10 Graphs

10.1 Graphs and Graph Models

Def. *1 A(n* ***undirected****)* ***graph****(or* ***ugraph****) G = (V, E) consists of*
V, a nonempty set of ***vertices*** *(or* ***nodes****) and E, a set of* ***edges****.*
Each edge has either ***one****? or two* ***vertices*** *associated with it,*
called ***endpoints****. An* ***edge*** *is said to* ***connect*** *its* ***endpoints****.*

infinite graph *infinite vertices and/**or** edges*
finite graph *finite vertices and edges*
But finite graph is considered in this text.

Let u, v ∈ V. Then we ***write*** *{u, v} ∈ E,* ***or*** *{u} ∈ E.*
The edge {u} is called the self-loop edge connectint the vertex u to u.
See Fig. 3 of p 618.

Def. *2 A **directed graph**(**digraph**, **graph**) on the set of **vertices** V is,*

$G = (V, E)$ where $E \subseteq V \times V$.

*A pair $(u, v) \in E$ is called **edge**(**arc**) and said to*

***starts** at the **vertex**(**node**) u and **ends** at the **vertex** v.*

*$(u, v) \neq (v, u)$ and $(u, u) \in E$ **self-loop***

***simple graph** **single edge** for each two vertices*

***multigraph** **multiple edges** for the same (two) vertices*

m different edges, $\{u, v\}$ connecting the vertices u and v.

We say that the edge $\{u, v\}$ has muliplicity of n.

*$G = (V, E, f)$ **Edge weighted** (**simple**) **graph simulates multigraph***

$f: E \to \mathbf{N}$ or $V \times V \to \mathbf{N}$

*Egde $\{u, v\}$ of **multiplicity** m, if $f(\{u, v\}) = m \in \mathbf{N}$.*

Edges *of* ***digraph*** *is a binary* ***relation*** *on V ($E \subseteq V \times V$).*

- ***Reflexivity*** *All vertices have a self-loop*
- ***Irreflexivity*** *No verteces have a self-loop*
- ***Symmetry*** *All edges are bidirectional*
- ***Antisymmetry*** *All edges are unidirectional, self-loop is allowded*
- ***Asymetry*** *All edges are unidirectional, self-loop is* **not** *allowded*
- ***Transitivity*** *All paths should have an (**extra**) edge*

Digraph simulates (undirected) graph *and* ***pseudograph***

- *$\{u, v\} \in E$ in ugraph $(u, v), (v, u) \in E$ in digraph symmetricity*
- *$\{u\} \in E$ in pseudograph $(u, u) \in E$ in digraph reflexivity*

Edge weighted graph simulates multigraph

Graph Models

Read the text book from p620 to p625.

10.2 Graph Terminology and Special Types of Graphs

Def. *1* ***Adjacency*** *Let G = (V, E) be an* ***ugraph****(undirected graph).*

If edge {u, v} ∈ E, then

Vertices u, v are ***adjacent****(****neighbors****,* ***connected****)*

Edge {u, v} ***connects****(is* ***incedent*** *with) vertices u and v.*

Vertices u and v are ***endopints*** *of the edge {u, v}.*

Def. 2 Neighbohood*: Let G = (V, E) be an* ***graph*** *and v ∈ V. Then*

$N(v) = \{u \in V \mid \{u, v\} \in E\}$, called ***neighbohood*** *of v*

the set of ***all*** *neighbors of v.*

Let $A \subseteq V$. Then we ***define*** *$N(A) = \cup_{v \in A} N(v)$.*

Def. 3 <u>Degree of a vertex</u>*: Let G = (V, E) be an* ***ugraph*** *and v ∈ V. Then*

deg(v) ∈ **N**. *is the* ***number*** *edges* ***incedent*** *with it*

(except a ***self-loop*** *counts twice)*

$deg(v) = |\{\{v, u\} \in E\}| + 2 \cdot |\{\{v\} \in E\}|$

If deg(v) = 0, v is called ***isolated****.*

If deg(v) = 1, v is called ***pendant****.(목걸이 , 팔찌)*

self-loop

Def. 3' <u>Degree of a vertex</u>*: Let G = (V, E) be a* ***digraph(directed graph)*** *and v ∈ V. Then*

deg(v) ∈ **N** *is the* ***number*** *edges* ***incedent*** *with it*

$deg(v) = |\{(u, v) \in E\}| + |\{(v, u) \in E\}|$

indegree ***outdegree***

Thm. 1 The Hand Shaking Theorem

Let G = (V, E) be an ***ugraph*** *with |E| = m. Then* $2m = \sum_{v \in V} deg(v)$.

