9 Graphs

9.1 Graphs and Graph Models

Definition *1 An* ***undirected graph****(****ugraph****,* ***simple graph****)* $G = (V, E)$ *where edge is a collection of* ***two elements subsets****(****unordered pair****) of V.*

$\{u, v\} \in E$ vs. $(u, v), (v, u)\} \in E.$

Edge {u, v} ***connectes*** *two* ***end points*** *u, v.*

Multigraph $G = (V, E, f)$

$f: E \to \mathbf{N}$

egde {u, v} of ***multiplicity*** *m,* $f(\{u, v\}) = m \in \mathbf{N}$.

Pseudograph $G = (V, E)$

Edge is a collection of ***one or two elements*** *subsets of V.*

$\{u, v\} \in E$ *connects u and v*

$\{u\} \in E$ *self-loop*

Definition *2 A* ***directed graph*****(*****digraph*****)** *G on the set of* ***vertices*** *V is, G = (V, E) where* $E \subseteq V \times V$. *A pair* $(u, v) \in E$ *is called* ***edge*****(*****arc*****)** *and said to* ***starts*** *at the* ***vertex*****(*****node*****)** *a and* ***ends*** *at the* ***vertex*** *b.*

Edge *is a binary* ***relation*** *on V.*

digraph and relation

Reflexivity	*All vertices have self-loop*
Irreflexivity	*No verteces have self-loop*
Symmetry	*All edges are bidirectional*
Antisymmetry	*All edges are unidirectional, self-loop is allowded*
Asymetry	*All edges are unidirectional, self-loop is* ***not*** *allowded*
Transitivity	*All paths should be an edge*

Definition *2.1 Let G = (V, E) be a digraph. Then a* ***path*** *is defined as a* ***sequence*** *of vertices* $a_o, a_1, \ldots, a_n$ *with* $n \geq 0$ *such that*

$(a_i, a_{i+1}) \in E, 0 \leq \forall i < n$. *The path* ***starts*** a_o *and* ***ends*** a_n.

The ***length*** *of the path is defined to be n.*

$a \in V$ *path of length 0*

A path is a ***cycle*** *if* $a_o = a_n$.

$a\, E^*\, b$ *there is a path from a to b*

$a\, E^+\, b$ *there is a positive length path from a to b*

Definition *2.2 A* ***cycle*** *is a positive lengh path that begins and ends* ***same*** *vertex. A* ***directed acyclic graph***(DAG) *is a directed graph with* ***no cycle.***

Lemma *If G is a DAG,* G^+ *is a* ***irreflexive partial order***.

9.2 Graph Terminology and Special Types of Graphs

Consider simple graph(ugraph)

Definition *1* ***Adjacency***

edge {u, v} ∈ E

*u, v are **adjacent**(**neighbors**, **connected**)*

*Edge {u, v} **connects**(is **incedent** with) vertices u and v.*

*Vertices u and v are **endopints** of edge {u, v}.*

Definition *2* ***Degree*** *of Vertex, v ∈ V, deg(v) ∈* **N**.

the number edges incedent with it(except self-loop counts twice)

*If deg(v) = 0, v is called **isolated**.*

*If deg(v) = 1, v is called **pendant**.*

Theorem *1 The* ***Handshaking Theorem***

$$\sum_{v \in V} deg(v) = 2|E|.$$

proof *Every edge contibutes two to the sum of the degrees.*

Theorem *2 Any undirected graph has* ***even*** *number of* ***vertices*** *of* ***odd*** *degree.*

Some Special Simple Graphs

Complete *graph, for* $n \geq 1$, *n vertices* K_n, *has*

an edge between ***every two vertices****.* $n(n-1)/2$ *edges*

Empty *graph has* ***no edge*** *at all.*

Line, L_n: $n+1$ *vertices and n edges. A* ***path*** *of length n.*

Cycle, *for* $n \geq 3$, C_n: *n vertices and n edges. A* ***cycle*** *of length n.*

Wheel, *for* $n \geq 3$, W_n: C_n *and n edges from a* ***hub*** *vertex and to* C_n.

$n+1$ *vertices,* $2n$ *edges.*

n-cube, *For* $n \geq 0$, Q_n, *with* 2^n *vertices and .*

$Q_0 = (\{v_0\}, \{\})$.

