14 Uncountable Sets(Denial of Self Recursion)

14.1 Countable sets

Def. *1 Two sets A and B are **isomorphic***

*with respect to the **bijective** function f,*

iff $\exists$ *a **bijection** f:* $A \leftrightarrow B$, *written* $A \cong_f B$ *or for short* $A \cong B$.

*Or A two sets A and B has **same cardinality**, written* $|A| = |B|$ *or* $|A| =_f |B|$.

Lem. *1 The set A can be **reproduced** with the set B and the function* f^{-1}*,*

*and The set B also can be **reproduced** with A and f.*

$$f: A \to B \wedge f^{-1}: B \to A \quad \equiv \quad f: A \leftrightarrow B$$

$$A = \{f^{-1}(b) \in A \mid b \in B\} = f^{-1}(B) \quad A \cong_{f^{-1}} B$$

$$B = \{f(a) \in B \mid a \in A\} = f(A) \quad B \cong_f A.$$

Def. *1 Let the set* $\aleph$ *= {0, 1, 2, …} be a set of* ***natural*** *numbers.*

Def. *2 A set S is said to be* ***countable***, *if has* ***same cardinality*** *with a* ***subset*** *of* $\aleph$. ***Uncountable***, *otherwise.*

Let a set S be ***countable***. *Then S is either* ***finite*** *or* ***countabley infinite***.

If the set S is ***countably infinite***, *we write* $|S| = \aleph$*(aleph).*

Fact *1-1 Let* $N_n = \{1, 2, \ldots, n\}$ *and* $A = \{a_1, a_2, \ldots, a_n\}$. *Then*

$|N_n| =_f |A| = n < \aleph$. $n \in \aleph$. $1 \le {}^{\forall} i \le n$: $f(i) = a_i$ and $f^{-1}(a_i) = i$.

$N_n \subset \aleph$. $\therefore$ N_n *is (**countable** and)* ***finite*** *of size* $n \in \aleph$.

Fact *1-2 Let* $N_{0,n} = \{0, 1, \ldots, n\}$ *and* $B = \{b_0, b_1, \ldots, b_n\}$. *Then*

$|N_n| =_f |A| = n < |N_{0,n}| = |B| = n+1$. $n+1 \in \aleph$.

$0 \le {}^{\forall} i \le n$: $f(i) = a_i$ and $f^{-1}(b_i) = i$. $\therefore$ $N_{0,n}$ *is* ***finite*** *of size* $n+1 \in \aleph$.

$N_{0,n} \subset N_n \subset \aleph$ and $|N_n| = |A| = n < |N_{0,n}| = |B| = n+1 \ll \aleph$.

Def. *3 Let* $\aleph_1 = \{1, 2, 3, \ldots\} = \{k+1 \in \aleph \mid k \in \aleph\}$ *be a set of* ***natural*** *numbers starting from 1. Then*

$\aleph_1 \subset \aleph$ *but* $\aleph_1 \cong_f \aleph$, $\forall i \in \aleph_1: f(i) = i-1$ *and* $\forall n \in \aleph: f^{-1}(n) = n+1$.

$\therefore$ $\aleph_1$ *is* ***countable*** *and* ***finite(countably infinite)***.

Fact *2-1 Let* $\aleph_E = \{0, 2, \ldots, 2n, \ldots\} = \{2k \in \aleph \mid k \in \aleph\}$ *be a set of* ***even*** *numbers. Then*

$\aleph_E \subset \aleph$ *but* $\aleph_E \cong_f \aleph$, $\forall e \in \aleph_E: f(e) = e/2$ *and* $\forall n \in \aleph: f^{-1}(n) = 2 \cdot n$.

$\therefore$ $|\aleph_E| = \aleph$ *is* ***countably infinite***.

Fact *2-2 Let* $\aleph_O = \{1, 3, \ldots, 2n+1, \ldots\} = \{2k+1 \in \aleph \mid k \in \aleph\}$ *be a set of* ***odd*** *numbers. Then*

$\aleph_O \subset \aleph$ *but* $\aleph_O \cong_f \aleph$, $\forall o \in \aleph_O: f(o) = (o-1)/2$ *and* $\forall n \in \aleph: f^{-1}(n) = 2 \cdot n+1$.