proof *Every edge contibutes two to the sum of the degrees.*

Thm *2 Let G = (V, E) be an* ***ugraph****. Then G has an* ***even*** *number of* ***vertices*** *of* ***odd*** *degree.*

proof *Let* V_e *be the set of vertices of* ***even*** *degree and* V_o *be the set of vertices of* ***odd*** *degree.*

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in Ve} deg(v) + \sum_{v \in Vo} deg(v)$$

$|\sum_{v \in Vo} deg(v)|$ *is even.* $\therefore$ $|V_o|$ *is even.*

Def. *4* ***<u>Adjacency</u>*** *Let G = (V, E) be a* ***digraph****. If edge (u, v)* $\in$ *E, then u is said to be* ***adjacent to*** *v and v is said to be* ***adjacent from*** *u. The vertex u is called the* ***initial*** *vertex of (u, v) and the vertex v is called the* ***terminal*** *or* ***end*** *vertex of (u, v).*

Def. 5 *Let G = (V, E) be a* ***<u>digraph</u>*** *and* $v \in V$.

<u>Out-degree</u> *of v, denoted as,* $deg^-(v)$, *is*

the number of edges with v as their ***termina*** *vertices.*

$deg^-(v) = |\{(u, v) \in E\}|$

<u>In-degree</u> *of v, denoted as,* $deg^+(v)$, *is*

the number of edges with v as their ***initial*** *vertices.*

$deg^+(v) = |\{(v, u) \in E\}|$

Thm. 3 *Let G = (V, E) be a* ***digraph****. Then*

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

Some Special Simple Graphs

Ex. 5 Complete graph, *for* $n \geq 1$, *n vertices* $K_n = (V_n, E_n)$, *has one edge between* ***each pair*** *of* ***distinct*** *vertices.*

$K_n = (V_n, E_n)$ where $V_n = \mathbf{N}_{1,n}$ and

$E_n = \{(i, j) \in (\mathbf{N} \times \mathbf{N}) | 1 \leq i < j \leq n\}$

K_n has $n(n-1)/2$ edges

Ex. 6 Cycle, *for* $n \geq 3$, $C_n = (V_n, E_n)$ *n vertices and n edges.*

$E_n = \{(i, i+1) \in (\mathbf{N} \times \mathbf{N})\} \cup \{(1, n)\}$

Ex. 7 Wheel, *for* $n \geq 3$, $W_n = (V_n, E_n)$

C_n *(a* ***cycle*** *of n vertices) and*

edges from a ***hub*** *vertex* (v_0) *and to* C_n.

$\therefore$ *n+1 vertices, 2n edges.*

$V_n = \mathbf{N}_{0,n}$ and $E_n = \{(i, j) \in (\mathbf{N} \times \mathbf{N}) | i \neq j\} \cup \{(0, j) \in (\mathbf{N} \times \mathbf{N})\}$

Ex. 8 n-cube, For $n \geq 0$, $Q_n = (V_n, E_n)$ with $|V_n| = 2^n$.

$V_n = \{b_n^i \in \{0, 1\}^n | 0 \leq i \leq 2^n-1\}$ b_n^i: n-bits binary string of $i \in \mathbf{N}$

$E_n = \{(b_n^i, b_n^j) | b_n^i, b_n^j \in \{0, 1\}^n$, b_n^i and b_n^j **differs only one** bit$\}$.

Construct Q_{n+1} using Q_n and Q'_n(copy of Q_n).

$Q_0 = (\{v_0^0\}, \{\})$. ***basis***

$Q_{n+1} = (V_{n+1}, E_{n+1})$ where

$V_n^Z = \{\mathbf{0} \cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading **zero**

$V_n^O = \{\mathbf{1} \cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading **one**

$V_{n+1} = V_n^Z \cup V_n^O = \{b_{n+1}^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^{n+1}-1\}$

$E_{n+1} = E_n^Z \cup E_n^O \cup \{(\mathbf{0} \cdot b_n^i, \mathbf{1} \cdot b_n^i) \in V_n^Z \times V_n^O | 0 \leq i \leq 2^n-1\}$

recursion

Bipartite Graph

G = (V, E) is bipartite, iff $V = V_1 \cup V_2$ *where* $V_1 \cap V_2 = \varnothing$ *and*

$\forall\{v_1, v_2\} \in E, v_1 \in V_1, v_2 \in V_2$.

Partition $\{V_1, V_2\}$ *is called* ***bipartition***.

