6 Discrete Probability

6.1 An Introduction to Discrete Probability

Experiment *a procedure that yields one of the given set of* ***outcomes***

Sample space *of an experiment universe of possible* ***outcoms***

Event *a subset of the sample space*

Definition *1 If S is a* ***finite*** *sample spcae of equally likely outcomes and E* ($\subseteq$ *S) is an* ***event****, the* ***probability*** *of E is,* $p(E) = |E|/|S|$.

We say the event E ***occurs*** *when* ***actual outcome*** *e is in E,* $e \in E$.

Theorem *1 Let E be an event in a sample space S. The probability of the event* $\overline{E}$, *the* ***complementary*** *event of E, is*

$$p(\overline{E}) = 1 - p(E).$$

Theorem *2 Let* E_1 *and* E_2 *be events in a sample space S. Then*

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

6.2 Probability Theory

Let S be **countable** *sample space and* $s \in S$ *be an outcome,* $p(s)$ *be the probability of the outcome.*

i) $\forall s \in S, 0 \le p(s) \le 1$.

ii) $\Sigma_{s \in S}\, p(s) = 1$.

Probability distribution *The function* $p: S \rightarrow [0, 1]$

Definition *1 Let S be a set with n elements. The* **uniform(random) distribution** *assigns the probability 1/n to each element of S.*

Definition *2 Let E be an event. Then*

$$p(E) = \Sigma_{s \in E}\, p(s).$$

Theorem *1 It* $E_1, E_2, \ldots$ *is a sequence of* **pairwise disjoint** *events,*

$$p(\cup_i E_i) = \Sigma_i\, p(E_i).$$

Definition *3 Let E and F be events with p(F) > 0. The* **conditional probability** *of E given F is*

$$p(E \mid F) = p(E \cap F)/p(F).$$

Definition *4 The event E and F* **independent** *if and only if*

$$p(E \cap F) = p(E)p(F).$$

Bernoulli Trial and the Binomial distribution

expereiments with two possible outcomes {***success***, ***failure***}

$$p + q = 1$$

Theorem *2 The probability of k successes in n* **indepedent** *Bernoulli trials, with probability of succee p and probability of failure q = 1 - p, is*

$$C(n, k)\, p^k q^{n-k}.$$

Random Variables

Definition *5 Let S be a sample space S and A be a set of values of random variable.* ***Random variable*** *X is a function from set of sample space(events) to some set of value A.*

$X: S \rightarrow A.$

Definition *6 The* ***distribution*** *of random variable X is a pair (s, p(X=s))*

$\forall s \in S$, *p(X=s) is the probability that X takes the value s.*

We write p(X=s) instead of pair(s, p(X=s)).

6.3 Bayes' Theorem

Theorem *1* ***Baye's Theorem*** *Suppose E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$.*

$$p(F \mid E) = p(E \mid F)p(F) / (p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F}))$$

$$= p(E \mid F)p(F) / p(E)$$

$$p(F \mid E)p(E) = p(E \mid F)p(F).$$

proof

$$p(F \mid E) = p(E \cap F)/p(E).$$

$$p(E \mid F) = p(E \cap F)/p(F).$$

$$p(F \mid E)p(E) = p(E \mid F)p(F) \underline{=} p(\underline{E} \cap F).$$

$$p(E) = p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})$$

6.4 Expected Value and Variance

Definition *1 The* ***expected value****(or* ***expactation****) of the random variable X(s) on the sample space S is equal to*

$$E(X) = \Sigma_{s\in S}\, p(s)X(s).$$

Theorem *1 The* ***expected value*** *of the random variable p(X=s), then*

$$E(X) = \Sigma_{s\in S}\, p(X=s)s.$$

Theorem *2 The expected number of successes when n independent Bernoulli trails are performed, where p is the probability of success of each trial, is np.*

proof $E(X) = \sum_{k=1}^{n} p(X=k)k = \sum_{k=1}^{n} kC(n, k)p^k q^{n-k}$

$= \sum_{k=1}^{n} nC(n-1, k-1)p^k q^{n-k} = np\sum_{k=1}^{n} C(n-1, k-1)p^{k-1} q^{n-k}$

$= np\sum_{j=0}^{n-1} C(n-1, j)p^j q^{n-1-j} = np(p+q)^{n-1} = np$

Theorem *3 Let X_1, X_2, …, X_n are random variables on S, and a, b* $\in$ **R**.

(i) $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$

(ii) $E(aX + b) = aE(X) + b$.

Average-Case Computational Complexity

Let S be the sample space of an algorithm. Then a $\in$ *S is a input data.*

$E(X) = \Sigma_{a\in S}\, p(a)X(a)$.

Geometric Distribution

Definition *2 A random variable X has a* ***geometric distribution*** *with parameter p, if* $p(X=k) = (1 - p)^{k-1}p$ *for k* $\in$ **N**.

Theorem *4 If the random variable X has the geometruc distribution with parameter p, then E(X) = p.*

proof *See Example 10*

Independent Random Variable

Definition *3 A random variable X and Y on a sapce S are* ***independent****, if*

$p(X(s) = r_1 \text{ and } Y(s) = r_2) = p(X(s) = r_1)p(Y(s) = r_2)$.

Theorem *5 If X and Y are* ***independent*** *random variables on a space S,*

$E(XY) = E(X)E(Y)$.

Definition *4 Let X be a random variable on a sapce S. The* ***variance*** *of X,*

V(X), is $V(X) = \Sigma_{s\in S} (X(s) - E(X))^2 p(s)$.

The standard deviation of X, $\sigma(X) = (V(X))^{1/2}$.

Theorem *6 If X is a random variables on a space S, then*

$V(X) = E(X^2)E(X)^2$.

Theorem *7 If X and Y are* ***independent*** *random variables on a space S,*

$V(X + Y) = V(X) + V(Y)$.