$$

conjunctive normal form

12.8 Propositional Calculus

*A **proposition** is a statement that may be either be **true**(T) or **false**(F)*

Consider a boolean lattice, ({F, T}, {F ≤ T})

*A **bolean algebra** ({F, T}, ∨, ∧, ¬) defined by a boolean lattice ({F, T}, {F ≤ T}).*

disjunction	∨
conjunction	∧
negation	¬

tautology	⇔ *T(1)*
contradiction	⇔ *F(0)*
disjunction	⇔ ***join***(∨, *lub*)
conjunction	⇔ ***meet***(∧, *glb*)
negation	⇔ ***complement***(¬)

condition	⇔ ***complement meet***	
$p \to q$	$\neg p \vee q$	$p \leq q.$
biconditional	⇔ ***complement meet join***	
$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$p = q.$
	$\equiv (\neg p \vee q) \wedge (p \vee \neg q)$	

12.9 Digital Network

$OR_n: \{0, 1\}^n \to \{0, 1\}$ *n-input OR gate*

$OR_n(x_1, \ldots, x_n) = x_1 \vee \ldots \vee x_n.$

$AND_n: \{0, 1\}^n \to \{0, 1\}$ *n-input AND gate*

$AND_n(x_1, \ldots, x_n) = x_1 \wedge \ldots \wedge x_n$.

$NOT_1: \{0, 1\} \rightarrow \{0, 1\}$ 1-input NOT gate

$NOT_1(x) = \neg x$.

Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a boolean function. Then f is equivalent to the composition of three gates.

Proof

*Let **disjuntive** normal form has $d(\leq 2^n)$ **minterms***

$1 \leq {}^\forall j \leq d$, we define minterm $i_j = AND_n(\hat{x}_1, \ldots, \hat{x}_n)$

where $1 \leq {}^\forall k \leq n$, $\hat{x}_k = x_k$, if $\tilde{x}_k = x_k$; $NOT_1(x_k)$ if $\tilde{x}_k = \neg x_k$.

Then f is equivalent to $OR_d(i_1, \ldots, i_m)$.

*Let **conjuntive** normal form has $c(\leq 2^n)$ **maxterms***

$1 \leq {}^\forall j \leq c$, we define maxterm $t_j = OR_n(\hat{x}_1, \ldots, \hat{x}_n)$

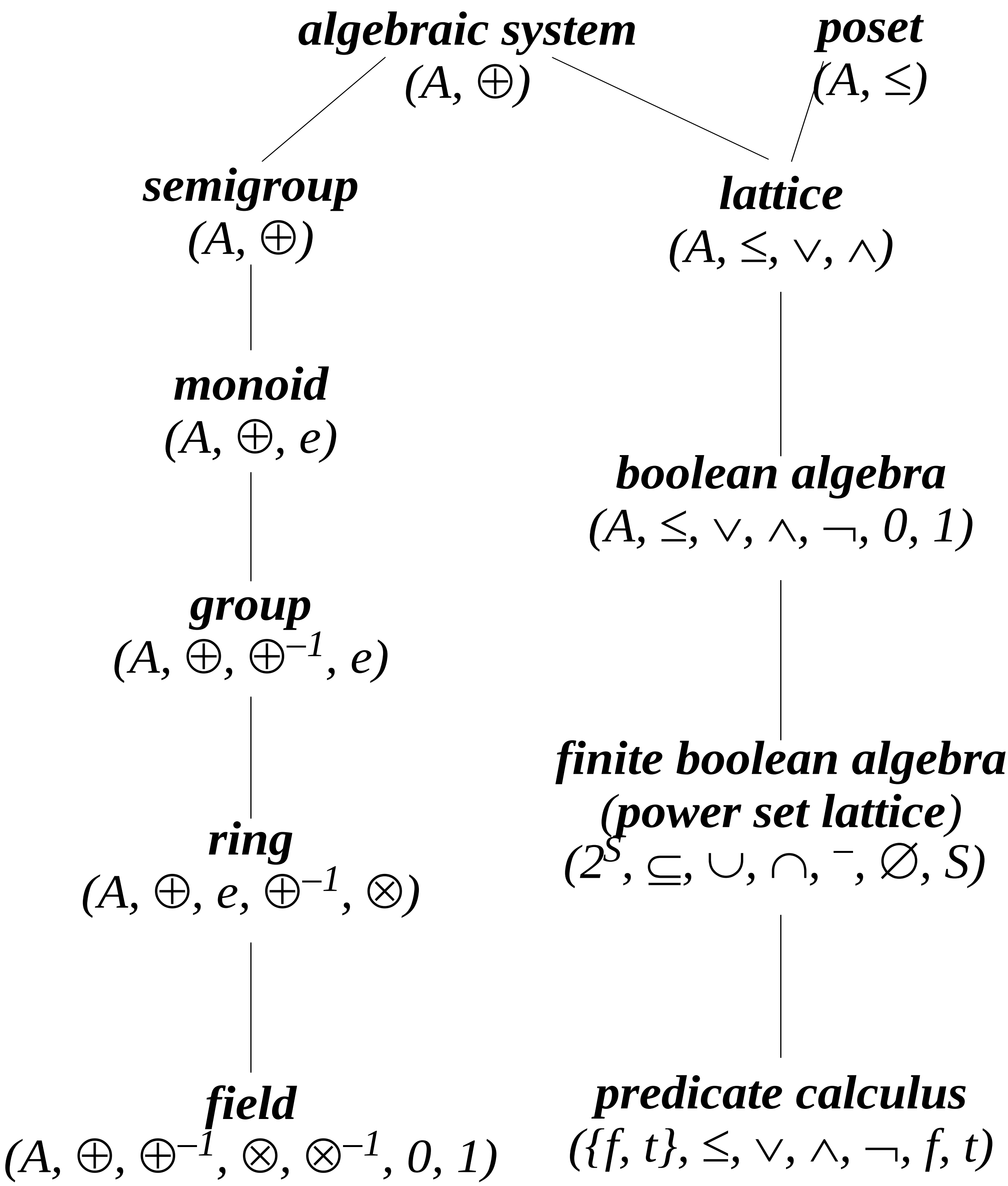
where $1 \leq {}^\forall k \leq n$, $\hat{x}_k = x_k$, if $\tilde{x}_k = x_k$; $NOT_1(x_k)$ if $\tilde{x}_k = \neg x_k$.

Then f is equivalent to $AND_c(i_1, \ldots, i_m)$.

Furthermore,

*f is equivalent to the composition of **three** gates, OR_2, AND_2, and NOT_1.*

*f is equivalent to the composition of **two** gates, NOR_2, $NAND_2$.*


algebraic system
$(A, \oplus)$
poset
$(A, \leq)$
semigroup
$(A, \oplus)$
lattice
$(A, \leq, \vee, \wedge)$
monoid
$(A, \oplus, e)$
boolean algebra
$(A, \leq, \vee, \wedge, \neg, 0, 1)$
group
$(A, \oplus, \oplus^{-1}, e)$
finite boolean algebra
(power set lattice)
$(2^S, \subseteq, \cup, \cap, ^{-}, \emptyset, S)$
ring
$(A, \oplus, e, \oplus^{-1}, \otimes)$
field
$(A, \oplus, \oplus^{-1}, \otimes, \otimes^{-1}, 0, 1)$
predicate calculus
$(\{f, t\}, \leq, \vee, \wedge, \neg, f, t)$