Complete Bipartite Graph, $K_{n,m} = (V_1 \cup V_2, E)$ *where* $V_1 \cap V_2 = \varnothing$ *and*

$|V_1| = n, |V_2| = m$, *and* $E = \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\}$

Thm. 4 *A simple graph, is bipartite ,if and only if,*

it is ***2-colorable****.(see Fig. 7, Fig. 8, and Sec. 10.8)*

Every ***path****,* ***tree****, and* ***even length cycles***(C_{2n}) *are bipartite.*

odd vetecies and even vertecies

Theorem *A graph is* ***bipartite****, if and only if,*

it contains ***no odd lengh cycle****.*

New Grapgs from old

Def. 7 *A* ***subgraph*** *of graph $G = (V, E)$ is $G' = (V', E')$,*
where $V' \subseteq V$ and $E' \subseteq E$.
A ***proper subgraph*** *of graph $G = (V, E)$ is $G'' = (V'', E'')$,*
where $V' \subset V$ and $E' \subset E$.

Def. 8 *A* ***subgraph*** *of graph $G = (V, E)$* ***induced*** *subvertices $V'(V' \subseteq V)$ is $G' = (V', E')$, where $E = \{(u, v) | u, v \in V'\}$.*

Def. *9 The* ***union*** *of two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is $G_{G_1 \cup G_2} = (V_1 \cup V_2, E_1 \cup E_2)$.*

Let G' = (V', E') is a subgraph of G = (V, E). Then
G is called ***spanning subgraph*** *of G, if V' = V.*

Let G' = (V', E') is a subgraph of G = (V, E). Then
G" = (V", E") is called the ***complement*** *of G' with respect to G*
where E" = E – E', and $V'' = \{a \in V \mid (a, b) \in E''\}$

The ***complement*** *of ugraph G = (V, E) where |V| = n,*
with respect to K_n*, denoted* $\overline{G}$*, is the (****absolute****)* ***complement*** *of G.*

10.3 Representing Graphs and Graph Isomorphism

Adjacecency List

Adjacecency Matrices

$|V| \times |V|$ ***boolean*** *matrix*

Isomorphism of Graphs

Def. *1 The graph* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are* ***isomorphic***,

iff $^\exists$*a* ***bijection*** $f: V_1 \leftrightarrow V_2$.э. $^\forall u ^\forall v \in V_1: (u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

Let $Q = (V, E)$ *a graph. A* ***bijection*** $f{:}V \rightarrow V'$ *where* $V \cap V' = \emptyset$. *Then*

$Q' = (\{f(v) \in V' | v \in V\}, \{\{f(v), f(u)\} | \{v, u\} \in E\})$

We may extend f domain and codomain of f from vertices to edges.

$f((v, u)) = (f(v), f(u))$.

$Q' = (f(V), f(E))$

If Q and Q' are ***isomorphic***,

Q' is called an ***isomorphic image*** *of Q.*

n-cube, *revisited.*

i) Q = ({v}, { }) be a ***0-cube****.*

ii) Let Q be an ***n-cude****. Then*

$Q' = (V \cup f(V), E \cup f(E) \cup \{(V, f(V))\}$ *is an (n+1)-cube.*

Graph isomorphism

Example 8, 9, 10

Difficult to check!

Necessary *conditions for graph* ***isomophisms***

$|V_1| = |V_2|$ *and* $|E_1| = |E_2|$.

The number of vertices with degree n is same in both graphs.

Subgraph H of one graph is isomorphic to a subgraph of the other

isomorphic simple cycle

10.4 Connectivity

Def. 1 *Let G = (V, E) be a graph. Then a* ***path*** *of* ***length*** *n(≥0) from u to v in G is a* ***sequence*** *of n* ***edges*** *Path(u, v) = $(e_1, e_2, \ldots, e_n)$ where*

$1 \le \forall i < n: e_i = (v_i^s, v_i^e), v_i^e = v_{i+1}^s, e_1^s = u, e_n^e = v.$

Consider two sequences of n edges and (n+1) vertices

$((v_0, v_1),(v_1, v_2),\ldots, (v_{n-1}, v_n))$ vs. $(v_0, v_1, \ldots, v_n)$ *where* $v_0 = u, v_n = v$.

What is the ***path*** *of length 0?* *()*

A path is a ***circuit*** *if u = v, for n ≥ 1.*

A path ***pass through*** *the vertices* $v_1, \ldots, v_{n-1}$ *or* ***traverse*** *along it.*

A path is ***simple****, if it contains no* ***edges*** *more than once.*

Def. 1 *Let G = (V, E) be a graph. Then a* ***path*** *of* ***length*** *n(≥0) from u to v in G is a* ***sequence*** *of (n+1)* ***vertices*** *Path(u, v) = $(v_0, v_1, \ldots, v_n)$ where*

$v_0 = u, v_n = v, \text{ and } 0 \le \forall i < n: (v_i, v_{i+1}) \in E.$

Note that Path(u, u) = (u) is a ***path*** *of length 0* ***without*** *edge.*

*A graph is **simple cycle** of **length** n, iff it is isomorphic to C_3.*

Def. *2.1 Let G = (V, E) be a digraph. Then a **path** is defined as a **sequence** of vertices $(a_0, a_1, \ldots, a_n)$ with $n \geq 0$ such that*

*$(a_i, a_{i+1}) \in E, 0 \leq \forall i < n$. The path **starts** a_o and **ends** a_n.*

*The **length** of the path is defined to be n.*

$(a) \in V$ path of length 0; $n = 0, a_0 = a$.

