# *1 The Foundation: Logic and Proofs*

## *1.1 Propositional Logic*

***Propositions( 명제 )***

*a declarative* ***sentence*** *that is either* ***true*** *or* ***false****,*
*but* ***not both nor neither***

*letters denoting propostions* *p, q, r, s, …*

**T***: true value*

**F***: false value*

***propositional calculus*** *or* ***propositional logic***

***compounded propositions***

***propositions*** *that are* ***formed*** *from* ***exisiting*** *propositions*
*using* ***logical operators(connectives)***

***Definition*** *1* ***negation(Not)***

*Let p be a proposition, ¬p is a (new)* ***compounded*** *proposition,*
*called* ***negation*** *of p, or "****not*** *p".*

*See truth table for negation of proposition*

*¬: negation* ***operator****(unary, prefix)*

***Definition*** *2* ***conjunction(And)***

*Let p and q be a propositions, p ∧ q is a (compounded) proposition,*
*called* ***conjunction*** *of p and q, or "p and q".*

***Definition*** *3* ***disjunction(and; inclusive or)***

*Let p and q be a propositions, p ∨ q is a proposition,*
*called* ***disjunction*** *of p and q, or "p or q".*

∧, ∨: ***conjunctive****,* ***disjunctive connectives****(binary, infix)*

***Definition*** *4* ***exclusive or***

*Let p and q be a propositions,* $p \oplus q$ *is a proposition,*

*called* **exclusive or** *of p and q, or "p xor q".*

***Definition*** *5* ***implication****(****conditional****)*

*Let p and q be a propositions,* $p \to q$ *is a proposition,*

*called* **implication** *of p and q, or "p implies q".*

*"if p, then q"   "p, only if q"   …*

*p is called* ***hypothesis****(****antecedent****;* ***premise****) of* $p \to q$

*q is called* ***conclusion****(****consequence****) of* $p \to q$

$p \to q$ *is* ***false*** *only in the case that p is true anf q is false.*

*logic says* ***nothing*** *when the* ***hypothesis*** *is* ***false***

$p \to q$ *is true when q is false (inclusive) or q is true*

$\therefore$ $p \to q$ *is* ***equivalent*** *to* $\neg p \vee q$*.*$\neg$

*Let $p \rightarrow q$ be a implication proposition. Then*

*$q \rightarrow p$ is a **converse** ( 역 ) of $p \rightarrow q$,*

*$\neg q \rightarrow \neg p$ is a **contrapositive** ( 대우 ) of $p \rightarrow q$, and*

*$\neg p \rightarrow \neg q$ is a **inverse** ( 이 ) of $p \rightarrow q$.*

*$p \rightarrow q$ and **contrapositive** $\neg q \rightarrow \neg p$ are equivalent.*

***converse** $q \rightarrow p$ and **inverse** $\neg p \rightarrow \neg q$ are equivalent.*

***Definition** 6 **biconditional**(**equivalence**)*

*Let $p$ and $q$ be a propositions, $p \leftrightarrow q$ is a proposition,*

*called **bicondtional** of $p$ and $q$, or "$p$, if and only if, $q$".*

*$p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.*

*$p \leftrightarrow q$ is equivalent to $\neg(p \oplus q)$.*

# Precedence of Logical Operators

| | |
|---|---|
| $\neg$ | *high* |
| $\wedge$ | |
| $\vee$ | |
| $\rightarrow$ | |
| $\leftrightarrow$ | *low* |

$$p \vee q \wedge \neg r \rightarrow s = (p \vee (q \wedge (\neg r))) \rightarrow s$$

*syntax grammar for propositions*

$$p = \mathbf{T} \mid \mathbf{F} \mid v \mid \neg p \mid p \wedge p \mid p \vee p \mid p \rightarrow p \mid p \leftrightarrow p$$

## Truth Table of Compound Statement

*n variables* $2^n$ *rows in the truth table*

## Translating English Sentences

## System Specifications

## Boolean Searches

## Logic Puzzle

## Logic and Bit Operations

| | |
|---|---|
| **T** | *1* |
| **F** | *0* |
| $\neg$ | *NOT* |
| $\wedge$ | *AND* |
| $\vee$ | *OR* |

## 1.2 Propositional Equivalences

***Def.*** *1 A (compound) proposition that is always true:* ***tautology***

*A proposition that is always false:* ***contradiction***

*Otherwise:* ***contingency****.*

## Logical Equivalences

***Def.*** *2 Two propositions p and q are* ***logically equivalent****, if $p \leftrightarrow q$ is a* ***tautology****, written, $p \equiv q$.*

*Remark: $p \equiv q$ vs $p \leftrightarrow q$.*

*$\equiv$ has the* ***lowest*** *precedence($\neg \wedge \vee \rightarrow \leftrightarrow \equiv$)*