$\therefore$ $|\aleph_O| = \aleph$ *is* ***countably infinite***.

Def. 4 Let $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \leftrightarrow_f \{0, -1, 1, -2, 2, \ldots\}$
$= \{k \in \mathbf{Z} \mid k \in \aleph_1\} \cup \{0\} \cup \{-k \in \mathbf{Z} \mid k \in \aleph_1\}$ be a set of ***integers***.

Then $\mathbf{Z} \supset \aleph$ but $\mathbf{Z} \cong_f \aleph$ where

$f: \mathbf{Z} \to \aleph, \forall i \in \mathbf{Z}$: **if** $i \geq 0 \to f(i) = 2 \cdot i \mid i < 0 \to f(i) = -(2 \cdot i + 1)$ **fi**.

$f^{-1}: \aleph \to \mathbf{Z}, \forall n \in \aleph$: **if** $n = 2 \cdot k \to f(n) = n/2 \mid n = 2 \cdot k + 1 \to f(n) = -\lfloor n/2 \rfloor$ **fi**.

$\therefore |\mathbf{Z}| = \aleph$ is ***countably infinite***.

Def. 5 Let $\mathbf{Z}_n^p = \{k \in \mathbf{Z} \mid k$ **mod** $n = p$ or $[k]_n = p\}$ be
a set ***integers*** of ***congruence*** n of ***modulus*** p. Then

$\mathbf{Z}_n^p \subset \mathbf{Z}$ but $\mathbf{Z}_n^p \cong_f \mathbf{Z}$. $\therefore |\mathbf{Z}_n^p| = |\mathbf{Z}| = \aleph$ is ***countably infinite***.

$f: \mathbf{Z}_n^p \to \mathbf{Z}, \forall m \in \mathbf{Z}_n^p: f(m) = m$ **div** n.

$f^{-1}: \mathbf{Z} \to \mathbf{Z}_n^p, \forall i \in \mathbf{Z}, f^{-1}(i) = i \cdot n + p$.

$\{\ldots, p-2n, p-n, p, p+n, , p+2n, \ldots\} \leftrightarrow_f \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Thm. 1 $\aleph\times\aleph = \{(i, j) \in \aleph\times\aleph \mid i, j \in \aleph\}$ *is* ***countably infinite***.

$(0, 0), (0, 1), (0, 2), \ldots, (0, n), \ldots$

$(1, 0), (1, 1), (1, 2), \ldots, (1, n), \ldots$

…

$(n, 0), (n, 1), (n, 2), \ldots, (n, n), \ldots$

…

$f: \aleph\times\aleph \rightarrow \aleph, \forall i, j \in \aleph: f(i, j) = (i+j)\cdot(i+j+1)/2 + j.$

$f^{-1}: \aleph \rightarrow \aleph\times\aleph, \forall n \in \aleph: f^{-1}(n) = ?$

$\therefore |\aleph\times\aleph| = |\aleph^2| = \aleph.$

Lem. 2 *Let Q be a set of* ***rational*** *numbers. Since* $Q \subset \aleph\times\aleph$, $|Q| = \aleph$.

Lem. 3 *Let* $k \in \aleph$. *Then* $\aleph\times\aleph\times\ldots\times\aleph = \aleph^k$ *is* ***countably infinite***.

$$|\aleph\times\aleph\times\ldots\times\aleph| = |\aleph^k| = \aleph.$$

$$|\aleph| = |\aleph_1| = |\aleph_E| = |\aleph_O| = |\mathbf{Z}| = |\mathbf{Z}_n^{\,p}| = |\aleph\times\aleph| = |Q| = |\aleph^k| = \aleph.$$

*Let V be a set of **symbols**, called **vocabulary(alphabet)**. We define a set of **strings** over V of length n($n \geq 0$).*

$V^0 =_B \{\lambda\}$ λ: *epmty string of lengh 0, **identity** for concatenation($\cdot$)*

$V^{n+1} =_R V^n \cdot V$

$x \in V^n$ *is a **string** over V whose length* $|x| = n$.

V^n *is a set of **string** over V and* $|V^n| = |V|^n$.

