

# 4 Induction and Recursion

## 4.1 Mathematical Induction

### The First Principle of Mathematical Induction

$P(0)$

*basis*

$\forall n \geq 0 P(n) \rightarrow P(n+1)$

*induction*

$\therefore \forall n \geq 0 P(n).$

*proof*

$P(0).$

$P(0) \rightarrow P(1).$

$P(1) \rightarrow P(2).$

...

*Q.E.D.*

*domino effect*

## 4.2 Strong Induction and Well Ordering

### The Second Principle of Mathematical Induction (Strong Induction)

$P(0)$	<i>basis</i>
$\forall n \geq 0 (\forall k: 0 \leq k \leq n P(k)) \rightarrow P(n+1)$	<i>induction</i>
$\therefore \forall n \geq 0 P(n).$	

Why *strong*?

$(\forall k: 0 \leq k \leq n P(k))$  is *stronger* than  $P(n)$ .

Example 4

### Generalized Induction

Let  $C \geq 1, c \geq 0$ .

$\forall n: c \leq n \leq c+C-1, P(n)$	<i>C-basis</i>
$\forall n: n \geq c, P(n) \rightarrow P(n+C)$	<i>C-step induction</i>
$\therefore \forall n \geq c P(n).$	

## Proofs Using Well Ordering Property

### *well ordering property*

*Every nonempty set of nonnegative integers has a least element.*

$$\forall S: \emptyset \subset S \subseteq \mathbf{N}, \exists m \in S, \forall n \in S, m \leq n.$$

*Assume  $S = \{n \mid \neg P(n)\} \supset \emptyset$  and  $m \in S$ , is the least element.*

*$m \neq 0$ , since  $P(0)$  by **basis**.*

*$m - 1 \in \mathbf{N}$ , but  $m - 1 \notin S$ .*

*$P(m - 1) \rightarrow P(m)$  is **contradicted by induction**.*

### 4.3 Recursive Definition and Structural Induction

#### Recursively Defined Function

$f: \mathbf{N} \rightarrow S$  (for any set  $S$ )

i) Define  $f(0)$ .

ii) For  $n > 0$ , define  $f(n)$  in terms of  $f(0), f(0), \dots, f(n-1)$ .

**Example** Define the series  $a_n := a^n$  recursively:

Let  $a_0 := 1$   $a^0 = 1$

For  $n > 0$ ,  $a_n := a \cdot a_{n-1}$ .  $a^n = a \cdot a^{n-1}$  for  $n > 0$

**Example 2**  $F(n) = n!$

$F(0) = 1$ .  $0! = 1$

$F(n) = n \cdot F(n-1)$  for  $n > 0$ .  $n! = n \cdot (n-1)!$  for  $n > 0$

**Definition 1** The Fibonacci series are defined by  $f_0 = 0$ ,  $f_1 = 1$ , and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

**Theorem**  $f_n < 2^n$ .

**Proof** Mathematical induction

**basis**  $f_0 = 0 < 2^0 = 1$  and  $f_1 = 1 < 2^1 = 2$ .

**induction** Assume  $\forall k < n, f_k < 2^k$ . (strong induction)

$$f_n = f_{n-1} + f_{n-2} < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2^n.$$

**Example**

## Recursively Defined Sets and Structures

**Definition 2** The set of  $\Sigma^*$  of strings over the alphabet  $\Sigma$  is defined:

**basis**  $\varepsilon \in \Sigma^*$  where  $\varepsilon$  is the empty string containing no symbols.

**recursion** If  $w \in \Sigma^*$  and  $a \in \Sigma$ , then  $wa \in \Sigma^*$ .

**Example 9** The length of a string  $|w|$ ,  $w \in \Sigma^*$ .

**basis**  $|\varepsilon| = 0$ .

**recursion** If  $w \in \Sigma^*$  and  $a \in \Sigma$ , then  $|wa| = |w| + 1$ .

**Definition** Set of natural numbers  $\mathbf{N}$ .

**basis**  $0 \in \mathbf{N}$ .

**recursion** If  $n \in \mathbf{N}$ , then  $n+1 \in \mathbf{N}$ .

**Example 10 Well-Formed Formulae for Compound Statement Form**

**basis**  $\mathbf{T}$ ,  $\mathbf{F}$ , and  $s$  where  $s$  propositional variables are w.f.f..

**recursion** If  $E$  and  $F$  are well-formed formulae,  $\neg E$ ,  $E \wedge F$ ,  $E \vee F$ ,  $E \rightarrow F$ ,  $E \leftrightarrow F$ , and  $(E)$  are well-formed formulae.