Let $Q_n = (V, E)$ *where* $V = \{v_0, v_1, \ldots v_k\}$ *where* $k = 2^n - 1$ *and*

$Q_n' = (V', E')$ *where* $V' = \{u_0, u_1, \ldots u_k\}$ *and* $V \cap V' = \emptyset$. *Then*

$Q_{n+1} = (V \cup V', E \cup E' \cup \{\{v_0, u_0\}, \{v_1, u_1\}, \ldots, \{v_k, u_k\}\})$

Bipartite Graph

$G = (V, E)$ *is bipartite, iff* $V = V_1 \cup V_2$ *where* $V_1 \cap V_2 = \emptyset$ *and*

$\forall \{v_1, v_2\} \in E, \exists v_1 \in V_1, v_2 \in V_2$.

Partition $\{V_1, V_2\}$ *is called* ***bipartition***.

Complete Bipartite Graph, $K_{n+m} = (V_1 \cup V_2, E)$

where $|V_1| = n, |V_2| = m$, *and* $E = \{\{v_1, v_2\} | v_1 \in V_1, v_2 \in V_2\}$

Definition *6 A* ***subgraph*** *of graph G = (V, E) is H = (V', E') where* $V' \subseteq V$ *and* $E' \subseteq E$.

Let G' = (V', E') is a subgraph of G = (V, E). Then G is called ***spanning subgraph*** *of G, if V' = V.*

Let G' = (V', E') is a subgraph of G = (V, E). Then G" = (V", E") is called the ***complement*** *of G' with respect to G where E" = E – E', and* $V'' = \{a \in V \mid \{a, b\} \in E''\}$

The ***complement*** *of ugraph G = (V, E) with respect to* K_n, *denoted* $\overline{G}$, *is the (**absolute**)* ***complement*** *of G.*

Definition *7 The union of two graph* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *is* $G = (V_1 \cup V_2, E_1 \cup E_2)$.

9.3 Representing Graphs and Graph Isomorphism

Isomorphism of Graphs

Definition 1 The graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**, iff $\exists$ a bijection .э. $\forall u \forall v \in V_1$: $\{u, v\} \in E_1$ iff $\{f(u), f(v)\} \in E_2$.

n-cube, *revisited.*

$Q_0 = (\{v_0\}, \{\})$.

Let $Q_n = (V, E)$, bijection $f: V \rightarrow V'$ where $V \cap V' = \varnothing$. Then

$Q_n' = (\{f(v) \in V' | v \in V\}, \{\{f(v), f(u)\} | \{v, u\} \in E\})$

We may extend f domain and codomain of f from vertices to edges.

$f(\{v, u\}) = \{f(v), f(u)\}$.

$Q_n' = (f(V), f(E))$

$Q_{n+1} = (V \cup f(V), E \cup f(E) \cup \{\{v, f(v)\} | v \in V\})$.

$= (V \cup f(V), E \cup f(E) \cup \{V, f(V)\}$.

Example 8, 9, 10

Necessary *conditions for graph* ***isomophisms***

$|V_1| = |V_2|$ *and* $|E_1| = |E_2|$.

The number of vertices with degree n is same in both graphs.

Subgraph H of one graph is isomorphic to a subgraph of the other

isomorphic simple cycle

9.4 Connectivity

Definition *1 Let G = (V, E) be a ugraph. Then a* ***path*** *is from u to v is defined as a* ***sequence*** *of vertices* $(v_o, v_1, \ldots, v_n)$ *with* $n \geq 0$ *such that*

$\{v_i, v_{i+1}\} \in E, 0 \leq \forall i < n, v_o = u, v_n = v.$

The ***length*** *of the path is defined to be* $n(\geq 0)$*.(number of edges)*

Note that u is a path of length 0 but not an edge.

A path is a ***cycle*** *if* $a_o = a_n$, $n \geq 1$.

A path ***traverses*** *the vertices along it.*

A path is ***simple***, *if it contains no* ***vertex*** *more than once.*

A graph, C, is ***simple cycle*** *of* ***length*** *n, iff it is isomorphic to* C_n *for some* $n \geq 3$.