We define $V^* = \cup_{i \in \aleph_0} V^i = \{\lambda\} \cup V \cup V^2 \cup \ldots$

*as an **universe** of strings over V.*

Let $x = b_1 b_2 \ldots b_n \in V^n (n \geq 0)$, $1 \leq \forall i \leq n$: $b_i \in V$. *Then*

$f(x) = f(b_1 b_2 \ldots b_n) = b_1|V|^n + b_2|V|^{n-1} + \ldots + b_{n-1}|V| + b_n$.

$\therefore |V^*| = \aleph$ *is **countably infinite**.*

14-2 Uncountable(Denial of Self Recursion)

14-2-1 Cantor's Diagonal Argument(1874, 1892)

Consider $f: \aleph \rightarrow \{0, 1\}$ and $2^{\aleph} = P(\aleph)$

infinite binary string **Power set** of **natural numbers**

Assme $2^{\aleph}$ is **countable**. Then $b_n \in 2^{\aleph}$ can be **enumerated**.

$b_0 = (b_{00}, b_{01}, \ldots, b_{0n}, \ldots) \in 2^{\aleph}$.

$b_1 = (b_{10}, b_{11}, \ldots, b_{1n}, \ldots) \in 2^{\aleph}$.

…

$b_n = (b_{n0}, b_{n1}, \ldots, b_{nn}, \ldots) \in 2^{\aleph}$. $i, j \in \aleph: b_{ij} \in \{0, 1\}$

…

Consider **diagonal** $b_d = (b_{00}, b_{11}, \ldots, b_{nn}, \ldots) \in 2^{\aleph}$.

Consider **complement** of b_d, $\overline{b}_d = (\overline{b}_{00}, \overline{b}_{11}, \ldots, \overline{b}_{nn}, \ldots)$ $\overline{b}_{ii} = \neg b_{ii}$.

$\overline{b}_d$ is an infinite binary string($\overline{b}_d \in 2^{\aleph}$) but $\overline{b}_d \notin 2^{\aleph}$.

$\therefore$ Assumption $2^{\aleph}$ is **countable**. is a **contradiction**. $\therefore$ $2^{\aleph}$ is **uncountable**.

14-2-2 Russel's Paradox*(1901, 1911)*

Let $R = \{x \mid x \notin R\}$ *in* $x = R$. *Then* $R = \{R \mid R \notin R\}$.

$\therefore R \in R \Leftrightarrow R \notin R$.

Contradiction*!*

14-2-3 Gödel's Incompleteness Theorem*(1931)*

Assume proof(theorem) $\rightarrow$ *{****proved****,* $\neg$***proved****} exists.*

Hilbert's dream(1929)

Let *Gödel(proof, theorem) =*

if *proof(theorem)* $\rightarrow$ $\neg$***proved*** | $\neg$*proof(theorem)* $\rightarrow$ ***proved fi***

in *proof = theorem = Gödel* ***then***

Gödel(Gödel, Gödel) ***proved*** $\Leftrightarrow$ *Gödel(Gödel)* $\neg$***proved***

$\wedge$ *Gödel(Gödel, Gödel)* $\neg$***proved*** $\Leftrightarrow$ *Gödel(Gödel)* ***proved***

Contradiction*!*

14-2-4 Halting problem

Let *Halting(program, data) =*

  if *program(data)* $\rightarrow$ $\neg$***stop*** | $\neg$*program(data)* $\rightarrow$ ***stop fi***

in *program = data = Halting* ***then***

  Halting(Halting, Halting) ***stop*** $\Leftrightarrow$ $\neg$*Halting(Halting)*

$\wedge$ *Halting(Halting, Halting)* $\neg$***stop*** $\Leftrightarrow$ *Halting(Halting).*

Contradiction*!*

Denial *of* ***self recursion****!*

14-2-5 A Barber problem

A ***barber cuts everyone*** *who* ***cannot cut himself****.*

Shall the ***barber*** *cut* ***himself****?*

Contradiction*!*

14-2-6 A memo written by a liar

*"This memo is written by me who is a **liar.**"*

*Is the memo **written by the liar** or **not**?*

Contradiction!

14-2-7 Heterlogical adverb(이종형용사)

monosyllabic(단음절)	***heterlogical***
polysyllabic(다음절)	***not heterlogical***
heterlogical	***heterlogical** or **not heterlogical***

Contradiction!