2 Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

Definition *1 A* ***set*** *is an* <u>*unordered*</u> ***collection*** *of* ***objects***.

Cantor's naive set theory

Russel's paradox

$S = \{x \mid x \notin x\}$

$x \in S$ iff $x \notin x$.

$S \in S$ iff $S \notin S$. ***contradictory***

Definition *2 An object in a set is called* ***element*** *or* ***members*** *of the set. A set is said to contain its elements.*

$a \in A$ "*a is an element of the set A*"

$a \notin A$ "*a is* **not** *an element of the set A*"

Two way of to define sets

i) To enumerate the elements

$A = \{a_1, a_2, \ldots, a_n\}$ *finite*

$A = \{a_1, a_2, \ldots\}$ *infinite*

ii) to specify condition with predicate

$A = \{x \mid P(x)\}$

$A = \{x \in U \mid P(x)\}$ *U:* ***universe****(data type)*

Some important sets in descrete mathematics

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ *set of* ***natural numbers***.

$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ *set of* ***integers.***

$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ *set of* ***positive natural numbers***.

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z} \text{ and } q \neq 0\}$ *set of* ***rational numbers***.

$\mathbf{R}$ *set of* ***real numbers.***

Definition *3 Two sets are* ***equal*** *if and only if they have* ***same elements****.*

$A = B$ *"Two set A and B are equal"*

$\{\}, \varnothing$ ***empty set*** *"a set that has no elements"*

note: $\{\} = \varnothing \neq \{\varnothing\}$.

Definition *4 The set A is said to* ***subset*** *of B, if and only if, every elements of A is also an elements of B, and denoted as* $A \subseteq B$.

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

Theorem *1 For any set A,*

(i) $\varnothing \subseteq S$ (ii) $S \subseteq S$.

(i) $\forall S\ \varnothing \subseteq S$ (ii) $\forall S\ S \subseteq S$

For two sets A and B, write $A \subset B$ *and say that A is a proper subset of B, if and only if,* $A \subseteq B$ *and(but)* $A \neq B (\equiv \neg A = B)$.

$$A \subset B \equiv A \subseteq B \wedge A \neq B$$

$$A = B \equiv A \subseteq B \wedge B \subseteq A \equiv \forall x(x \in A \rightarrow x \in B) \wedge \forall x(x \in B \rightarrow x \in A)$$
$$\equiv \forall x(x \in A \leftrightarrow x \in B)$$

Definition *5 Let S be a set. If there are exactly n elements in S where n is a nonnegetive integer, we say that S is finite set and that n is the* ***cardinality*** *of the set S, and denoted as |S|.*

Definition *6 A set is said to be* ***finite****, if the cardinality of set is finite. A set is said to be* ***infinite*** *it it is not finite.*

The Power Set

Definition *7 Given a set S, the power set of S, denoted by P(S), is set of all subsets of S.*

$$P(S) \equiv \{A | A \subseteq S\}$$
$$|P(S)| = 2^{|S|}.$$

Cartesian Products

Definition *8 The* ***ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ *is the* ***ordered collection****, that has* a_1 *as its first element,* a_2 *as its second element, …,* a_n *as its n-th element.*

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n), \text{ iff, } a_1 = b_1 \text{ for } i = 1, 2, \ldots, n.$$

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \equiv a_1 = b_1 \wedge a_2 = b_2 \wedge \ldots \wedge a_n = b_n.$$

We call ***ordered pair****, for 2-tuples.*
The ordered pairs $(a, b) = (c, d)$*, iff,* $a = c \wedge b = d$.
Note that $(a, b) \neq (b, a)$

Definition *9 Let A and B be sets. The* ***Cartesian product****, denoted by* $A \times B$*, is …*

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}.$$

If $R \subseteq A \times B$, the R is called a ***relation*** *from A to B.*

$A \times B \neq B \times A$.

$|A \times B| = |A| \times |B|$

We write a R b, if $(a, b) \in R$.

Two aspects of the relation

i) subset of $A \times B$, $(a, b) \in R$.

$$R \subseteq A \times B$$

ii) infix binary boolean operation, a R b,

$$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$$

Definition *10 The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by*

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

2.2 Set Operations

Definition *1 Union*

$$A \cup B = \{x | x \in A \vee x \in B\}$$

Definition *2 Intersection*

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

Definition *3 Two sets A and B are called* **disjoint**, *iff, their intersection …*

$$A \cap B = \emptyset.$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

principles of set of inclusion-exclusion

Definition *4 Difference*

$$A - B = \{x | x \in A \wedge x \notin B\}$$

Let universe of discourse be U, U is called ***universe****.*

Definition *5 Let U be a universe. The complement of the set A, denoted $\overline{A}$, is called the complement of A with respect to (w.r.t.) U is …*

$$\overline{A} = U - A = \{x \mid x \in U \wedge x \notin A\} = \{x \in U \mid x \notin A\}$$

Four regions in the Venn diagram for two sets A and B.

