8 Relations

8.1 Relations and Their Properties

Definition 1 Let A and B be two set.

A **binary relation** R from A to B is subset of $A \times B$.

$R \subseteq A \times B$.

A: **domain** of the relation R.

B: **range(codomain)** of the relation R.

Let $a \in A$, $b \in B$, Then $(a, b) \in R$ or $(a, b) \notin R$.

If $(a, b) \in R$, we also write a R b and we say a is **related to** b by R.

If $(a, b) \notin R$, we also write $a \not R b$ and a is **not related to** b by R.

Two notations

$(a, b) \in R$ relation R is a **set**

$R \subseteq A \times B$

a R b relation R is a infix **boolean** binary **operator**

$R: A \times B \to \{true, false\}$

Function as Relation

$f: A \to B$ vs $R \subseteq A \times B$

$f(a) = b$ vs $R(a) = \{b_1, b_2, \ldots, b_n\}$

Function f is a special kind of relation

$\forall a \in A \exists_1 b \in B. \quad \therefore f(a) = \{b\}$

Some Relation R may be a function.

$\forall a \in A \exists_1 b \in B.(a, b) \in R.$

Function is a relation

you can write $(a, b) \in f$ *or* $a\, f\, b$ *insteand of* $f(a) = b$

Let $f: A_1 \times A_2 \times \ldots \times A_n \to B_1 \times B_2 \times \ldots \times B_m$.

$((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_m)) \in f$

$(a_1, a_2, \ldots, a_n)\, f\, (b_1, b_2, \ldots, b_m)$

$f((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_m)$ or

$f(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m)$.

Relation on a Set

Definition *2 Let A be a set and* $R \subseteq A \times A$. *R is called a relation* ***on*** *A.*

Relation on A is a ***directed graph*** *on* ***vertices*** *A and* ***edges*** *R.*

Properties of Relations

Definition *3 A relation R is* ***reflexive****, if* $\forall a \in A$, $a\, R\, a$.

A relation R is ***irreflexive****, if* $\forall a \in A$, $a \not R\, a$.

A relation is ***not both*** *reflexive and irreflexive.(disjoint)*

Definition *4 A relation R is* ***symmetric****, if* $a\, R\, b \Rightarrow b\, R\, a$.

A relation R is ***asymmetric****, if* $a\, R\, b \Rightarrow b \not R\, a$.

A relation R is ***antisymmetric****, if* $(a\, R\, b \wedge b\, R\, a) \Rightarrow (a = b)$.

or if $(a\, R\, b \wedge a \neq b) \Rightarrow b \not R\, a$.

If a relation is a ***asymmetric*** *then it is also* ***antisymmetric****.(subset)*

Definition *5 A relation R is* ***transitive****, if* $a\, R\, b \wedge b\, R\, c \Rightarrow a\, R\, c$.

Combining Relations

Let $R_1, R_2 \subseteq A \times B$. Consider $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$.

***Definition** 6 Let $R \subseteq A \times B$, $S \subseteq B \times C$. Then **composition** of R and S, denoted as $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.*

***Definition** 7 Let $R \subseteq A \times A$. Then for $n \in \mathbf{N}^+$,*

$R^1 = R$ ***basis***

$R^{n+1} = R^n \circ R$. ***induction***

***Definition** 6.5 Let A be a set. We define **identity** relation*

$id_A = \{(a, a) \in A \times A \mid a \in A\}$

***Colorally** 0.5 Let $R \subseteq A \times B$. Then*

*$R \circ id_A = id_A \circ R = R$. id_A is a **identity** element for **composition**.*

***Definition** 7.5 Let $R \subseteq A \times A$. Then for $n \in \mathbf{N}$,*

$R^0 = id_A$ ***basis*** $(x^0 = 1)$

$R^{n+1} = R^n \circ R$. ***induction***

Theorem *1 Let* $R \subseteq A \times A$. *R is* ***transitive***, *if and only if,* $R^n \subseteq R$ *for* $\forall n \in \mathbf{N}^+$.

Proof:

1. (if) $R^n \subseteq R$ for $\forall n \in \mathbf{N}^+ \rightarrow R$ is transitive.

$\forall (a, b) \in R, \forall (b, c) \in R, (a, c) \in R^2$.

Since $R^2 \subseteq R$, $(a, c) \in R$.

$\therefore$ R is transitive.

2.(only if) R is transitive $\rightarrow R^n \subseteq R$ for $\forall n \in \mathbf{N}^+$.

basis Trivial for $n = 1$.

induction Assume $R^n \subseteq R$ and R is ***transitive*** for some $n \in \mathbf{N}^+$.

$\forall (a, b) \in R^{n+1}, \exists c \in A$.э. $(a, c) \in R \wedge (c, b) \in R^n$. $(R^{n+1} = R^n \circ R)$

Since $R^n \subseteq R$, $(c, b) \in R$, and since R is transitive, $(a, b) \in R$.

$\therefore R^{n+1} \subseteq R$.