*Example 2 $\neg(p \vee q) \equiv \neg p \wedge \neg q$*

*Example 3 $p \rightarrow q \equiv \neg p \vee q$*

## Algebraic rules of logical equivalences

*Proof of logical equivalences*

*truth tables n variables, $2^n$ raws in the table*

*algebraic rules of logical equivalences*

# Logical equivalences laws

| | | |
|---|---|---|
| $p \vee \mathbf{F} \equiv p$ | $p \wedge \mathbf{T} \equiv p$ | *Identity laws* |
| $p \vee \mathbf{T} \equiv \mathbf{T}$ | $p \wedge \mathbf{F} \equiv \mathbf{F}$ | *Domination laws* |
| $p \vee p \equiv p$ | $p \wedge p \equiv p$ | *Idempotent laws* |
| $\neg(\neg p) \equiv p$ | | *Double negation law* |
| $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ | *Commutative laws* |
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | *Associative laws* |
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | *Disributive laws* |
| $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | *De Morgan's laws* |
| $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ | *Absorption laws* |
| $p \vee \neg p \equiv \mathbf{T}$ | $p \wedge \neg p \equiv \mathbf{F}$ | *Negation laws* |

*Since disjunction*($\vee$) *and conjunction*($\wedge$) *are* ***associative***,

$p \vee q \vee r$ *and* $p \wedge q \wedge r$ *are* ***well defined***.

*Let* $p_1, p_2, \ldots, p_n$ *be n propositions. Then*

$p_1 \vee p_2 \vee \ldots \vee p_n$ *and* $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ *are* ***well defined***.

***Extened De Morgan's law***

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n.$$

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \equiv \neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n.$$

***Logical equivalences involving conditional statements***

$p \to q \equiv \neg p \vee q$ $\qquad$ $\neg(p \to q) \equiv p \wedge \neg q$

$p \to q \equiv \neg q \to \neg p$ $\qquad$ *contrapositive*

$p \vee q \equiv \neg p \to q$ $\qquad$ $p \wedge q \equiv \neg(p \to \neg q)$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ $\qquad$ $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ $\qquad$ $(p \to q) \vee (r \to q) \equiv (p \wedge r) \to q$

## *Logical equivalences incolving biconditional statements*

$p \leftrightarrow q \equiv q \leftrightarrow p$ *commutative(symetric)*

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ *definition of biconditional*

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ *symetricity of biconditional*

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$*disjuntive normal form(truth table)*

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$ *De Mogran's law for bicondi.*

## *1.3 Predicates and Quantifiers*

***Predicate****: a* ***proposition*** *with* ***variable***

*A predicate P(x) has the proposition P and the variable x*

*Example 1 Let P(x) denotes "x > 3". Then*

*P(4) denotes "4 > 3" is T. P(2) denote "2 > 3" is F.*

*Let* $x_1, x_2, \ldots, x_n$ *be n variables. Then*

$P(x_1, x_2, \ldots, x_n)$ *is the value of* ***propositional function****( 조건명제 ) P*

*at the n-tuple*$(x_1, x_2, \ldots, x_n)$*, and*

*P is also called* ***predicate****.*

# Quantifiers

*A **predicate** is **not** a **proposition** only if, **variables** are **not fixed**.*
*If **all** the variables are fixed, the predicate **becomes** a propositions.*
*How can we fix variables?*

*Consider **universe of discourse**(**domain**) for each **variable**.*

*If P(x) is true for **all** values of x in the **universe of discourse**,*

$\forall xP(x)$ *is true;*

*otherwise* $\forall xP(x)$ *is false.*

$\therefore$ $\forall xP(x)$ *becomes a **proposition**.*

***predicate calculus***

***Definition** 1 **universal quantifier***

*$\forall xP(x)$ is a proposition such that*

*"P(x) for all values in the **domain**."*

*$\forall$ is called **universal quantifier**.*

*We read $\forall xP(x)$ as "for all x P(x)".*

*An element for which $\forall xP(x)$ is **false** is called a **counterexample** of $\forall xP(x)$.*

*Let a set $\{x_1, x_2, \ldots, x_n\}$ be the domain(**finite**). Then*

$$\forall xP(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n).$$

***Definition** 2 **existential quantifier***

*$\exists xP(x)$ is a proposition such that*

*"There **exists** an element x in the domain such that P(x)."*

*$\exists$ is called **existential quantifier**.*

$$\exists xP(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n).$$

## Binding variables

*A variable is said to be* ***bound****, if the variable binds to*

*(1) quantifiers(∀, ∃) or*

*(2) specific value(in the domain), and*

*it is said to be* ***free****, otherwise.*

***scope*** *of* ***quantifier***

*the* ***part of logical expression*** *to which the quantifier is applied*

*Example*

$$\exists \underline{x}(\underline{P(x) \land R(x)}) \lor \forall \underline{xR(x)} \equiv \exists \underline{x}(\underline{P(x) \land R(x)}) \lor \forall \underline{yR(y)}.$$