i) $A \cap B$ *ii)* $\overline{A} \cap B$ *iii)* $A \cap \overline{B}$ *iv)* $\overline{A} \cap \overline{B}$

membership table

Four cases for relations on two set in the Venn Diagram

i) $(A \cap \overline{B} = \emptyset) \wedge (\overline{A} \cap B = \emptyset)$ $\equiv A = B$, ***equal***

ii) $(A \cap \overline{B} = \emptyset)$ *or* $(\overline{A} \cap B = \emptyset)$ $\equiv A \subseteq B$ *or* $B \subseteq A$, ***subset***

iii) $(A \cap B = \emptyset)$ $\equiv A \cap B = \emptyset$, ***disjoint***

iv) otherwise ***incomparable***

Set Indetities

$A \cup \varnothing = A$	$A \cap U = A$	*identity laws*
$A \cup U = U$	$A \cap \varnothing = \varnothing$	*domination laws*
$A \cup A = A$	$A \cap A = A$	*idempotent laws*
$\overline{(A)} = A$		*double complement law*
$A \cup B = B \cup A$	$A \cap B = B \cap A$	*commutative laws*
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	*associative ...*
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	*distibutive ...*
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	*De Morgan's laws*
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	*absorption laws*
$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$	*complement laws*

See Table 6(logical equivalences) of Section 1.2

($\vee$, $\wedge$, $\neg$, **T**, **F**) *vs* ($\cup$, $\cap$, , *U*, $\varnothing$).

boolean algebra
complete lattice

Two prove A = B

i) $\forall x(x \in A \rightarrow x \in B)$ $\quad$ $A \subseteq B$, and
ii) $\forall x(x \in B \rightarrow x \in A)$ $\quad$ $B \subseteq A$.

Membership table

with n set variables, 2^n membership regions
similar to truth table in logic

Generalized Union and Intersections

Definition 6

$$A_1 \cup A_2 \cup \ldots \cup A_n = \cup_{i=1}^{n} A_i$$

Definition 7

$$A_1 \cap A_2 \cap \ldots \cap A_n = \cap_{i=1}^{n} A_i$$

*Note that $\cup$ and $\cap$ are **associative**.*

2.3 Functions

Definition *1 Let A and B be nonempty sets.*

*A **function**(**mapping**, **transformation**) f from A to B is*

*an assignment of exactly **one** element of B to **each**(**all**) element of A.*

We write f(a) = b, if b is the unique element of B assigned to a of A.

We write $f: A \rightarrow B$.

***total**: to all elements of A(domain)*

***unique**: exactly one elements of B(codomain)*

$f: A \rightarrow B$ *is a relation from A to B,* $f \subseteq A \times B$.

If f(a)=b, we write $(a, b) \in f$ *or* $(a, f(a)) \in f$.

$f = \{(a, f(a)) \mid {}^{\forall}a \in A\}$

$B^A = \{f: A \rightarrow B\}$

$|B^A| = |B|^{|A|}$

Let $f: A \rightarrow A$, *then f is called a function on A.*

Definition *2 Let f: A → B.*

*A is a **domain** of f,B is a **codomain(range)** of f.*

*If f(a)=b, b is the **image** of a and a is the **preimage** of b.*

*Two functions f and g are said to be **equal** … if f = g.(**set equivalence**).*

Definition *3 Let* $f_1, f_2: A \to \mathbf{R}$. $f_1+f_2, f_1 f_2: A \to \mathbf{R}$ *is defined by*

$(f_1+f_2)(x) = f_1(x) + f_2(x)$ *and* $f_1 f_2(x) = f_1(x)f_2(x)$.

Definition *4 Let f: A → B and S ⊆ A. Then the imange of S under f is*

$f(S) = \{t \in B | \exists s \in S\ (t = f(s)\}$ *or*

$= \{f(s) \in B | \forall s \in S\}$ *for short.*

The range of f = f(A).

One-to-One and Onto function

Definition *5 Let f: $A \rightarrow B$. f is **one-to-one**(1:1) or **injective**, iff*

$\forall a \in A \forall b \in A(a \neq b \rightarrow f(a) \neq f(b))$, *or logically equvalenty*

$\forall a \in A \forall b \in A(f(a)=f(b) \rightarrow a=b)$.

*A injective function is called **injection**.*

Definition *6 Let f: $A \rightarrow B$ and $(A, \leq)$ and $(B, \leq)$ are **posets**(See 8.6),*

*if $x, y \in A$ and $x < y$, $f(x) \leq f(y)$, f is called **increasing***

*$f(x) < f(y)$, f is called **strictly increasing***

*$f(x) \geq f(y)$, f is called **decreasing***

*$f(x) > f(y)$, f is called **strictly decreasing***

Definition *7 Let f: $A \rightarrow B$. f is **onto** or **surjective**, iff*

$\forall b \in B \exists a \in A(f(a) = b)$, *or* $f(A) = B$.

*A surjective function is called **surjection** or **correspondence**.*

Definition *8 Let f: $A \rightarrow B$. f is* ***one-to-one correspondence****, or* ***bijection****, if f is both* ***one-to-one****(****injective****) and* ***onto****(****surjective****).*

Theorem *1 Let A and B are sets.*

$|A| \leq |B|$, *if there is a* ***injection*** $f: A \rightarrow B$.

