

5 Counting

5.1 The Basics of Counting

Product and Sum Rules

Suppose two task,

m number of ways to do task 1

n number of ways to do task 2

Do **both** task 1 **and** task 2:

mn ways

$$|A \times B| = |A| \cdot |B|.$$

Product Rule

Do **either** task 1 **or** task 2, **but not both**:

$m+n$ ways

$$|A \cup B| = |A| + |B|, \text{ if } A \cap B = \emptyset.$$

Sum Rule

Principle of Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

5.2 The Pigeonhole Principle

Theorem 1 *The pigeonhole principle*

*If $k + 1$ or more objects are placed into k box,
there is at least one or more box containing two or more objects.*

Dirichlet drawer principle

Corollary 1 *If $f: A \rightarrow B$ and $|A| \geq |B| + 1$, then f is not one-to-one.*

Theorem 2 *The generalized pigeonhole principle*

*If N objects are placed into k box,
there is at least one or more box containing $\lceil N/k \rceil$ objects.*

proof by contradiction, no box contains $\lceil N/k \rceil$ objects.

$$k(\lceil N/k \rceil - 1) < k((N/k + 1) - 1) = N$$

$N = 59$ students, $k = 12$ months,

$$\lceil N/k \rceil = \lceil 4.91... \rceil = 5 \text{ students/month}$$

5.3 Permutations and Combinations

$$1 \leq r \leq n, P(n, r) = n(n-1) \dots (n-r+1)$$

$$0 \leq r \leq n, P(n, r) = n! / (n-r)!$$

$$0 \leq r \leq n, C(n, r) = P(n, r) / P(r, r) = n! / r!(n-r)!$$

$$C(n, r) = C(n, n-r)$$

5.4 Binomial Coefficient

$$\begin{aligned} (x+y)^n &= C(n, 0)x^n + C(n, 1)x^{n-1}y + \dots + C(n, n-1)x^1y^{n-1} + C(n, n)y^n. \\ &= \sum_{j=0}^n C(n, n-j)x^{n-j}y^j. \end{aligned}$$

Corollary 1 $\sum_{k=0}^n C(n, k) = 2^n$. for n

Corollary 2 $\sum_{k=0}^n (-1)^k C(n, k) = 0$.

Pascal's Identity and Triangle

$$C(n+1, k) = C(n, k-1) +$$

5.5 Generalized Permutation and Combinations

5.6 Generating Permutation and Combinations

Let $a_1a_2\dots a_n$ and $b_1b_2\dots b_n$ be permutation of $\{1, 2, \dots, n\}$.

We say $a_1a_2\dots a_n < b_1b_2\dots b_n$,

if $1 \leq \exists k \leq n \ .\exists. a_1 = b_1, a_2 = b_2, \dots, a_{k-1} = b_{k-1}, a_k < b_k$.

permutation $a_1a_2\dots a_n \in \{1, \dots, n\}^n$.

$<$ is a total order

Next permutation of $a_1 \dots a_n$?

Find largest j $\exists. a_j < a_{j+1}$.

Find smallest integer a_k in a_{j+1}, \dots, a_n but $a_k > a_j$.

Interchange a_k and a_j .

Increasing order from $j+1$ -th to n -th position.

procedure *nextPermutation*($a_1 \dots a_n \in \{1, \dots, n\}^n \neq n(n-1) \dots 21$)

$j := n-1$; **do** $a_j > a_{j+1} \rightarrow j := j - 1$ **od**

$k := n$; **do** $a_j > a_k \rightarrow k := k - 1$ **od**

$a_k, a_j := a_j, a_k$;

$r, s := n, j+1$; **do** $r > s \rightarrow a_r, a_s := a_s, a_r$; $r, s := r - 1, s+1$ **od**

procedure *next bit string*($b_1 \dots b_n \in \{0, 1\}^n \neq 111 \dots 11$)
 $i := n$; **do** $b_i = 1 \rightarrow b_i := 0$; $i := i - 1$ **od**; $b_i := 1$.

Next r -combination of $\{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$?

Find $a_i \exists. a_i \neq n - r + i$.

replace a_i with $a_i + 1$.

replace a_j 's with $a_i + j - i$.

procedure *next r-combination*($\{a_1, a_2, \dots, a_n\} \subseteq \{1, 2, \dots, n\}$
 $\neq \{n-r+1, n-r+2, \dots, n\}$ and $a_1 < a_2 < \dots < a_n$)
 $i := r$; **do** $a_i = n - r + i \rightarrow i := i - 1$ **od**; $a_i := a_i + 1$;
for $j := i + 1$ **to** r **do** $a_j := a_i + j - i$ **od**