8 Relations

8.1 Relations and Their Properties

Definition *1 Let A and B be two set.*

A ***binary relation*** *R from A to B is subset of $A \times B$.*

$$R \subseteq A \times B.$$

A: ***domain*** *of the relation R.*

B: ***range(codomain)*** *of the relation R.*

Let $a \in A$, $b \in B$, Then $(a, b) \in R$ or $(a, b) \notin R$.

If $(a, b) \in R$, we also write $a\, R\, b$ and we say a is ***related to*** *b by R.*

If $(a, b) \notin R$, we also write $a \not R\, b$ and a is ***not related to*** *b by R.*

Two notations

$(a, b) \in R$	*relation R is a* ***set*** $R \subseteq A \times B$
$a\, R\, b$	*relation R is a infix* ***boolean*** *binary* ***operator*** $R: A \times B \rightarrow \{true, false\}$

Function as Relation

$f: A \rightarrow B$ *vs* $R \subseteq A \times B$

$f(a) = b$ *vs* $R(a) = \{b_1, b_2, \ldots, b_n\}$

Function f is a special kind of relation

$\forall a \in A \exists_1 b \in B. \quad \therefore f(a)=\{b\}$

Some Relation R may be a function.

$\forall a \in A \exists_1 b \in B.(a, b) \in R.$

Function is a relation

you can write $(a, b) \in f$ *or* $a\ f\ b$ *insteand of* $f(a) = b$

Let $f: A_1 \times A_2 \times \ldots \times A_n \rightarrow B_1 \times B_2 \times \ldots \times B_m.$

$((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_m)) \in f$

$(a_1, a_2, \ldots, a_n)\ f\ (b_1, b_2, \ldots, b_m)$

$f((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_m)$ *or*

$f(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m).$

Relation on a Set

Definition *2 Let A be a set and $R \subseteq A \times A$. R is called a relation* ***on*** *A.*

Relation on A is a ***directed graph*** *on* ***vertices*** *A and* ***edges*** *R.*

Properties of Relations

Definition *3 A relation R is* ***reflexive****, if $\forall a \in A$, $a\ R\ a$.*

A relation R is ***irreflexive****, if $\forall a \in A$, $a \not R\ a$.*

A relation is ***not both*** *reflexive and irreflexive.(disjoint)*

Definition *4 A relation R is* ***symmetric****, if $a\ R\ b \Rightarrow b\ R\ a$.*

A relation R is ***asymmetric****, if $a\ R\ b \Rightarrow b \not R\ a$.*

A relation R is ***antisymmetric****, if $(a\ R\ b \wedge b\ R\ a) \Rightarrow (a = b)$.*

or if $(a\ R\ b \wedge a \neq b) \Rightarrow b \not R\ a$.

If a relation is a ***asymmetric*** *then it is also* ***antisymmetric****.(subset)*

Definition *5 A relation R is* ***transitive****, if $a\ R\ b \wedge b\ R\ c \Rightarrow a\ R\ c$.*

Combining Relations

Let $R_1, R_2 \subseteq A \times B$. *Consider* $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$.

Definition *6 Let* $R \subseteq A \times B$, $S \subseteq B \times C$. *Then* ***composition*** *of R and S, denoted as* $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.

Definition *7 Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}^+$,

$R^1 = R$ ***basis***

$R^{n+1} = R^n \circ R$. ***induction***

Definition *6.5 Let A be a set. We define* ***identity*** *relation*

$id_A = \{(a, a) \in A \times A \mid a \in A\}$

Colorally *0.5 Let* $R \subseteq A \times B$. *Then*

$R \circ id_A = id_A \circ R = R$. id_A *is a* ***identity*** *element for* ***composition***.

Definition *7.5 Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}$,

$R^0 = id_A$ ***basis*** $(x^0 = 1)$

$R^{n+1} = R^n \circ R$. ***induction***

Theorem *1 Let* $R \subseteq A \times A$. *R is* ***transitive****, if and only if,* $R^n \subseteq R$ *for* $\forall n \in \mathbf{N}^+$.