8.2 n-ary Relations and Their Applications

Definition *1 Let $A_1, A_2, \ldots, A_n$ be sets.$R \subseteq A_1 \times A_2 \times \ldots \times A_n$ is a n-ary relation on Let $A_1, A_2, \ldots, A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the* ***domain*** *of the relations, and n is called its* ***degree****.*

8.3 Representing Relations

$R: A \times B \rightarrow \{0, 1\}$

boolean matrix

$R \subseteq A \times A.$

Definition *1 A directed graph, digraph G = (V, E) consistes of a set V of **vertices**, and a set* $E \subseteq V \times V$ *of **edges(arcs)**.*

8.4 Closure of Relations

Let $R \subseteq A \times A$. R may or may not have some property **P** *={reflextive, symmetric, transitive}. If $\forall T \subseteq A \times A$ with property* **p** $\in$ **P** *and $R \subseteq T$, $S \subseteq T$, then S is called the* **p *closue*** *of R.*

Reflexive *closure of R*

$R \cup id_A$.

Symetric *closure of R*

$R \cup R^{-1}$.

Definition *1 A* ***path*** *from a to b in the directed grapg G = (V, E).*

$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n) \in E$, *and* $x_0 = a$, $x_n = b$.

The path is denoted by $(x_0, x_1, x_2, \ldots, x_n)$ and has ***length*** *n.*

A path of length $n \geq 1$ and

begins and ends at the same vertex is called ***cycle.***

Theorem *1 Let* $R \subseteq A \times A$*. There is a path of length* $n \geq 1$ *from a to b, if and only if,* $(a, b) \in R^n$.
proof *Easy for induction*

Definition *1 A* ***connetivity*** *relation* $R^+ = \{(a, b) \mid (a, b) \in R^n, \forall n \geq 1\}$

$R^+ = \cup_{i \in \mathbf{N}+} R^i = R^1 \cup R^2 \cup \ldots$

transitive *closure of R.*

$R^* = \cup_{i \in \mathbf{N}} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots$

reflexive and transitive *closure of R.*

Warshall's algorithm $O(n^3)$
Depth first search $O(n^2)$

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Algorithm Depth first search
S: stack of Vertex; n(Vertex) array of Depth;
procedure Traverse(x: Vertex; d: Depth);
    push x onto S; n(x) := d;                     R*(x) := {x};
    for y ∈ Vertex where x R y do
        if n(y) = 0 then Traverse(y, d+1) fi;
        n(x) := min(n(x), n(y));                  R*(x) := R*(x) ∪ R*(y)
        od;
    if n(x) = d then repeat
        y = pop of S; n(y) := infinite;           R*(y) := R*(x)
        until  y = x
    fi
end procedure Traverse
for x ∈ Vertex do n(x) := 0 od;
for x ∈ Vertex where n(x) = 0 do Traverse(x, 1) od
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8.5 Equivalence Relations

Definition *1 Let $R \subseteq A \times A$. R is called **equivalence** relation,*
if it is reflexive, symmetric, and transitive.

Definition *3 Let $R \subseteq A \times A$ be an **quivalence** relation.*
*$[a]_R = \{b |\ a\ R\ b\}$ is called the **quivalence class** of a w.r.t. R.*

*If $b \in [a]_R$, b is called the **representative** of the equivalent class.*

*Note that $a \in [a]_R$, since R is **reflexive**.*

Theorem *1 Let $R \subseteq A \times A$ be an **quivalence** relation. Three statments are logically equivalent*

i) $a\ R\ b$ ii) $[a]_R = [b]_R$ iii) $[a]_R \cap [b]_R \neq \emptyset$.

proof

1) i) $\rightarrow$ ii)

$\forall c \in [a]_R$, $a\ R\ c$, $a\ R\ b$, $b\ R\ a$. $\therefore$ $b\ R\ c$, $c \in [b]_R$. $\therefore$ $[a]_R \subseteq [b]_R$.

$\forall c \in [b]_R$, $b\ R\ c$, $a\ R\ b$, $\therefore a\ R\ c$, $c \in [a]_R$. $\therefore [b]_R \subseteq [a]_R$.

2) ii) $\rightarrow$ iii)

Assume $[a]_R = [b]_R$. $[a]_R \cap [b]_R \neq \emptyset$, since $a \in [a]_R$ is not empty.

3) iii) $\rightarrow$ i)

Suppose $[a]_R \cap [b]_R \neq \emptyset$, $[a]_R \neq \emptyset$, and $[b]_R \neq \emptyset$.

$\exists c \in [a]_R \wedge c \in [b]_R$. $a\ R\ c$, $b\ R\ c$. $c\ R\ b$, $\therefore a\ R\ b$.

Lemma *1.5 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation.*

$\cup_{a \in A} [a]_R = A$. $a \in [a]_R$.

$\therefore$ $\{[a]_R \subseteq A \mid a \in A\}$ is a **partition** of A.

