10 Graphs

10.1 Graphs and Graph Models

Def. *1 An (**undirected**) **graph**(or **ugraph**) G = (V, E) consists of*
*V, a nonempty set of **vertices** (or **nodes**) and E, a set of **edges**.*
*Each edge has either **one**? or two **vertices** associated with it,*
*called **endpoints**. An **edge** is said to **connect** its **endpoints**.*

infinite graph *infinite vertices and/**or** edges*
finite graph *finite vertices and edges*
But finite graph is considered in this text.

Let u, v $\in$ *V. Then we **write** {u, v}* $\in$ *E, **or** {u}* $\in$ *E.*
The edge {u} is called the self-loop edge connectint the vertex u to u.
See Fig. 3 of p 618.

Def. *2 A* ***directed graph(digraph, graph)*** *on the set of* ***vertices*** *V is,*

$G = (V, E)$ where $E \subseteq V \times V$.

A pair $(u, v) \in E$ *is called* ***edge(arc)*** *and said to*

starts *at the* ***vertex(node)*** *u and* ***ends*** *at the* ***vertex*** *v.*

$(u, v) \neq (v, u)$ *and* $(u, u) \in E$ ***self-loop***

simple graph ***single edge*** *for each two vertices*

multigraph ***multiple edges*** *for the same (two) vertices*

m different edges, $\{u, v\}$ *connecting the vertices u and v.*

We say that the edge $\{u, v\}$ *has muliplicity of n.*

$G = (V, E, f)$ ***Edge weighted (simple) graph simulates multigraph***

$f: E \to \mathbf{N}$ or $V \times V \to \mathbf{N}$

Egde $\{u, v\}$ *of* ***multiplicity*** *m, if* $f(\{u, v\}) = m \in \mathbf{N}$.

Edges *of* ***digraph*** *is a binary* ***relation*** *on V($E \subseteq V \times V$).*

- ***Reflexivity*** *All vertices have a self-loop*
- ***Irreflexivity*** *No verteces have a self-loop*
- ***Symmetry*** *All edges are bidirectional*
- ***Antisymmetry*** *All edges are unidirectional, self-loop is allowded*
- ***Asymetry*** *All edges are unidirectional, self-loop is* **not** *allowded*
- ***Transitivity*** *All paths should have an (**extra**) edge*

Digraph simulates (undirected) graph *and* ***pseudograph***

- *$\{u, v\} \in E$ in ugraph $(u, v), (v, u) \in E$ in digraph symmetricity*
- *$\{u\} \in E$ in pseudograph $(u, u) \in E$ in digraph reflexivity*

Edge weighted graph simulates multigraph

Graph Models

Read the text book from p620 to p625.

10.2 Graph Terminology and Special Types of Graphs

Def. *1* **<u>Adjacency</u>** *Let G = (V, E) be an* ***ugraph****(undirected graph).*

If edge $\{u, v\} \in E$*, then*

Vertices u, v are ***adjacent****(****neighbors****,* ***connected****)*

Edge {u, v} ***connects*** *(is* ***incident*** *with) vertices u and v.*

Vertices u and v are ***endpoints*** *of the edge {u, v}.*

Def. 2 **<u>Neighborhood</u>***: Let G = (V, E) be an* ***graph*** *and* $v \in V$*. Then*

$N(v) = \{u \in V \mid \{u, v\} \in E\}$*, called* ***neighborhood*** *of v*

the set of ***all*** *neighbors of v.*

Let $A \subseteq V$*. Then we* ***define*** $N(A) = \cup_{v \in A} N(v)$*.*

Def. 3 <u>Degree of a vertex</u>: *Let G = (V, E) be an* ***ugraph*** *and $v \in V$. Then*

deg(v) $\in$ **N**. *is the* ***number*** *edges* ***incident*** *with it*

(except a ***self-loop*** *counts twice)*

$deg(v) = |\{\{v, u\} \in E\}| + 2 \cdot |\{\{v\} \in E\}|$

If deg(v) = 0, v is called ***isolated***.

If deg(v) = 1, v is called ***pendant***.(목걸이 , 팔찌)

self-loop

Def. 3' <u>Degree of a vertex</u>: *Let G = (V, E) be a* ***(digraph) graph***$(E \subseteq V \times V)$ *and $v \in V$. Then*

deg(v) $\in$ **N** *is the* ***number*** *edges* ***incident*** *with it*

$deg(v) = |\{(u, v) \in E\}| + |\{(v, u) \in E\}|$

indegree ***outdegree***

$dev(V) = 2 \cdot |E|$ (***Thm***. *1*).