Definition *3 Two vertices u and v are to said to be* ***connected****,*
if there is a path between them. A graph is said to be ***connected****,*
if there ***every*** *pair of vertices are connected.*
We write u connected v, if u and v are connected.

$$connected \subseteq V \times V \qquad \text{binary relation on } V$$

Lemma *0.1 The binary relation* ***connected*** *on V*
is an ***equivalence*** *relation*

Definition *3.2 The* ***quivalence class*** *defined by the equivalence relation*
connected *is called* ***connected component.***

$P = \{[v]_{connected} | \, v \in V\}$ ***partition*** *of V.*

$|P|$ = *number of conncted components*

Definition *3.3 Two vertices in a graph are* ***k-connected*** *if they* ***remain connected*** *in any subgraph by* ***deleting*** *k–1 edges.*

A graph is ***k-connected*** *if every pair of vertices are k-connected.*

simple cycle	*2-connected*
K_n	*(k–1)-connected*

Theorem *1 If vertex v is connected to vertex u in a graph, there is a* ***simple path*** *from u to v.*

proof *Consider* ***minimum*** *length path from u to v.*

$v_o, v_1, \ldots, v_k$ *where* $v_o=u$, $v_k=v$ *with* $k \geq 2$*(if* $k \leq 1$*, simple) is simple.*

Assume it is not simple, $0 \leq \exists i < \exists j < n$ *.϶.* $v_i = v_j$*. Not minimum path!*

Corollary *1.1 For any path of* ***length*** *k in graph, there is a* ***simple*** *path of length at* ***most*** *k with the same endpoints.*

Theorem *1.1 Every graph G = (V, E) has*
***at least** $|V| - |E|$ **connected components**.*

proof *P(n): G = (V, E) with $|E| = n$ has at least $|V| - n$ C. C.*
***base**: $|E| = 0$, $|V|$ connected components.*
***induction**: Consider G = (V, E) with n+1 edges.*

- ***Remove** an edge {u, v} and call the resulting graph G'.*
- *G' has at least $|V| - n$ connected components.(I. H.)*
- ***Add back** the edge {u, v}.*
- *case 1: u and v are in the **same** C. C. **Same** number of components*
  - *G has at least $|V| - n > |V| - (n+1)$ components.*
- *case 2: u and v are in the **diffrent** C. C. One **less** component*
  - *G has at least $|V| - n - 1 = |V| - (n+1)$ components.*

***Corollary**1.2 The **connected** graph with n vertices has **at least** n-1 edges.*

proof $1 \geq |V| - |E|$ $|E| \geq |V| - 1$

10.1 Introduction Trees

Definition *10.1 A **tree** is an **acyclic connected** graph.*

*A vertex of **degree one** is called **leaf**.*

*A **forest** is a set of trees.*

Theorem *Every **tree** has following properties:*

*1. Any **connected subgraph** is a **tree**.*

***2**. There is a **unique (simple) path** between **every pair of vetices**.*

*3. **Adding** an **edge** between two vertices create a **(simple) cycle.***

*4. **Removing** any **edge disconnects** the tree.*

*5. If tree has at **least two vertices**, then it has **at least two leaves**.*

***6**. The number of **vertices** is **one larger** than that of **edges**.*

proof *1. Any subgraph of acyclic graph subgraph is also **acyclic.***

*2. There is **at least one** (simple) path, since connected and acyclic.*

*Assume **two different** simple paths from u to v.*

Assume x be the first vertex where the path ***diverge****,*

y be the next vertex they ***share****.*

There is a ***cycle*** *from x to y and then y to x.*

3. *Additional edge {u, v} ∪ (simple)* ***path*** *from u to v = (simple)* ***cycle***
4. *Remove {u, v}.* ***Unique*** *simple path was (u, v). ∴Not connected*
5. *Let* $(v_1, \ldots, v_m)$ *be the* ***longest*** *simple path in the tree. Then* $m \geq 2$.

   $2 < \forall i \leq m, \{v_1, v_i\} \notin E$, *since* $(v_1, \ldots, v_i, v_1)$ *is a* ***cycle****.*

   $\{u, v_1\} \notin E$ *where u is not in the path, since* $(v_1, \ldots, v_m)$ *is the* ***longest****.*

   ∴ v_1 *is a* ***leaf****. By* ***symmetric*** *argumant* v_m *is a* ***second*** *leaf.*
6. *Induction on number of vertices.*

   n=1, no edge. 0 + 1 = 1. O.K.

   Consider (n+1)-vertex tree T and let v be a ***leaf*** *of T.*

   Deleting v and its incident edge gives a smaller ***tree****.*

   $(|E| - 1) = (|V| - 1) + 1$

   Adding v and its incident edge gives a larger ***tree****.* ∴ $|E| = |V| + 1$.

Theorem *Every* ***connected graph*** *has* ***spanning tree****.*

proof *Let T be a* ***connected spanning*** *subgraph of G with the* ***smallest*** *number of* ***edges****.*

Suppose T has a cycle $(v_0, v_1, \ldots, v_n, v_0)$.