Definition *1.5 Let S be a set. The* ***partition*** *of S, $\{A_i \mid i \in I\}$ I: index set, is*

i) $A_i \neq \emptyset$, $i \in I$. ***nonempty***

ii) $A_i \cap A_j = \emptyset$, when $i \neq j$. ***disjoint***

iii) $\cup_{i \in I} A_i = A$. ***exhaustive***

Theorem *2 Let R be a equivalent relation on A.*
Then the ***equivalent class*** *of R form a* ***partion*** *of A.*
Conversely, given a ***partition*** $\{A_i \mid i \in I\}$ *of the set A,*
there is an ***equivalent relation*** *R that has the set* A_i, $i \in I$,
as its ***equivalent class****.*

Realtion $R \subseteq A \times A$ $O(n^2)$ *where* $|A| = n$.
Equivalent *relation* $R \subseteq A \times A$ $O(n)$

8.6 Partial Ordering

Definition *1 Let $R \subseteq A \times A$. R is called* ***(ir)reflexive partial order****,*
if it is ***(ir)reflexive****,* ***antisymmetric****, and* ***transitive****.*
(A, R) is called ***partially odreded*** *set or* ***poset****.*

Example *1 2 3 $(\mathbf{Z}, \leq)$, $(\mathbf{Z}^+, |)$, $(2^S, \subseteq)$ are posets.*

Definition *2 Let $(A, \leq)$ be poset and $a, b \in A$. Then*
The elements a and b are ***comparable*** *if either $a \leq b$ or $b \leq a$.*
The elements a and b are ***incomparable*** *if* ***neither*** *$a \leq b$* ***nor*** *$b \leq a$.*

Definition *3 Let $(S, \leq)$ be poset. If $\forall a, b \in S$, a and b are* ***comparable****,*
S is called ***totally ordered set****,* ***linearly ordered set****, or* ***chain****.*
$\leq$ is called ***total order*** *or* ***linear order****.*

Definition *4 (S, ≤) is a* ***well-ordered*** *set, if it is a* ***poset****, ≤ is a* ***total order****, and every nonempty subset of S has a* ***least*** *emelent.*

Theorem *1* ***Principle of Well-Ordered Induction***

Let (S, ≤) be a well-ordered set. $\forall x \in S, P(x)$, *if*

$\forall y \in S, \forall x \in S$.э. $x < y$: $P(x) \rightarrow P(y)$.

proof *Suppose* $\exists y \in S, \neg P(y)$.

$A = \{x \in S \mid \neg P(x)\} \neq \emptyset$.

Let $a \in A$ *be the least element.*

$\exists a \in A, \forall x \in S$.э. $x < a$: $P(x) \rightarrow P(a)$.

Contradiction $\exists y \in S, \neg P(y)$.

Lexicographic order

Let $(A_1, \leq_1)$ *and* $(A_2, \leq_2)$ *be* ***well-ordered*** *sets.*

We define $(A_1 \times A_2, \leq)$ *lexicographic order*

$(a_1, a_2) \leq (b_1, b_2)$, *if* $(a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2))$

Hasse Diagram

Let (A, ≤) be a poset. We say a **covers** *b, if* $a < b \wedge \neg\exists c \in A, a \leq c \wedge c \leq b$.
(A, covers) is a Hasse Diagram.

Maximal and Minimal Elements

Let (A, ≤) be a poset.
We say a is a **maximal**, *if* $\neg\exists b \in A, a < b$.
We say a is a **minimal**, *if* $\neg\exists b \in A, b < a$.
We say a is a **greatest element**, *if* $\forall b \in A, b \leq a$.
We say a is a **least elements**, *if* $\forall b \in A, a \leq b$.
Let $S \subseteq A$. $u \in A$ *is called* **upper bound** *of S, if* $\forall a \in S, a \leq u$.
$l \in A$ *is called* **lower bound** *of S, if* $\forall a \in S, l \leq a$.
$x \in A$ *is called the* **least upper bound** *of S, if* $\forall a \in S, a \leq x$,
$x \leq z$, *if* $\forall z \in$ *upper bound of S.*
$y \in A$ *is called the* **greatest lower bound** *of S, if* $\forall a \in S, y \leq a$,
$z \leq y$, *if* $\forall z \in$ *lower bound of S.*

Lattice

A poset (A, ≤) is called ***lattice****. If every pair of elements has both a least upper bound, and a greatest lower bound.*

($\mathbf{Z}$, ≤) is a lattice. *max, min:* $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$

($\mathbf{Z}^+$, |) is a lattice. *lcm, gcd:* $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$

(2^S, ⊆) is a lattice. $\cup, \cap$: $2^S \times 2^S \rightarrow 2^S$

Topological Sorting

Lemma *1 Every* ***finite nonempty*** *poset (A, ≤) has* ***at least one minimal*** *elements.*

Algorithm *1* ***Topological sorting***

Construct a total ordering $<_t$ *from a partial ordering ≤.*

$a <_t b$*, if and only if, a ≤b or a and b are* ***incomparable****.*