Thm. 1 The Hand Shaking Theorem

*Let G = (V, E) be an **ugraph** with |E| = m. Then $2m = \sum_{v \in V} deg(v)$.*

proof *Every edge contributes twice to the sum of the degrees.*

Thm *2 Let G = (V, E) be an **ugraph**. Then G has*

*an **even** number of **vertices** of **odd** degree.*

proof *Let V_e be the set of vertices of **even** degree and*

*V_o be the set of vertices of **odd** degree.*

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in V_e} deg(v) + \sum_{v \in V_o} deg(v)$$

$|\sum_{v \in V_o} deg(v)|$ is even. $\therefore$ $|V_o|$ is even.

Def. *4* ***<u>Adjacency</u>*** *Let G = (V, E) be a **digraph**. If edge (u, v) ∈ E, then*

*u is said to be **adjacent** to v and v is said to be **adjacent** from u.*

*The vertex u is called the **initial** vertex of (u, v) and*

*the vertex v is called the **terminal** or **end** vertex of (u, v).*

Def.* 5** *Let G = (V, E) be a* ***digraph *and* $v \in V$.

In-degree *of v, denoted as,* $deg^{-}(v)$, *is*

the number of edges with v as their ***terminal*** *vertices.*

$$deg^{-}(v) = |\{(u, v) \in E\}|$$

Out-degree *of v, denoted as,* $deg^{+}(v)$, *is*

the number of edges with v as their ***initial*** *vertices.*

$$deg^{+}(v) = |\{(v, u) \in E\}|$$

Thm.* 3** *Let G = (V, E) be a* ***digraph*. Then*

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|.$$

Some Special Simple Graphs

***Ex. 5 Complete graphs**, for $n \geq 1$, n vertices $K_n = (V_n, E_n)$, has one edge between **each pair** of **distinct** vertices.*

$K_n = (V_n, E_n)$ where $V_n = \mathbf{N}_{1,n} = \{1, 2, \ldots, n\}$ and

$E_n = \{(i, j) \in (\mathbf{N}_{1,n} \times \mathbf{N}_{1,n}) \mid 1 \leq i < j \leq n\}$ vs. $1 \leq i \neq j \leq n$ vs. $1 \leq i, j \leq n$

K_n has $n(n-1)/2$ edges or $n(n-1)$ edges or n^2 edges.

***Ex. 5 Empty graphs**, $G = (V, \emptyset)$.*

***Ex. 6.0 Lines**, $L_n = (\mathbf{N}_{0,n}, E_n^L)$ $n+1$ vertices and n edges.*

$E_n^L = \{(i, i+1) \mid i \in \mathbf{N}_{0,n-1}\}$.

***Ex. 6 Cycles**, for $n \geq 3$, $C_n = (\mathbf{N}_{1,n}, E_n^C)$ n vertices and n edges.*

$E_n^C = \{(i, i+1) \mid i \in \mathbf{N}_{1,n-1}\} \cup \{(n, 1)\}$.

Ex. 7 Wheels, *for* $n \geq 3$, $W_n = (\mathbf{N}_{0,n}, E_{2n}^{W})$.

C_n *(a* ***cycle*** *of* n *vertices) and a* ***hub*** *vertex (0).*

edges from a ***hub*** *vertex (0) to the* ***cycle***(C_n).

$\therefore$ $n+1$ *vertices,* $2n$(*cycle* + *hub*) *edges.*

$E_{2n}^{W} = \{(i, i+1) \in \mathbf{N}_{1,n-1}\} \cup \{(n, 1)\} \cup \{(0, i) \in (\{0\} \times \mathbf{N}_{1,n})\}$.

$= E_n^{C} \cup \{(0, i) \in (\{0\} \times \mathbf{N}_{1,n})\}$.

Ex. 8 n-cube, For $n \geq 0$, $Q_n = (V_n, E_n)$ with $|V_n| = 2^n$.

$V_n = \{b_n^i \in \{0, 1\}^n | 0 \leq i \leq 2^n-1\}$ b_n^i: n-bits binary string of $i \in \mathbf{N}$

$E_n = \{(b_n^i, b_n^j) | b_n^i, b_n^j \in \{0, 1\}^n$, b_n^i and b_n^j **differs only one** bit$\}$.

Construct Q_{n+1} using Q_n and Q'_n(copy of Q_n).