Proof:

1\. (if) $R^n \subseteq R$ for $\forall n \in \mathbf{N}^+ \rightarrow R$ is transitive.

$\forall (a, b) \in R, \forall (b, c) \in R, (a, c) \in R^2$.

Since $R^2 \subseteq R, (a, c) \in R$.

$\therefore$ R is transitive.

2\.(only if) R is transitive $\rightarrow R^n \subseteq R$ for $\forall n \in \mathbf{N}^+$.

basis Trivial for $n = 1$.

induction Assume $R^n \subseteq R$ and R is ***transitive*** for some $n \in \mathbf{N}^+$.

$\forall (a, b) \in R^{n+1}, \exists c \in A$.϶. $(a, c) \in R \wedge (c, b) \in R^n$. ($R^{n+1} = R^n \circ R$)

Since $R^n \subseteq R, (c, b) \in R$, and since R is transitive, $(a, b) \in R$.

$\therefore R^{n+1} \subseteq R$.

8.2 n-ary Relations and Their Applications

Definition *1 Let $A_1, A_2, \ldots, A_n$ be sets.* $R \subseteq A_1 \times A_2 \times \ldots \times A_n$ *is a n-ary relation on Let $A_1, A_2, \ldots, A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the* ***domain*** *of the relations, and n is called its* ***degree****.*

8.3 Representing Relations

R: A × B → {0, 1}

boolean matrix

$R \subseteq A \times A$.

Definition *1 A directed graph, digraph G = (V, E) consistes of a set V of* ***vertices****, and a set* $E \subseteq V \times V$ *of* ***edges(arcs)****.*

8.4 Closure of Relations

Let $R \subseteq A \times A$. R may or may not have some property **P** *={reflextive, symmetric, transitive}. If $\forall T \subseteq A \times A$ with property* **p** $\in$ **P** *and $R \subseteq T$, $S \subseteq T$, then S is called the* **p *closue*** *of R.*

Reflexive *closure of R*

$R \cup id_A$.

Symetric *closure of R*

$R \cup R^{-1}$.

Definition *1 A* ***path*** *from a to b in the directed grapg G = (V, E).*

$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n) \in E$, and $x_0 = a$, $x_n = b$.

The path is denoted by $(x_0, x_1, x_2, \ldots, x_n)$ and has ***length*** *n.*

A path of length $n \geq 1$ and

begins and ends at the same vertex is called ***cycle****.*

Theorem *1 Let $R \subseteq A \times A$. There is a path of length $n \geq 1$ from a to b, if and only if, $(a, b) \in R^n$.*
proof *Easy for induction*

Definition *1 A* ***connetivity*** *relation $R^+ = \{(a, b) | (a, b) \in R^n, \forall n \geq 1\}$*

$$R^+ = \cup_{i \in \mathbf{N}^+} R^i = R^1 \cup R^2 \cup \ldots$$

transitive *closure of R.*

$$R^* = \cup_{i \in \mathbf{N}} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots$$

reflexive and transitive *closure of R.*

Let A and C be sets $R \subseteq A \times A$, *and f, g:* $A \to 2^C$.*(set valued functions)*

$f(a) = \{c \in C \mid c \in g(a)\} \cup \{c \in C \mid a\ R\ b, c \in f(b)\}$.

$f(a) = \{c \in g(a)\} \cup \{c \in f(b) \mid a\ R\ b\}$.

$f(a) = g(a) \cup \cup_{a\ R\ b} f(b)$. (***recursive*** *definition of f) Then*

$f(a) = \{c \in C \mid c \in g^*(a)\}$.