## Negations

$\forall x P(x)$ *where* $P(x)$

*"Every student in this class has taken a course in calculus."*

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

*"It is* ***not*** *the case that every student in this class has taken a course in calculus."*

*is* ***logically equivalnt*** *to*

*"There is a student in this class who has* ***not*** *taken a course in calculus."*

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

*"It is not the case that there is a student who has* ***not*** *taken a course in the calculus"*

*"Every student in this class has* ***not*** *taken class"*

***Remarks**: DeMorgan's Law*

*Let $\{x_1, x_2, \ldots, x_n\}$ be a set of discourse. Then*

$\neg\forall x P(x) \equiv \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)$

$\equiv \exists x \neg P(x).$

$\neg\exists x P(x) \equiv \neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n).$

$\equiv \forall x \neg P(x).$

***Translating from English into Logical expressions***

***Examples from Lewis Carroll***

***Alice in Wondeland***

***Logic Programming***

# *1.4 Nested Quantifier*

$\forall x \exists y (x+y=0)$

## *The Order of Quantifiers*

*Example 15 Let Q(x, y) denotes "x+y=0"*

$\forall x \exists y (x+y=0)$ *vs* $\exists y \forall x (x+y=0)$

## *Translating Statements involving Nested Quantifiers*

## *Translating Sentences into Logical Expressions*

## *Negating Nested Quantifier*

*Example 12 Negate* $\forall x \exists y (xy=1)$

$\neg \forall x \exists y (xy=1) \equiv \exists x \neg \exists y (xy=1) \equiv \exists x \forall y (\neg xy=1) \equiv \exists x \forall y (xy \neq 1)$.

## *1.5 Rules of Inference*

***Proof***: ***valid arguments*** *that*

*establish the truth of mathematical statements*

***argument*** *a sequence of statement that ends with a* ***conclusion***

***valid*** *the* ***conclusion*** *must follow from the truth of*

*the* ***preceding statements*** *or* ***premises of the argument***

*An* ***argument*** *is* ***valid****, if and only if,*

*it is impossible for* ***all premises*** *to be* ***true*** *and* ***conclusion*** *to be false*

*Rules of inference*

***deducing*** *new statements from statements we already have.*

*propositional logic*

*Incorrect reasoning*

***fallacies***

*rules of inference for* ***qualified*** *statements*

## Valid Arguments in Propositional Logic

*Definition 1* ***argument*** *a sequence of propositions preceeding* ***premeses*** *and finally a* ***conclusion***.

***argument form***

$(p_1 \land p_2 \land \ldots \land p_n) \to q$

***valid*** *argument from*

$(p_1 \land p_2 \land \ldots \land p_n) \to q$ *is* ***tautology***.

## Modus ponen

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array} \qquad [p \wedge (p \rightarrow q)] \rightarrow q$$

## Modus tollen

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array} \qquad [\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$$

## Hypothetical syllogism

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array} \qquad [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

## Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p \quad} \\ \therefore q \end{array} \qquad [(p \vee q) \wedge \neg p] \rightarrow q$$

## Addidion

$$\begin{array}{l} \underline{p \quad} \\ \therefore p \vee q \end{array} \qquad p \rightarrow (p \vee q)$$

## Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array} \qquad (p \wedge q) \rightarrow p$$

## Conjunction

$$\begin{array}{l} p \\ \underline{q \quad} \\ \therefore p \wedge q \end{array} \qquad [(p) \wedge (q)] \rightarrow (p \wedge q)$$

## Resolution

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array} \qquad [(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

*Example 6 Prove that four hypotheses*

| | |
|---|---|
| *(H1) "It is not sunny and its cold."* | $\neg sunny \land cold$ |
| *(H2) "We will swim only if it is sunny."* | $swim \rightarrow sunny$ |
| *(H3) "If we do not swim, then we will canoe."* | $\neg swim \rightarrow canoe$ |
| *(H4) "If we canoe, then we will be home early,"* | $canoe \rightarrow early$ |

*Concludes*

| | |
|---|---|
| *"We will be home early"* | $early$ |

*proof*

| | |
|---|---|
| 1. $\neg sunny \land cold$ | *Hypothesis(H1)* |
| 2. $\neg sunny$ | *Simplefication using (1)* |
| 3. $swim \rightarrow sunny$ | *Hypothesis(H2)* |
| 4. $\neg swim$ | *Modus tollen using (2) and (3)* |
| 5. $\neg swim \rightarrow canoe$ | *Hypothesis(H3)* |
| 6. $canoe$ | *Modus ponen using (4) and (5)* |
| 7. $canoe \rightarrow early$ | *Hypothesis(H4)* |
| 8. $early$ | *Q.E.D.* |