Suppose we remove the edge $\{v_n, v_0\}$.

If arbitrary vertices x and y has a ***path not*** *containing the edge* $\{v_n, v_0\}$,

x and y has a path containing ***that path****.*

If arbitrary vertices x and y has a path ***containing*** *the edge* $\{v_n, v_0\}$,

x and y has a path containing the ***path*** $(v_0, v_1, \ldots, v_n)$.

This is a ***contradiction*** *that T has the* ***smallest*** *number of edges and* ***connected****.*

$\therefore$ *T is acyclic.*

T is a tree.

9.8 Graph Coloring

Definition *1 A graph G is k-**colorable**, if each vertex can be assigned one of k colors so that **adjacent vertices** get the **different colors**.*

map coloring problem

*The smallest number of colors are called **chromatic number** of G.*

$\chi(G)$, $\chi(K_n) = n$.

Theorem *1 The **chromatic** number of **planar** graph is no greater than 4.*

Theorem *2 A graph with maximum degree at most k is (k+1)-colorable.*

proof *Induction on number of **vertices**. P(n).*

basis *1-vertex graph, maximum degree 0 and 1-colorable. P(1) is true.*

induction *Let G be (n+1)-**vertex** graph with maximum degree at most k*

Remove *a vertex v and its incident edges.*

G' has n vertices and max. deg. at most k. ∴ G' is (k+1)-colorable(IH).

v has at most k adjacent vertices. ∴ G is (k+1)-colorable((k+1) – k = 1).

Bipartite graph

Every ***bipartite*** *graph is 2-colorable.*
Every ***path***, ***tree***, *and* ***even length cycles*** *are bipartite.*

Theorem *A graph is* ***bipartite*** *if and only if*
it contains ***no odd lengh cycle***.

9.7 Planar Graph

Definition *1 A graph is* ***planar****, if it can be drawn in the* ***plane without edge crossing****.*

K_4 is planar and Q_3 is planar, but $K_{3,3}$ is not planar.

Region*: planar graph divides regions*

Tree *has* ***one*** *region*

Removing an edge *on the* ***cycle*** *merges* ***two*** *regions into* ***one****.*

Theorem *1 Let G be a* ***connected planar graph*** *with v* ***vertices*** *and e* ***edges****. Let r be the number of* ***regions*** *in planar reprsentation of G. Then*

$$r = e - v + 2.$$

proof *Induction on number of edges, P(e).*

basis *e=0, v=1, r=1.* $\therefore$ *1 = 0 – 1 + 2 = 1. O.K.*

induction *Consider a connected planar graph G with e+1 edges.*

1. If G is ***acyclic****. G is* ***tree****.* $\therefore$ *r = 1, e – v + 2 = –1 + 2 =1.* $\therefore$ *O.K.*

2. If G is not acyclic, G has at least one ***cycle****, C.*

Consider {u, v} ***in*** *C and a* ***spanning tree****, T. .э. {u, v} is* ***not*** *in T.*

There exisits such an edge {u, v} because T is acyclic.

Remove *the* ***edge*** *{u, v} from G, it is called as G'.*

G' ***has one less regions****, since removing an edge {u, v} on the cycle C.*

G' is connected planar graph and has e edges.

$\therefore$ *r = e – v + 2 in G'* *induction hypothesis*

In G, r+1 regions, e+1 edges, and v vertices. $\therefore$ *(r +1) = (e+1) – v + 2.*

Corollary *1 If connected planar graph G with e edges and v vertices where* $v \geq 3$*. Then* $e \leq 3v - 6$.

proof *Consider degree of region number of boundary edges*
Sum of the degree of regions is 2e.
Minimum number of degree of each region r is 3r(triangle)

$\therefore\ 2e \geq 3r = 3(e - v + 2)$ $\therefore\ e \leq 3v - 6$.

Corollary *2 If G is a connected planar graph, then G has a vertex not exceeding five.*

proof *If G has one or two vertices, the result is true.*
If G has more that three vertices, $e \leq 3v - 6$*. So* $2e \leq 6v - 12$.
If every vertex has more than or equal 6, $2e \geq 6v$*(Handshaking Theorem)*
But it contradict with $2e \leq 6v - 12$.