*Abstract syntax*

$$E ::= \mathbf{T} \mid \mathbf{F} \mid s \mid \neg E \mid E \wedge E \mid E \vee E \mid E \rightarrow E \mid E \leftrightarrow E \mid (E)$$

**Example 11 Well-Formed Formulae for Expression**

$$E ::= n \mid v \mid E + E \mid E - E \mid E * E \mid E / E \mid E \uparrow E \mid (E)$$

where  $n$  is integer,  $v$  is a variable, and  $\uparrow$  denotes exponential.

**Definition 4** The set of **rooted trees** with root  $r$ .

**basis** A single vertex  $r$  is a rooted tree with root  $r$ .

**recursion** Suppose  $T_1, T_2, \dots, T_n$  are disjoint rooted trees with roots  $r_1, r_2, \dots, r_n$ , respectively. Adding a **new** node  $r$  and edges to each of the vertices  $r_1, r_2, \dots, r_n$  is also rooted tree with root  $r$ .

A **directed graph (digraph)**  $G = (V, E)$

where  $V$  is a set of **vertices** and  $E \subseteq V \times V$  is a set of **edges**.

The rooted tree  $T = (V, E, r)$  is a digraph  $(V, E)$  with root  $r \in V$ , and ....

**Definition 4.5** The set of **rooted trees** with root  $r$  is defined as follows:

**basis**  $(\{r\}, \emptyset, r)$  is a rooted tree.

**recursion** If  $(V_1, E_1, r_1), (V_2, E_2, r_2), \dots, (V_n, E_n, r_n)$  are rooted trees

where  $1 \leq i < j \leq n: V_i \cap V_j = \emptyset$ . Then **disjoint**

$(\cup_{i=1}^n V_i \cup \{r\}, \cup_{i=1}^n E_i \cup \{(r, r_1), (r, r_2), \dots, (r, r_n)\}, r)$  is a rooted tree.



**Definition 5** The set of **extended binary trees**(EBT) with root  $r$ .

**basis** An empty set is an extended binary tree.

**recursion** If  $T_1$  and  $T_2$  are disjoint EBT. Adding a new node  $r$  and edges to each of the roots of left subtree  $T_1$  and right subtree  $T_2$  is an EBT with root  $r$  when these subtrees are not empty.

**Definition 6** The set of **full binary trees**(FBT) with root  $r$ .

**basis** A full binary tree with root  $r$ .

**recursion** If  $T_1$  and  $T_2$  are disjoint FBT. Adding a new node  $r$  and edges to each of the roots of left subtree  $T_1$  and right subtree  $T_2$  is an FBT with root  $r$ .

**Definition 6** The set of **full binary tree**(FBT) with root  $r$  is defined as ...

**basis**  $(\{r\}, \emptyset, r)$  is a FBT.

**recursion** If  $T_1 = (V_1, E_1, r_1)$  and  $T_2 = (V_2, E_2, r_2)$  are FBT's where ...

$T = T_1 \cdot T_2 = (V_1 \cup V_2 \cup \{r\}, E_1 \cup E_2 \cup \{(r, r_1), (r, r_2)\}, r)$  is a FBT.

## Structural Induction

**Example 12** Let  $3 \in S$ , and let  $x+y \in S$ , if  $x, y \in S$ . Let  $A = \{n \in \mathbf{Z}^+ \mid (3|n)\}$

Prove  $A = S$

i) Prove  $A \subseteq S$ .