$f(a) = g^*(a)$. (***iterative*** *definition of f)*

Warshall's algorithm	$O(n^3)$
Depth first search	$O(n^2)$

Algorithm *Depth first search*

```
S: stack of Vertex; n(Vertex) array of Depth;
procedure Traverse(x: Vertex; d: Depth);
    push x onto S; n(x) := d;                    f(x) := g(x);
    for y ∈ Vertex where x R y do
        if n(y) = 0 then Traverse(y, d+1) fi;
        n(x) := min(n(x), n(y));                 f(x) := f(x) ∪ f(y)
        od;
    if n(x) = d then repeat
        y = pop of S; n(y) := infinite;          f(y) := f(x)
        until  y = x
    fi
end procedure Traverse
for x  ∈ Vertex do n(x) := 0 od;                 f(x) := {};
for x  ∈ Vertex where n(x) = 0 do Traverse(x, 1) od
```

8.5 Equivalence Relations

Definition *1 Let $R \subseteq A \times A$. R is called* ***equivalence*** *relation,*
if it is reflexive, symmetric, and transitive.

Definition *3 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation.*
$[a]_R = \{b \mid a\ R\ b\}$ *is called the* ***quivalence class*** *of a w.r.t. R.*
If $b \in [a]_R$, b is called the ***representative*** *of the equivalent class.*
Note that $a \in [a]_R$, since R is ***reflexive****.*

Theorem *1 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation. Three statments are logically equivalent*
i) $a\ R\ b$ ii) $[a]_R = [b]_R$ iii) $[a]_R \cap [b]_R \neq \emptyset$.

proof

1) i) $\rightarrow$ ii)
$\forall c \in [a]_R$, $a\ R\ c$, $a\ R\ b$, $b\ R\ a$. $\therefore$ $b\ R\ c$, $c \in [b]_R$. $\therefore$ $[a]_R \subseteq [b]_R$.

$\forall c \in [b]_R$, $b\ R\ c$, $a\ R\ b$. $\therefore$ $a\ R\ c$, $c \in [a]_R$. $\therefore$ $[b]_R \subseteq [a]_R$.

2) ii) $\rightarrow$ iii)

Assume $[a]_R = [b]_R$, $a\ R\ b$. $\therefore$ $a, b \in [a]_R \cap [b]_R \neq \emptyset$.

3) iii) $\rightarrow$ i)

Suppose $[a]_R \cap [b]_R \neq \emptyset$, $[a]_R \neq \emptyset$ and $[b]_R \neq \emptyset$.

$\exists c \in [a]_R \wedge c \in [b]_R$. $a\ R\ c$, $b\ R\ c$. $c\ R\ b$, $\therefore$ $a\ R\ b$.

Lemma *1.5 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation.*

$\cup_{a \in A} [a]_R = A$. $a \in [a]_R$.

$\therefore$ $\{[a]_R \subseteq A \mid a \in A\}$ is a ***partition*** of A.

Definition *1.5 Let S be a set. The* ***partition*** *of S, $\{A_i \mid i \in I\}$ I: index set, is*

i) $A_i \neq \emptyset$, $i \in I$. ***nonempty***

ii) $A_i \cap A_j = \emptyset$, when $i \neq j$. ***disjoint***

iii) $\cup_{i \in I} A_i = A$. ***exhaustive***

Theorem *2 Let R be a equivalent relation on A.*

*Then the **equivalent classes** of R form a **partion** of A.*

*Conversely, given a **partition** $\{A_i | i \in I\}$ of the set A,*

*there is an **equivalent relation** R that has the set A_i, $i \in I$,*

*as its **equivalent class**.*

Realtion $R \subseteq A \times A$ $O(n^2)$ where $|A| = n$.