Example *5* K_5 *is not planar.*
$v = 5, e = 10. 3v - 6 = 9.$

9.5 Euler and Hamilton Path

***Definition** 1 Let G be a graph.*
*An **Euler path** is a path containing **every edge** of G.*
*An **Euler circuit** is a circuit containing **every edge** of G.*

***Theorem** 1 A connected graph has **Euler circuit** iff each vertex has **even degree**.*
***proof** (→) The circuit contributes 2 to the degree of each node*
(←) Algorithm 1

***Algorithm** 1 Constructing Euler Circuit*
*Begins with **arbitrary** node.*
*Construct a **cycle** form the vertex to the vertex.*
*Repeat for each **remaining** subgraph,*
***inserting** the **new cycle** into the **original** one.*

Theorem *2 A connected graph has* ***Euler path*** *iff it has exaclty* ***two*** *verti-ces of* ***odd degree****.*
proof *One is the start, the other is the end.*

Definition *2 Let G be a graph.*
An ***Hamilton path*** *is a path traverse each vertex ib G exaclty* ***once****.*
An ***Hamilton circuit*** *s a circuit traverse each vertex ib G exaclty* ***once****.*

K_n *has a Hamilton circuit circuit*($n \geq 3$)*.*

If a graph has a vertex of ***degree one****, there is* ***no Hamilon*** *circuit.*
Exactly ***two edges*** *incident to a vertex are in Hamaiton circuit.*

Trvaling Saleman
NP complete

9.6 Shortest-Path Problem

*Let $G = (V, E)$ be a graph, and $f: E \rightarrow \mathbf{R}^+$ be a **cost** of the edges.*

*Edge **weighted** graph*

Iterative** and **recursive extended** definition of weight of **path.

Let $(v_0, v_1, \ldots, v_n) \in E^n (= V^{n+1})$ be a path of length $n \geq 0$. Then we define

if $n = 0$, $f^*(v_0, v_0) = 0$.

if $n \geq 1$, $f^*(v_0, v_n) = \sum_{j=0}^{n-1} f(v_j, v_{j+1})$

$f^*(u, u) = 0.$

$f^*(u, v) = f(u, x) + f^*(x, v)$ *or* $f^*(u, x) + f(x, v).$

We may use f instead of f^*, since $f \subseteq f^*$.

$f: E^* \rightarrow \mathbf{R}^+$ ***extend** the domain of f from E to E^*.*

Shortest path problem

Definition $min_f(u, v) = (v_0, v_1, \ldots, v_n)$ $n \geq 0$.϶. $m \geq 0$

$$f(v_0, v_1, \ldots, v_n) \leq f(u_0, u_1, \ldots, u_m) \text{ .϶. } \forall (u_0, u_1, \ldots, u_m) \in V^*$$

$$\text{where } v_0 = u = u_0 \text{ and } v_n = v = u_m.$$

$u, v \in V$, find a shortest path such that $f(u, v)$ is minimum.

Definition *Let $W \subseteq V$, $u, v \in V$. $L_f^W(u, v) = (u, w_1, \ldots w_n, v)$.϶.*

$$f(u, w_1, \ldots w_n, v) \leq f(u, x_1, \ldots, x_m, v) \text{ .϶. } \forall (u, x_1, \ldots, x_m, v) \in V^*$$

$$n, m \geq 0,\ 1 \leq \forall i \leq n,\ w_i \in W,\ 1 \leq \forall i \leq m,\ x_i \in W.$$

Theorem *Let $W \subseteq V$, $u \in V$, and $x \notin W$.϶. $L_f^W(u, x)$ is the* **minimum**.

$$\forall y \notin W,\ L_f^{W \cup \{x\}}(u, y) = \min(L_f^W(u, y), L_f^W(u, x) + f(x, y)).$$

procedure *Shortest path*$(G = (V, E), f: E \to \mathbf{R}^+, u, v \in V)$

for $w \in V$ ***do*** $L(w) :=$ *Infinite* ***od***; $L(u), W := 0, \emptyset$;

Initialization $\forall y \in V, L(y) = L_f^{\{\}}(u, y)$.

do $v \notin W \to$

Loop invariance $\forall y \in V, L(y) = L_f^{W}(u, y)$.

$x := x \notin W \wedge min(L(x)); W := W \cup \{x\};$

Loop invariance, invalidated $\forall y \in V, L(y) = L_f^{W-\{x\}}(u, y)$.

do $y \notin W \wedge (x, y) \in E \to L(y) := min(L(y), L(x) + f(x, y))$ ***od***

od

After the loop termination $\forall y \in V, L(y) = L_f^{W}(u, y) \wedge v \in W$.

We should also prove that $L_f^{W}(u, v) = min_f(u, v)$, *if* $v \in W$.

But it is trivial, since we add the shortest path vertex x to W.