*base: Let  $n = 3 \in A$ ,  $3 \in S$  by basis def'n of  $S$ .*

*induction: if  $n \in A$ , then  $n \in S$ .*

*Consider  $n + 3$ .  $n+3 \in S$ , since  $3 \in S$ .*

ii) Prove  $S \subseteq A$ ,

*base:  $3 \in S$ ,  $3|3$ ,  $3 \in A$ .*

*recursion: Consider  $n = x+y \in S$ .*

*$\forall x < n, \forall y < n: x, y \in A$ .      *strong induction hypothesis**

*$\therefore 3 \mid x \wedge 3 \mid y$ .       $3 \mid (x+y)$ .*

*$x+y \in A$ .*

**Definition 7** Height of a full balanced tree.

**basis**  $h(\{r\}, \emptyset, r) = 0$ .

**recursion**  $h(T_1 \cdot T_2) = 1 + \max(h(T_1), h(T_2))$ .

**Definition** Number of nodes of a full balanced tree.

**basis**  $n(\{r\}, \emptyset, r) = 1$ .

**recursion**  $n(T_1 \cdot T_2) = 1 + n(T_1) + n(T_2)$ .

**Theorem 2** Let  $T$  be a full binary tree. Then  $n(T) \leq 2^{h(T)+1} - 1$ .

**basis step**  $h(\{r\}, \emptyset, r) = 0, n(\{r\}, \emptyset, r) = 1. \therefore 1 \leq 2^1 - 1 = 1$ .

**recursion step** Assume  $n(T_1) \leq 2^{h(T_1)+1} - 1, n(T_2) \leq 2^{h(T_2)+1} - 1$ .

$$\begin{aligned} n(T) &= 1 + n(T_1) + n(T_2) && \leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \\ &&& \leq 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 && = 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1 \\ &&& = 2^{h(T)+1} - 1 \end{aligned}$$

*Lexicographic ordering(Example 15)*

Let  $< \subseteq \mathbf{N} \times \mathbf{N}$ .  $(x_1, y_1) < (x_2, y_2)$ , if  $(x_1 < x_2) \vee ((x_1 = x_2) \wedge (y_1 < y_2))$ .

Every subset of  $\mathbf{N} \times \mathbf{N}$  has a **least element(well-ordering)**

**Define**  $a_{m,n}: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$  **recursively.**

**basis**  $a_{0,0} = 0$ .

**recursion**  $a_{m,n} = a_{m-1,n} + 1$  if  $(n = 0) \wedge (m > 0)$

$a_{m,n-1} + n$  if  $(n > 0)$

	0	1	2	3	4	...	$m$	$m+1$
0	0	1	3	6	10	...	$K+m$	$K+m+1$
1	1	4	6	9	...		$K+m-1$	$K+1+m$
2	2	3	5	...		$K+m-2$	$K+2+m-1$	
3	3	4	...		$K+m-3$			
...	...							
$m$	$m$	$K+m+1$						
$m+1$	$m+1$							

**Solution** We can prove  $a_{m,n} = m + n(n+1)/2$ , if  $a_{i,j} < a_{m,n}$ ,  $(i, j) < (m, n)$ .

**basis**  $a_{0,0} = 0 + 0 \cdot 1/2 = 0$ .  $a_{0,0}$  and  $(0, 0)$  is a least element in  $\mathbf{N}$  and  $\mathbf{N} \times \mathbf{N}$ .

**recursion** Suppose  $(0, 0) < \forall (i, j) < (m, n): a_{i,j} = i + j(j+1)/2$

*case 1) If  $(n = 0) \wedge (m > 0)$   $(m-1, 0) < (m, 0)$ (I.H.)*

$$a_{m-1,0} = m - 1 + 0(0+1)/2 = m - 1 \quad \text{Induction hypothesis}$$

$$a_{m,0} = a_{m-1,0} + 1 = m - 1 + 1 = m. \quad \text{Recursive def. of } a_{m,0}$$

