1) 　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　1

4/6 Partial order and lattice

Equivalence relation $\Leftrightarrow$ partition

Theorem 1. Let $R \subseteq A \times A$ be an equivalence relation. (ref. symm. trans.) 　$[a]_R = \{b \mid aRb\}$

Following three statements are (logically) equivalent. 　$a \in [a]_R$ ($\because$ ref.)

i) aRb 　　ii) $[a]_R = [b]_R$ 　　iii) $[a]_R \cap [b]_R \neq \emptyset$.

proof) aRb $[a]_R = [b]_R$

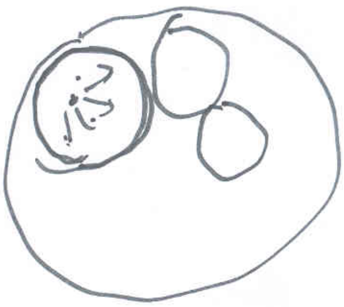
1) i) $\rightarrow$ ii)

$\forall c \in [a]_R$, $aRc \wedge aRb$, bRa ($\because$ ref) cRa (x), bRc (Tran), $c \in [b]_R$, $[a]_R \subseteq [b]_R$

$\forall c \in [b]_R$, bRc, aRb. aRc 　$c \in [a]_R$, $[b]_R \subseteq [a]_R$

2) ii) $\rightarrow$ iii) 　$[a]_R \cap [b]_R = [a]_R = [b]_R$, $a \in [a]_R$.

3) iii) $\rightarrow$ i)



if $a \not R b$, $[a]_R \cap [b]_R = \emptyset$

if aRb, $[a]_R = [b]_R \neq \emptyset$

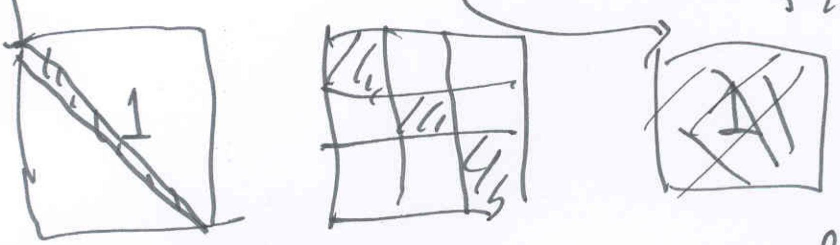
$\bigcup_{a \in A} [a]_R = A$ ($\because$ R is ref.)

$\therefore$ equiv rel. partition

$R \subseteq A \times A$ $= \{a_1 \cdots a_n\}$ 　partition 　singleton set.

equi rel. 　id_A $\longrightarrow$ $\{\{a_1\}, \{a_2\}, \cdots, \{a_n\}\}$

〃 　　$A \times A$ $\longrightarrow$ $\{\{a_1, a_2, \cdots, a_n\}\}$



$a_i \in A$

$a_i \not\subseteq A$

$k \leq n$

Partition of $A = \{A_1, A_2, \cdots, A_k\}$ 　　$A_i \not\subseteq A$

$A_i \subseteq A$

2/8.6 Partial Order

Def. Let $R \subseteq A \times A$, R ~~is~~ partial order

R : reflexive, ~~antisymmetric~~ asymmetric, transitive

(irreflexive) $(aRb \wedge bRa) \rightarrow (a=b)$

(A, R) partially ordered set or poset

$(\mathbb{Z}, \leq)$ $(\mathbb{Z}^+, |)$

$(\mathbb{Z}, <)$

↳ irreflexive partial order poset

$\forall a, b \in A$, aRb or bRa or $a \not R b$, $b \not R b$

aRb or bRa: comparable; $a \not R b$, $b \not R b$: incomparable

$3 \leq 5$.

$(\mathbb{Z}, \leq)$ $\forall a, b \in \mathbb{Z}$, aRb or bRa total order / linear order

... -4 -3 -2 -1 0 1 2 3 ...

But $S = \{a, b, c\}$ $\{a,c\} \not\subseteq \{b\}$, $\{a,c\} \not\supseteq \{b\}$

Def. Well ordered set $(S, \leq)$

1. total order
2. $\forall$ subset of S have a least element.

※ least element of $A \subseteq S$.

$a \in A$ is a least element, if $\forall b \in A$, $a \leq b$.

$\exists a \in A$, $\forall x \in S$, $a < x$. $P(a) \rightarrow P(x)$.