$Q_0 = (\{v_0^0\}, \{\})$. ***basis***

$Q_{n+1} = (V_{n+1}, E_{n+1})$ where

$V_n^Z = \{\mathbf{0} \cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading **zero**

$V_n^O = \{\mathbf{1} \cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading **one**

$V_{n+1} = V_n^Z \cup V_n^O = \{b_{n+1}^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^{n+1}-1\}$

$E_{n+1} = E_n^Z \cup E_n^O \cup \{(\mathbf{0} \cdot b_n^i, \mathbf{1} \cdot b_n^i) \in V_n^Z \times V_n^O | 0 \leq i \leq 2^n-1\}$

recursion

Bipartite Graph

G = (V, E) is bipartite, iff $V = V_1 \cup V_2$ *where* $V_1 \cap V_2 = \varnothing$ *and*

$\forall\{v_1, v_2\} \in E, v_1 \in V_1, v_2 \in V_2$.

Partition $\{V_1, V_2\}$ *is called* ***bipartition***.

Complete Bipartite Graph, $K_{n,m} = (V_1 \cup V_2, E)$ *where* $V_1 \cap V_2 = \varnothing$ *and*

$|V_1| = n$, $|V_2| = m$, *and* $E = \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\}$

$|E| = |V_1| \cdot |V_2| = n \cdot m$.

Thm. 4 *A simple graph, is bipartite ,if and only if,*

it is ***2-colorable***.*(see Fig. 7, Ex. 11(p 632), and Sec. 10.8)*

Every ***path***, ***tree***, *and* ***even length cycles***(C_{2n}) *are* ***bipartite***.

odd vetecies and even vertecies

Thm *A graph is* ***bipartite***, *if and only if, it contains* ***no odd lengh cycle***.

New Graphs from old

Def. 7 *A* ***subgraph*** *of graph $G = (V, E)$ is $G' = (V', E')$,*
where $V' \subseteq V$ and $E' \subseteq E$.
A ***proper subgraph*** *of graph $G = (V, E)$ is $G'' = (V'', E'')$,*
where $V' \subset V$ and $E' \subset E$.

Def. 8 *A* ***subgraph*** *of graph $G = (V, E)$*
induced *by a* ***subvertices*** *$V'(V' \subseteq V)$ is*
$G_{V'} = (V', E')$, where $E' = \{(u, v) | u, v \in V'\}$.

Removing or Adding edges of a Graph

$G - e = (V, E - \{e\})$ $\quad$ $G + e = (V, E \cup \{e\})$

Def. *9 The* ***union*** *of two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is*
$G_{G_1 \cup G_2} = (V_1 \cup V_2, E_1 \cup E_2)$.

Let G' = (V', E') is a subgraph of G = (V, E). Then
G' is called ***spanning subgraph*** *of G, if V' = V.*

Let G' = (V', E') is a subgraph of G = (V, E). Then
G" = (V", E") is called the ***complement*** *of G' with respect to G*
where E" = E – E', and V" = {a ∈ V| (a, b) ∈ E" or (b, a) ∈ E"}

The ***complement*** *of ugraph G = (V, E) where |V| = n,*
with respect to K_n*, denoted* $\overline{G} = (\overline{V}, \overline{E})$*, is*
*the (****absolute****)* ***complement*** *of G (w.r.t.* K_n*).*

10.3 Representing Graphs and Graph Isomorphism

Adjacecency List

Adjacecency Matrices

$|V| \times |V|$ ***boolean*** *matrix*

Isomorphism of Graphs

Def. *1 The graph* $G = (V, E)$ *and* $G' = (V', E')$ *are* ***isomorphic***,
iff $\exists$ *a* ***bijection*** $f: V \leftrightarrow V'$.э. $\forall u \forall v \in V: (u, v) \in E \Leftrightarrow (f(u), f(v)) \in E'$.

Let $Q = (V, E)$ *a graph. A* ***bijection*** $f: V \to V'$ *where* $V \cap V' = \emptyset$. *Then*

$Q' = (V', E') = (\{f(u) \in V' | u \in V\}, \{(f(u), f(v)) \in E' | (u, v) \in E\})$

We may extend f domain and codomain of f from vertices to edges.

$f((u, v)) = (f(u), f(v))$.

$Q' = (f(V), f(E))$ $Q = (f^{-1}(V'), f^{-1}(E'))$

If Q and Q' are ***isomorphic***,
Q' is called an ***isomorphic image*** *of Q and vice versa.*

n-cube, *revisited.*

i) Q = ({v}, { }) be a ***0-cube****.*

ii) Let Q be an ***n-cude****. Then*

$Q' = (V \cup f(V), E \cup f(E) \cup \{(V, f(V))\}$ *is an (n+1)-cube.*

What are f and f^{-1}*?*

Graph isomorphism

Example 8, 9, 10

Difficult to check!