Equivalent *relation* $R \subseteq A \times A$ $O(n)$

8.6 Partial Ordering

Definition *1 Let $R \subseteq A \times A$. R is called* ***(ir)reflexive partial order****,*
if it is ***(ir)reflexive****,* ***antisymmetric****, and* ***transitive****.*
(A, R) is called ***partially odreded*** *set or* ***poset****.*

Example *1 2 3 $(\mathbf{Z}, \leq)$, $(\mathbf{Z}^{+}, |)$, $(2^{S}, \subseteq)$ are posets.*

Definition *2 Let $(A, \leq)$ be poset and $a, b \in A$. Then*
The elements a and b are ***comparable*** *if either $a \leq b$ or $b \leq a$.*
The elements a and b are ***incomparable*** *if* ***neither*** *$a \leq b$* ***nor*** *$b \leq a$.*

Definition *3 Let $(S, \leq)$ be poset. If $\forall a, b \in S$, a and b are* ***comparable****,*
S is called ***totally ordered set****,* ***linearly ordered set****, or* ***chain****.*
$\leq$ is called ***total order*** *or* ***linear order****.*

Definition *4 (S, ≤) is a* ***well-ordered*** *set, if it is a* ***poset****, ≤ is a* ***total order****, and every nonempty subset of S has a* ***least*** *emelent.*

Theorem *1* ***Principle of Well-Ordered Induction***

Let (S, ≤) be a well-ordered set. $\forall x \in S, P(x)$, *if*

$\forall y \in S, \forall x \in S$.э. $x < y: P(x) \to P(y)$.

proof *Suppose* $\exists y \in S, \neg P(y)$.

$A = \{x \in S | \neg P(x)\} \neq \emptyset$.

Let a ∈ A be the least element.

$\exists a \in A, \forall x \in S$.э. $x < a: P(x) \to P(a)$.

Contradiction $\exists y \in S, \neg P(y)$.

<u>Lexicographic order</u>

Let $(A_1, \leq_1)$ *and* $(A_2, \leq_2)$ *be* ***well-ordered*** *sets.*

We define $(A_1 \times A_2, \leq)$ *<u>lexicographic order</u>*

$(a_1, a_2) \leq (b_1, b_2)$, *if* $(a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2))$

Hasse Diagram

Let (A, $\leq$) be a poset. We say a ***covers*** *b, if $a < b \wedge \neg\exists c \in A, a \leq c \wedge c \leq b$.*
(A, covers) is a Hasse Diagram.

Maximal and Minimal Elements

Let (A, $\leq$) be a poset.
We say a is a ***maximal****, if $\neg\exists b \in A, a < b$.*
We say a is a ***minimal****, if $\neg\exists b \in A, b < a$.*
We say a is a ***greatest element****, if $\forall b \in A, b \leq a$.*
We say a is a ***least elements****, if $\forall b \in A, a \leq b$.*
Let $S \subseteq A$. $u \in A$ is called ***upper bound*** *of S, if $\forall a \in S, a \leq u$.*
$l \in A$ is called ***lower bound*** *of S, if $\forall a \in S, l \leq a$.*
$x \in A$ is called the ***least upper bound*** *of S, if $\forall a \in S, a \leq x$,*
$x \leq z$, if $\forall z \in$ upper bound of S.
$y \in A$ is called the ***greatest lower bound*** *of S, if $\forall a \in S, y \leq a$,*
$z \leq y$, if $\forall z \in$ lower bound of S.

Lattice

A poset (A, ≤) is called ***lattice****. If every pair of elements has both a least upper bound, and a greatest lower bound.*

($\mathbf{Z}$, ≤) is a lattice. *max, min:* $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$

($\mathbf{Z}^+$, |) is a lattice. *lcm, gcd:* $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$

(2^S, ⊆) is a lattice. $\cup, \cap$: $2^S \times 2^S \rightarrow 2^S$

Topological Sorting

Lemma *1 Every* ***finite nonempty*** *poset (A, ≤) has*
at least one minimal *elements.*

Algorithm *1* ***Topological sorting***

Construct a total ordering $<_t$ *from a partial ordering ≤.*

$a <_t b$, *if and only if, a ≤b or a and b are* ***incomparable***.