$|A| \geq |B|$, *if there is a* ***surjection*** $f: A \rightarrow B$.

$|A| = |B|$, *if there is a* ***bijection*** $f: A \rightarrow B$.

Definition *8.1 Let A and B are sets. We say cardinality of A and B are same,* $|A| = |B|$, *if there is a* ***bijection*** $f: A \rightarrow B$.

Definition *8.2 Let A and B are sets. and there is a* ***bijection*** $f: A \rightarrow B$. *We say that A and B are isomorphic with respect to f,* $A \cong_f B$.

Definition *9 Let $f: A \to B$ and f be one-to-one correspondence. The* ***inverse*** *function of f, denoted $f^{-1}: B \to A$, is defined by*

$$f^{-1} = \{(b, a) \mid a \in A, f(a) = b \in B\} \text{ or } f^{-1}(b) = a \text{ when } f(a) = b.$$

Definition *10 Let $g: A \to B$ and $f: B \to C$. The composition of f and g, denoted by $f \circ g: A \to C$, is defined by*

$$(f \circ g)(a) = f(g(a)) \text{ or } f \circ g = \{(a, c) \mid f(a) = b, g(b) = c\}.$$

Identity function(relation) *on A*

$$\iota_A = \{(a, a) \mid a \in A\} \qquad \text{or } \forall a \in A\ \iota_A(a) = a.$$

Let $f: A \to A$. Then

$$f \circ \iota_A = \iota_A \circ f = f.$$

Definition *11 Let f: A → B. The* ***graph*** *of the function f is defined by*

$$f = \{(a, b) \mid a \in A, f(a) = b \in B\}$$

Definition *12 The* ***floor*** *and* ***ceiling*** *funtion:* $\lfloor \cdot \rfloor \lceil \cdot \rceil : \mathbf{R} \to \mathbf{Z}$,

$\lfloor x \rfloor = n \in \mathbf{Z}$, *n is the* ***largest integer*** *such that* $n \le x$.

$\lceil x \rceil = n \in \mathbf{Z}$, *n is the* ***smallest integer*** *such that* $n \ge x$.

2.4 Sequences and Summations

Sequence $\{a_n\}$

a: **N** → **R**. *We write* a_n *instead of* $a(n)$.

Let $s: \{1, 2, \ldots, n\} \to V$.

We write s = boy or s = (b, o, y)

instead of s(1) = b, S(2) = o, s(3) = y.

s is called the finite ***string*** *over V of length n.*

V is called the ***vocabulary*****(*****alphabet*****)** *of string s.*

Some Useful Sequences

$n^2, n^3, n^4,$

$2^n, 3^n, n!$

Cardinality

Definition *4 The sets A and B have the same* ***cardinality****, if and only if, there is a one-to-one correspondence from A to B.*

Definition *5 A set is either* ***finite*** *or* ***same cardinality*** *with* $\mathbf{Z}^+$*(positive integers) is called* ***countable****. A set that is not countable is called* ***uncountable****. When an infinite set S is countable, we denote cardinality of S as* $\aleph_0$*(aleph null).* $|S| = \aleph_0$.

$\mathbf{N} \supset \mathbf{Z}^+$ *but* $|\mathbf{N}| = |\mathbf{Z}^+|$

$\mathbf{N} \subset \mathbf{Z}$ *but* $|\mathbf{N}| = |\mathbf{Z}|$

$\mathbf{N} \subset \mathbf{Q}$ *but* $|\mathbf{N}| = |\mathbf{Q}|$

$\mathbf{N} \subset \mathbf{R}$ *but* $|\mathbf{N}| < |\mathbf{R}|$

Cantor Didgonalization Argument*(1879)*

Consider $f: \mathbf{N} \to \{0, 1\}$

f is called ***infinite binary*** *string*

Consider cardinality of $\{0, 1\}^{\mathbf{N}} \cong 2^{\mathbf{N}}$.

Assume $|2^{\mathbf{N}}| = \aleph_0$. *Then we can* ***enumerate*** *binary strings*

$A_0 = (a_{00}, a_{01}, a_{02}, \ldots)$

$A_1 = (a_{10}, a_{11}, a_{12}, \ldots)$

$A_2 = (a_{20}, a_{21}, a_{22}, \ldots)$

…

Consider $A = (a_0, a_1, a_2, \ldots)$ *where* $a_n = 0$ *if* $a_{nn} = 1$; $a_n = 1$, *if* $a_{nn} = 0$.

$\forall n \in \mathbf{N}\ A \neq A_n$. *But* $A \in 2^{\mathbf{N}}$.

$\therefore$ *The assumption* $|2^{\mathbf{N}}| = \aleph_0$ *was wrong.*

$\therefore |2^{\mathbf{N}}| > \aleph_0$.

$2^{\mathbf{N}}$ *is* ***uncountable***.

Example 21