Necessary *conditions for graph* ***isomophisms***

$|V_1| = |V_2|$ *and* $|E_1| = |E_2|$.

The number of vertices with degree n is same in both graphs.

Subgraph H of one graph is isomorphic to a subgraph of the other

isomorphic simple cycle

10.4 Connectivity

Def. 1 *Let G = (V, E) be a graph. Then a **path** of **length** n(≥0) from u to v in G is a **sequence** of **n edges*** $Path^E(u, v) = (e_1, e_2, \ldots, e_n)$ *where*

$1 \le {}^{\forall}i \le n: e_i = (v_i^s, v_i^e), v_i^e = v_{i+1}^s, e_1^s = u, e_n^e = v.$

Consider two sequences of n edges and (n+1) vertices

$((v_0, v_1),(v_1, v_2),\ldots, (v_{n-1}, v_n))$ *vs.* $(v_0, v_1, \ldots, v_n)$ *where* $v_0 = u, v_n = v$.

*What is the **path** of length 0?* *() vs (v)*

*A path is a **circuit** if u = v, for n ≥ 1.*

*A path **pass through** the vertices* $v_1, \ldots, v_{n-1}$ *or **traverse** along it.*

*A path is **simple**, if it contains no **edges** more than once.*

Def. 1 *Let G = (V, E) be a graph. Then a **path** of **length** n(≥0) from u to v in G is a **sequence** of **(n+1) vertices*** $Path^V(u, v) = (v_0, v_1, \ldots, v_n)$ *where*

$v_0 = u, v_n = v$, *and* $0 \le {}^{\forall}i < n: (v_i, v_{i+1}) \in E$.

Note that $Path^V(u, u) = (u)$ *is a **path** of length 0 **without** edge.*

Def. *2 Let G = (V, E) be a digraph. Then the graph G is* ***simple****, if every* ***path*** *is simple.*

$u\ E^*\ v$ *there is a path from u to v*

$u\ E^+\ v$ *there is a positive length path from u to v*

Def. *2.1A* ***directed acyclic graph****(DAG) is a directed graph with* ***no cycle****.*

Lemma *If G is a DAG,* G^+ *is a* ***irreflexive partial order****.*

Connectedness in Directed Graph

Def. *3 Two vertices u and v are to said to be* ***connected****,*
if there is a path between them.

A graph is said to be ***connected****,*
if there ***every*** *pair of distinct vertices are connected.*

We write u connects v or u is connected to v, if u and v are connected.

$\textbf{connects} \subseteq V \times V$ *binary relation on V*

Lemma *0.1 The binary relation* ***connects*** *on V*
is an ***equivalence*** *relation*

Def. *3.2 The* ***quivalence class*** *defined by the equivalence relation*
connects *is called* ***connected component****.*

$P = \{[v]_{connects} \mid v \in V\}$ ***partition*** *of V.*

$|P|$ = *number of conncted components*

Def. *3.3 Two vertices in a graph are* ***k-connected*** *if they* ***remain connected*** *in any subgraph by* ***deleting*** *k–1 edges.*

A graph is ***k-connected*** *if every pair of vertices are k-connected.*

simple cycle *2-connected*

K_n *(k–1)-connected*

Thm *1 There is a* ***simple path*** *between every pair of distinct vertices of a connected graph.*

proof *Consider* ***minimum*** *length path from u to v.*

$v_o, v_1, \ldots, v_k$ *where* $v_o=u$, $v_k=v$ *with* $k \geq 2$*(if* $k \leq 1$*, simple) is simple.*

Assume it is not simple, $0 \leq \exists i < \exists j < n$.э. $v_i = v_j$. *Not minimum path!*

Corollary *1.1 For any path of* ***length*** *k in graph, there is a* ***simple*** *path of length at* ***most*** *k with the same endpoints.*