$$a_{m,0} = m + 0(0+1)/2 = m.$$

*case 2) If  $(n > 0)$   $(m, n-1) < (m, n)$*

$$a_{m,n-1} = m + n(n-1)/2 \quad a_{m,n} = m + n(n-1)/2 + n = m + n(n+1)/2.$$

## 4.4 Recursive Algorithms

*Recursive functions*

*Recursive sets*

*Recursive algorithm*

**Algorithm 1 Recursive Algorithm for Computing  $n!$**

**function**  $factorial(n: \mathbf{N}): \mathbf{N};$

**if**  $n=0 \rightarrow factorial(n) := 1$

  |  $n \geq 1 \rightarrow factorial(n) := n \cdot factorial(n-1)$

**fi**

**Algorithm 2 Recursive Algorithm for Computing  $a^n$**

**function**  $power(a: \mathbf{Z}^+, n: \mathbf{N};): \mathbf{N};$

**if**  $n=0 \rightarrow power(a, n) := 1$

  |  $n \geq 1 \rightarrow power(a, n) := a \cdot power(a, n-1)$

**fi**

**Compute  $b^n \bmod m$  where  $m \geq 2$ ,  $n \geq 0$ ,  $1 \leq b < m$ .**

$$b^n \bmod m = (b \cdot (b^{n-1} \bmod m)) \bmod m.$$

$$b^0 \bmod m = 1.$$

$$O(n)$$

*But*

$$b^n \bmod m = (b^{n/2} \bmod m)^2 \bmod m.$$

**Algorithm 3 Recursive Modular Exponentiation**

**function**  $mpower(b, n, m: \mathbf{N}, n: \mathbf{N};): \mathbf{N};$

**if**  $n=0 \rightarrow mpower(b, n, m) := 1$

**|**  $(2 \mid n) \rightarrow mpower(b, n, m) := mpower(b, n/2, m)^2 \bmod m$

**|**  $(2 \nmid n) \rightarrow mpower(b, n, m) :=$

$(mpower(b, \lfloor n/2 \rfloor, m)^2 \bmod m \cdot b \bmod m) \bmod m$

**fi**

**function (recursive) fibonacci**( $n \in \mathbf{N}$ )  $\in \mathbf{N}$ ;

**if**  $n = 0 \rightarrow \text{fibonacci} := 0$ ;

  |  $n = 1 \rightarrow \text{fibonacci} := 1$ ;

  |  $n \geq 2 \rightarrow \text{fibonacci}(n-1) + \text{fibonacci}(n-2)$

**fi**

**Theorem** *fibonacci*( $n$ ) performs  $f_{n+1} - 1$  additions

**basis** *fibonacci*(0):  $f_{0+1} - 1 = f_1 - 1 = 1 - 1 = 0$  additions.

*fibonacci*(1):  $f_{1+1} - 1 = f_2 - 1 = 1 - 1 = 0$  additions.

**recursion** Assume *fibonacci*( $n-1$ ):  $f_n - 1$ ; *fibonacci*( $n-2$ ):  $f_{n-1} - 1$  addi.

*fibonacci*( $n$ ):  $(f_n - 1) + (f_{n-1} - 1) + 1 = f_n + f_{n-1} - 1 = f_{n+1} - 1$  additions.

**function (iterative) fibo**( $n \in \mathbf{N}$ )  $\in \mathbf{N}$ ;

**if**  $n = 0 \rightarrow \text{fibo} := 0$

  |  $n = 1 \rightarrow \text{fibo} := 1$

  |  $n \geq 2 \rightarrow x, y := 0, 1$ ;

**for**  $i:=2$  **to**  $n$  **do**  $\text{fibo}, x, y := x+y, y, \text{fibo}$  **od** **fi**



**function**  $\text{findFib}(a, b, i, n \in \mathbf{N}) \in \mathbf{N}$ ;  
 { Assert :  $(a = f_{i-2}) \wedge (b = f_{i-1}) \wedge (2 \leq i \leq n)$  }  
 if  $i = n \rightarrow \text{findFib} := b$ ;  
 |  $i \neq n \rightarrow \text{findFib} := \text{findFib}(b, a+b, i+1, n)$   
 fi

**function**  $\text{fastFib}(n \in \mathbf{N}) \in \mathbf{N}$ ;  
 if  $n = 0 \rightarrow \text{fastFib} := 0$   
 |  $n = 1 \rightarrow \text{fastFib} := 1$   
 |  $n \geq 2 \rightarrow \text{findFib}(0, 1, 2, n)$   
 fi

*Storage(assignment statement) for speed efficiency!*

<i>recursive fibonacci</i>	$O(2^n)$
<i>iterative fibonacci</i>	$O(n)$
<i>fast recursive fibonacci</i>	$O(n)$