## Resolution

$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

$((p \vee q) \wedge (\neg p \vee q)) \rightarrow q$

$((p \vee q) \wedge (\neg p)) \rightarrow q$

## Fallacies

$((p \rightarrow q) \wedge q) \rightarrow p$ — ***fallacy*** *of* ***affirming*** *the* ***conclusion***

$((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ — ***fallacy*** *of* ***denying*** *the* ***hypothesis***

*Logic says nothing when hypotheses are false!*

## Rules of inferences for qualified Statements

***Universal instantiation***

$\underline{\forall x P(x)}$

$\therefore P(c)$

***Universal generalization***

$\underline{P(c) \text{ for a arbitrary } c}$

$\therefore \forall x P(x)$

***Existential instantiation***

$\underline{\exists x P(x)}$

$\therefore P(c) \text{ for some element } c$

***Existential generalization***

$\underline{P(c) \text{ for some element } c}$

$\therefore \exists x P(x)$

# *1.6 Introduction to Proofs*

## *Some Terminologies*

***Theorm****: A statement that has been proven to be* ***true****.*

***Axiom: Assumption*** *to be true (often unproven)*
*defining the* ***structures*** *about which we are reasoning.*

***Rules of inference****:* ***Patterns*** *of logically valid* ***deductions***
*from* ***hypotheses to conclusion.***

***Lemma****: A* ***minor*** *theorem used as a* ***stepping stone***
*to prove a major theorem*

***Corollary****: A* ***minor*** *theorem proven*
*as an* ***easy consequence*** *of a major theorem*

***Conjecture****: A statement whose thuth value has not been proven.*
*(A conjecture may be widely believed to be true, regardless)*

***Theory****: The set of* ***all theorems*** *that*
*can be proven from a* ***given*** *set of* ***axioms***

## Direct Proof

$\forall x(P(x) \to Q(x))$

$P(c) \to Q(c)$ *universal generalization* ($\Uparrow$)

$p \to q$ *propositional calculus*

*Example 1 "If n is odd ingetger, then* $n^2$ *is odd"*

*proof* $\forall n(O(n) \to O(n^2))$ *where O(n) is "n is odd"*

$n = 2k + 1$ $O(n)$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ $O(n^2)$

$O(n) \to O(n^2)$ *implication(conjunction)*

$\forall n(O(n) \to O(n^2))$ *universal generalization*

## Proof by Contraposition

$p \to q \equiv \neg q \to \neg p$

*Example 3 "If n is an ingeter and 3n + 2 is odd, then n is odd"*

*proof* $3n + 2 = 2k + 1, n = ?$

*If n is even, then 3n+2 is even.*

$n = 2k$, $3n+2 = 6k + 2 = 2(3k + 1)$ *is even.*

## Vacuous proof

If $p = \mathbf{F}$, $p \to q$ is a tautology.

See Section 4.1 Mathematical induction

## Trivial proof

If $q = \mathbf{T}$, $p \to q$ is a tautology.

See Section 4.1 Mathematical induction

## Proofs by Contradiction

If $\neg p \to q$, $q = \mathbf{F}$, $p$ is a tautology.

If $\neg p \to (r \wedge \neg r)$, $p$ is a tautology.

If $\neg p \to \mathbf{F}$, $p$ is a tautology.

*Example 9*

*p = "At least four of any 22 days must fall on the same day of the week"*

*¬p = "At most three of 22 days ..."*

*r = "22 days are chosen"*

$\neg p \rightarrow (r \land \neg r)$, *p is a tautology.*

*Example 10 Prove that* $\sqrt{2}$ *is irrational.*

*p = "*$\sqrt{2}$ *is irrational"* *¬p = "*$\sqrt{2}$ *is rational"*

$\sqrt{2} = a/b$, *a and b are integers.*

$2 = a^2/b^2$ $2b^2 = a^2$.

$a^2$ *is even, a is also even.* $a = 2c$.

$a^2 = 4c^2 = 2b^2$. $b^2 = 2c^2$.

$b^2$ *is even. b is even.*

$\sqrt{2} = a/b$, *a and b are* ***even*** *integers*

## Proof of Equivalence

$(p \leftrightarrow q) \equiv [(p \rightarrow q) \wedge (q \rightarrow p)]$

$(p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n) \equiv [(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge \ldots \wedge (p_n \rightarrow p_1)]$

## Counter Example

*To prove $\forall x P(x)$ is false*

*an example x P(x) is false*

## Mistakes in Proof

*Example 16 divide by zero*

*Example 17 $(p \rightarrow q)$ does not implies $(q \rightarrow p)$*

*Example 18 $(p \rightarrow q)$ does not implies $(\neg p \rightarrow \neg q)$*

# 1.7 Proof Methods and Stratagy

## Exhaustive Proof and Proof by Case

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$$