Thm *1.1 Every graph G = (V, E) has*

***at least** |V| – |E| **connected components**.*

proof *P(n): G = (V, E) with |E| = n has at least |V| – n C. C.*

***base**: |E| = 0, |V| connected components.*

***induction**: Consider G = (V, E) with n+1 edges.*

*   **Remove** an edge {u, v} and call the resulting graph G'.*

*   G' has at least |V| – n connected components.(I. H.)*

*   **Add back** the edge {u, v}.*

*   case 1: u and v are in the **same** C. C. **Same** number of components*

*   G has at least |V| – n > |V| – (n+1) components.*

*   case 2: u and v are in the **diffrent** C. C. One **less** component*

*   G has at least |V| – n – 1 = |V| – (n+1) components.*

Cor. *1.2 The **connected** graph with n vertices has **at least** n-1 edges.*

proof $1 \geq |V| - |E|$ $|E| \geq |V| - 1$

10.5 Euler and Hamilton Path

Def. *1 Let G be a graph.*

An ***Euler path*** *is a path containing* ***every edge*** *of G.*

An ***Euler circuit*** *is a circuit containing* ***every edge*** *of G.*

Thm. *1 A connected graph has* ***Euler circuit***
iff each vertex has ***even degree***.

proof *(→) The circuit contributes 2 to the degree of each node*
(←) Algorithm 1

Alg. *1 Constructing Euler Circuit*

Begins with ***arbitrary*** *node.*

Construct a ***cycle*** *form the vertex to the vertex.*

Repeat for each ***remaining*** *subgraph,*
inserting *the* ***new cycle*** *into the* ***original*** *one.*

Thm. *2 A connected graph has* ***Euler path***
iff it has exaclty ***two*** *vertices of* ***odd degree***.
proof *One is the start, the other is the end.*

Def. *2 Let G be a graph.*
An ***Hamilton path*** *is a path traverse each vertex in G exaclty* ***once***.
An ***Hamilton circuit*** *s a circuit traverse each vertex in G exaclty* ***once***.

K_n *has a Hamilton circuit circuit*($n \geq 3$).

If a graph has a vertex of ***degree one***, *there is* ***no Hamilon*** *circuit.*
Exactly ***two edges*** *incident to a vertex are in Hamaiton circuit.*

Trvaling Saleman
NP complete

10.6 Shortest-Path Problem

Let $G = (V, E)$ *be a graph, and* $f: E \to \mathbf{R}^+$ *be a* ***cost*** *of the edges. Then* (V, E, f) *is an edge* ***weighted*** *graph or multigraph(**N**) in this text.*

Iterative *definition(extension) of* ***cost*** *of* ***path***.

Let $(v_0, v_1, \ldots, v_n) \in E^*$ *be a path of length* $n \geq 0$. *Then we define*

if $n = 0$, $f^*(v_0, v_0) = 0$.

if $n \geq 1$, $f^*(v_0, v_n) = \sum_{j=0}^{n-1} f(v_j, v_{j+1})$

Recursive *definition(extension) of* ***cost*** *of* ***path***.

$f^*(u, u) = 0$.

$f^*(u, v) = f(u, x) + f^*(x, v)$ *or* $f^*(u, x) + f(x, v)$.

We may use f instead of f^*, *since* $f \subseteq f^*$.

$f: E^* \to \mathbf{R}^+$ ***extend*** *the domain of f from E to* E^*.

Shortest path problem

Let $G = (E, V, f)$ be an edge weighted graph.

Definition $min_f(u, v) = (u, v_1, \ldots, v_n, v) \in E^* \; n \geq 0$.э.

$$f(u, v_1, \ldots, v_n, v) \leq f(u, u_1, \ldots u_m, v) \text{ .э. } \forall m \geq 0 \; \forall (u, u_1, \ldots, u_m, v) \in E^*.$$

Find $u, v \in V$, find a shortest path such that $f(u, v)$ is minimum.

Definition *Let $W \subseteq V$, $u, v \in V$.* $L_f^W(u, v) = (u, w_1, \ldots w_n, v)$.э.

$$f(u, w_1, \ldots w_n, v) \leq f(u, x_1, \ldots, x_m, v) \text{ .э. } \forall m, (u, x_1, \ldots, x_m, v) \in E^*$$

$$n, m \geq 0, 1 \leq \forall i \leq n, w_i \in W, 1 \leq \forall i \leq m, x_i \in W.$$

Note that u, v may be in W or not.

***Thm**. Let $W \subseteq V$, $u \in V$, and $x \notin W$.э. $L_f^W(u, x)$ is the **minimum**.*

$$\forall y \notin W, L_f^{W \cup \{x\}}(u, y) = min(L_f^W(u, y), L_f^W(u, x) + f(x, y)).$$

procedure *Shortest path* $G = (V, E \subseteq V \times V, f: E \to \mathbf{R}^+)$

for $w \in V$ ***do*** $L(w) :=$ *Infinite* ***od***; $L(u), W := 0, \emptyset$;

Initialization $\forall y \in V, L(y) = L_f^{\{\}}(u, y)$.

do $v \notin W \to$

Loop invariance $\forall y \in V, L(y) = L_f^{W}(u, y)$.