The Merge Sort

**function**  $sort(L = l_1, l_2, \dots, l_n \in \mathbf{N}^n) \in \mathbf{N}^n$ ;

**if**  $n > 1 \rightarrow m := \lfloor n/2 \rfloor$ ;  $L = merge(sort(l_1, \dots, l_m), sort(l_{m+1}, \dots, l_n))$

  |  $n \leq 1 \rightarrow sort := L$ ;

**fi**

$O(\log n)$

**function**  $merge(A = (a_1, \dots, a_m) \in \mathbf{N}^m; B = (b_1, \dots, b_n) \in \mathbf{N}^n) \in \mathbf{N}^{m+n}$ ;

  { Assert  $(a_1 \leq a_2 \leq \dots \leq a_m) \wedge (b_1 \leq b_2 \leq \dots \leq b_n)$  }

**if**  $m = 0 \rightarrow merge := B$

  |  $n = 0 \rightarrow merge := A$

**elsif** |  $a_1 \leq b_1 \rightarrow merge := (a_1, merge((a_2, \dots, a_m); B))$

  |  $b_1 \leq a_1 \rightarrow merge := (b_1, merge(A; (b_2, \dots, b_n)))$

**fi**

$O(n)$

**function** *iterative merge*( $A = a_1, \dots, a_m \in \mathbf{N}^m; B = b_1, \dots, b_n \in \mathbf{N}^n$ )  $\in \mathbf{N}^{m+n}$ ;

$L = \text{empty list}; i, j, k, := 1, 1, 1;$

**do**  $(i \leq m) \wedge (j \leq n) \rightarrow$  **if**  $i > m \rightarrow L_k := b_j; j, k := j + 1, k + 1$

|  $j > n \rightarrow L_k := a_i; i, k := i + 1, k + 1$

|  $a_i \leq b_j \rightarrow L_k := a_i; i, k := i + 1, k + 1$

|  $a_i \geq b_j \rightarrow L_k := b_j; j, k := j + 1, k + 1$

**fi**

**od**;

*iterative merge* :=  $L$ .

$O(n \log n)$

**do**  $k \leq m+n \rightarrow$  **if**  $(i > m) \vee (a_i \geq b_j) \rightarrow L_k := b_j; j, k := j + 1, k + 1$

|  $(j > n) \vee (a_i \leq b_j) \rightarrow L_k := a_i; i, k := i + 1, k + 1$

**fi**

**od**

## 4.5 Program Correctness

**Definition 1** A Program  $S$  is said to be **partially correct**, with respect to the **initial assertion**  $p$  and the **final assertion**  $q$ , if the **initial condition**  $p$  is true and  $S$  **terminates**, then the **final assertion** is true, written  $p\{S\}q$  and it is called **Hoare triple**.

**Example 1**  $(x=1) \{y:=2; z:=x+y\} (z=3)$   
 $\mathbf{T} \{y:=2; z:=x+y\} (z=x+2)$

*Sequence of statement*

$$\begin{array}{l} p \{S_1\} q \\ q \{S_2\} r \\ \hline \therefore p \{S_1; S_2\} r \end{array}$$

**If statement**

$$\frac{(p \wedge B) \{S\} q \quad (p \wedge \neg B) \rightarrow q}{\therefore p \{ \text{if } B \text{ then } S \} q}$$

**If then else statement**

$$\frac{(p \wedge B) \{S_1\} q \quad (p \wedge \neg B) \{S_2\} q}{\therefore p \{ \text{if } B \text{ then } S_1 \text{ else } S_2 \} q}$$

**while** statement

$$\frac{(p \wedge B) \{S\} p}{\therefore p \{\mathbf{while} \ B \ \mathbf{do} \ S\} (p \wedge \neg B)}$$

**loop invariance**

**initialize**  $p$

**update**  $p$

**terminate**  $(p \wedge \neg B)$

**procedure** multiply( $m, n$ : integer)

**if**  $n < 0$  **then**  $a := -n$  **else**  $a := n$ ;

$k := 0$ ;  $x := 0$ ;

**do**  $k < a \rightarrow$

$x := x + n$ ;  $k++$

**od**

**if**  $n < 0$  **then**  $prod := -x$  **else**  $prod := x$ ;  $prod = mn$

$m, n \in \mathbf{Z}$

$a = |n|$

$x = mk$

$x = mk$

$x = mk \wedge k = a \therefore x = ma = m|n|$