5/2 Recurrence relations

$$a_n = f(a_{n-1}, \dots, a_0)$$

Ex. $\underline{a_n = a_{n-1} - a_{n-2}}$ $n \geq 2$, $a_1 = ?$ $a_0 = ?$

degree 2

$a_0 = 0$
$a_1 = 1$
$a_2 = 1$
$a_3 = 0$
$a_4 = -1$

Ex.3 $P_n = 1.05 P_{n-1}$ $n \geq 1$, ~~$n=0$~~ $P_0 = ?$ degree 1

$$P_n = 1.05 P_{n-1}$$
$$= 1.05 \cdot 1.05 P_{n-2} = 1.05^2 P_{n-2}$$
$$\vdots$$
$$= 1.05^k P_{n-k} \quad k = n$$
$$= \underline{1.05^n P_0}$$

Ex. 4. Fibonacci number

Ex. 5. Tower of Hanoi

degree 1, non

$$\boxed{H_n = 2H_{n-1} + 1}$$
$$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + \overset{3\ (\times)}{2+1}$$
$$= 2^2(2H_{n-3} + 1) + 2 + 1$$
$$= 2^3 H_{n-3} + 2^2 + 2^1 + 2^0$$
$$\vdots$$
$$= \underline{2^{n-k} H_k + 2^{n-k-1} + \dots + 2^0}$$
$$= 2^{n-k}(2H_{k-1} + 1) + 2^{n-k-1} + \dots + 2^0$$
$$= \underline{2^{n-k+1} H_{k-1} + 2^{n-k} + 2^{n-k-1} + \dots + 2^0}$$
$$= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \dots + 2^0$$

$$= 2^{n-1} + \dots + 2^0 \quad (\because H_1 = 1)$$
$$= \frac{2^n - 1}{2 - 1}$$
$$= 2^n - 1$$

7.2 Linear homogeneous Recurrence relation of degree k.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad n \geq k$$

$c_k \neq 0, \quad c_i \in \mathbb{R}$

k - initial cond. $a_0, a_1, \ldots a_{k-1}$

$a_k \leftarrow f(a_0 \ldots a_{k-1})$
$a_{k+1} \leftarrow f(a_1 \ldots a_k$
$a_{k+2} \leftarrow f(a_2 \ldots a_{k+1}$

Assume $a_n = r^n$

$$\frac{r^n}{r^{n-k}} = \frac{c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}}{r^{n-k}}$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r^1 - c_k r^0 = 0$$

characteristic equation of degree k.

Consider a recurrence relation of degree 2.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

$$r^2 - c_1 r - c_2 = 0$$

$r_1 \neq r_2$ .. two roots

$$\boxed{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n} \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

We know $r_1^2 = c_1 r_1 + c_2$ $\quad r_2^2 = c_1 r_2 + c_2$

$$c_1 a_{n-1} + c_2 a_{n-2}$$
$$= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$
$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$

($c_1 r_1 + c_2 = r_1^2$, $c_1 r_2 + c_2 = r_2^2$)

$$= \alpha_1 r_1^n + \alpha_2 r_2^n = a_n$$

Ex. 3.5 small Tower of Hanoi ($H_n = 2H_{n-1} + 1$) $\boxed{H_n = 2^n - 1}$ $\boxed{n=64}$

$h_n = 2h_{n-1}$ $h_1 = 1$

C.E. $r - 2 = 0$ $\therefore r = 2$

$h_n = \alpha \cdot 2^n$ $\therefore h_1 = \alpha \cdot 2^1 = 1$ $\therefore \alpha = \frac{1}{2}$

$= 2^{n-1}$

Ex 4

$f_n = f_{n-1} + f_{n-2}$ $f_0 = 0, f_1 = 1$ 326 백만원

C.E. $r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2}$

$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

$f_0 = \alpha_1 + \alpha_2 = 0$

$f_1 = \alpha_1 \frac{1+\sqrt{5}}{2} + \alpha_2 \frac{1-\sqrt{5}}{2} = 1$

$\alpha_1 = \frac{1}{\sqrt{5}}$ $\alpha_2 = -\frac{1}{\sqrt{5}}$

$\therefore f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

$f_0 = 0$

$f_1 = 1$

$f_2 = \ldots$

$f_3 = \ldots$

~~10^3~~ 6

10^3 $2^{10} = 1024$

10^6 $\simeq 10^3$

10^9 $2^{60} = (10^3)^6$

10^{12} 2^{10} $\boxed{123,456,789}$

10^{15}

$10^{18} = ((10)^4)^4$

10^2 만

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$\boxed{1,2345,6789}$