$x := x \notin W \wedge min(L(x)); W := W \cup \{x\};$

Loop invariance, invalidated $\forall y \in V, L(y) = L_f^{W-\{x\}}(u, y)$.

do $y \notin W \wedge (x, y) \in E \to L(y) := min(L(y), L(x) + f(x, y))$ ***od***

Loop invariance, validated $\forall y \in V, L(y) = L_f^{W}(u, y)$.

od

After the loop termination $\forall y \in V, L(y) = L_f^{W}(u, y) \wedge v \in W$.

We should also prove that $L_f^{W}(u, v) = min_f(u, v)$, *if* $v \in W$.

But it is trivial, since we add the shortest path vertex x to W.

10.7 Planar Graph

Def*. 1 A graph is* ***planar****, if it can be drawn in the* ***plane without edge crossing****.*

K_4 *is planar and* Q_3 *is planar, but* $K_{3,3}$ *is not planar.*

Region*: planar graph divides regions*

Tree *has* ***one*** *region*

Removing an edge *on the* ***cycle*** *merges* ***two*** *regions into* ***one****.*

Thm *1 Let G be a* ***connected planar graph*** *with v* ***vertices*** *and e* ***edges****.*
Let r be the number of ***regions*** *in planar reprsentation of G. Then*

$r = e - v + 2.$

proof *Induction on number of edges, P(e).*
basis *e=0, v=1, r=1.* $\therefore$ *1 = 0 – 1 + 2 = 1. O.K.*
induction *Consider a connected planar graph G with e+1 edges.*
1. If G is ***acyclic****. G is* ***tree****.* $\therefore$ *r = 1, e – v + 2 = –1 + 2 =1.* $\therefore$ *O.K.*
2. If G is not acyclic, G has at least one ***cycle****, C.*
Consider {u, v} ***in*** *C and a* ***spanning tree****, T. .э. {u, v} is* ***not*** *in T.*
There exisits such an edge {u, v} because T is acyclic.
Remove *the* ***edge*** *{u, v} from G, it is called as G'.*
G' ***has one less regions****, since removing an edge {u, v} on the cycle C.*
G' is connected planar graph and has e edges.
$\therefore$ *r = e – v + 2 in G'* *induction hypothesis*
In G, r+1 regions, e+1 edges, and v vertices. $\therefore$ *(r +1) = (e+1) – v + 2.*

Col. *1 If connected planar graph G with e edges and v vertices where* $v \geq 3$. *Then* $e \leq 3v - 6$.

proof *Consider degree of region number of boundary edges*

Sum of the degree of regions is 2e.

Minimum number of degree of each region r is 3r(triangle)

$\therefore\ 2e \geq 3r = 3(e - v + 2)$ $\therefore\ e \leq 3v - 6$.

Col. *2 If G is a connected planar graph, then G has a vertex not exceed-ing five.*

proof *If G has one or two vertices, the result is true.*

If G has more that three vertices, $e \leq 3v - 6$. *So* $2e \leq 6v - 12$.

If every vertex has more than or equal 6, $2e \geq 6v$*(Handshaking Theorem)*

But it contradict with $2e \leq 6v - 12$.

Ex. *5* K_5 *is not planar.*

$v = 5,\ e = 10.\ 3v - 6 = 9.$

10.8 Graph Coloring

Def. *1 A graph G is k-**colorable**, if each vertex can be assigned one of k colors so that **adjacent vertices** get the **different colors**.*

map coloring problem

*The smallest number of colors are called **chromatic number** of G, written as $\chi(G)$. $\chi(K_n) = n$.*

Thm. *1 The **chromatic** number of **planar** graph is no greater than 4.*

Thm. *2 A graph with maximum degree at most k is (k+1)-colorable.*

proof *Induction on number of **vertices**. P(n).*

basis *1-vertex graph, maximum degree 0 and 1-colorable. P(1) is true.*

induction *Let G be (n+1)-**vertex** graph with maximum degree at most k*

Remove *a vertex v and its incident edges.*

G' has n vertices and max. deg. at most k. $\therefore$ G' is (k+1)-colorable(IH).

v has at most k adjacent vertices. $\therefore$ G is (k+1)-colorable($(k+1) - k = 1$).