*A path is a **cycle** if $a_0 = a_n$.*

$a\ E^* b$ there is a path from a to b

$a\ E^+ b$ there is a positive length path from a to b

Def. *2.2 A **cycle** is a positive lengh path that begins and ends **same** vertex. A **directed acyclic graph**(DAG) is a directed graph with **no cycle**.*

Lemma *If G is a DAG, G^+ is a **irreflexive partial order**.*

Def. *3 Two vertices u and v are to said to be* ***connected****,*
if there is a path between them.
A graph is said to be ***connected****,*
if there ***every*** *pair of distinct vertices are connected.*
We write u connects v or u is connected to v, if u and v are connected.

$connected \subseteq V \times V$ *binary relation on V*

Lemma *0.1 The binary relation* ***connected*** *on V is an* ***equivalence*** *relation*

Def. *3.2 The* ***quivalence class*** *defined by the equivalence relation* ***connected*** *is called* ***connected component.***

$P = \{[v]_{connected} | v \in V\}$ ***partition*** *of V.*

$|P|$ = *number of conncted components*

Def. *3.3 Two vertices in a graph are **k-connected** if they **remain connected** in any subgraph by **deleting** k–1 edges.*

*A graph is **k-connected** if every pair of vertices are k-connected.*

simple cycle *2-connected*

K_n *(k–1)-connected*

Thm *1 If vertex v is connected to vertex u in a graph,*

*there is a **simple path** from u to v.*

proof *Consider **minimum** length path from u to v.*

$v_o, v_1, \ldots, v_k$ *where* $v_o=u$, $v_k=v$ *with* $k \geq 2$*(if* $k \leq 1$*, simple) is simple.*

Assume it is not simple, $0 \leq \exists i < \exists j < n$ *.϶.* $v_i = v_j$*. Not minimum path!*

Corollary *1.1 For any path of **length** k in graph, there is a **simple** path of length at **most** k with the same endpoints.*

Thm *1.1 Every graph G = (V, E) has*

***at least** |V| – |E| **connected components**.*

***proof** P(n): G = (V, E) with |E| = n has at least |V| – n C. C.*

***base**: |E| = 0, |V| connected components.*

***induction**: Consider G = (V, E) with n+1 edges.*

***Remove** an edge {u, v} and call the resulting graph G'.*

G' has at least |V| – n connected components.(I. H.)

***Add back** the edge {u, v}.*

*case 1: u and v are in the **same** C. C. **Same** number of components*

G has at least |V| – n > |V| – (n+1) components.

*case 2: u and v are in the **diffrent** C. C. One **less** component*

G has at least |V| – n – 1 = |V| – (n+1) components.

***Corollary**1.2 The **connected** graph with n vertices has **at least** n-1 edges.*

proof $1 \geq |V| - |E|$ $|E| \geq |V| - 1$

10.5 Euler and Hamilton Path

***Def**. 1 Let G be a graph.*
*An **Euler path** is a path containing **every edge** of G.*
*An **Euler circuit** is a circuit containing **every edge** of G.*

***Thm**. 1 A connected graph has **Euler circuit** iff each vertex has **even degree**.*
***proof** (→) The circuit contributes 2 to the degree of each node*
(←) Algorithm 1

***Alg**. 1 Constructing Euler Circuit*
*Begins with **arbitrary** node.*
*Construct a **cycle** form the vertex to the vertex.*
*Repeat for each **remaining** subgraph,*
***inserting** the **new cycle** into the **original** one.*

Thm. *2 A connected graph has* ***Euler path*** *iff it has exaclty* ***two*** *vertices of* ***odd degree***.
proof *One is the start, the other is the end.*

Def. *2 Let G be a graph.*
An ***Hamilton path*** *is a path traverse each vertex ib G exaclty* ***once***.
An ***Hamilton circuit*** *s a circuit traverse each vertex ib G exaclty* ***once***.

K_n *has a Hamilton circuit circuit*$(n \geq 3)$.

If a graph has a vertex of ***degree one***, *there is* ***no Hamilon*** *circuit.*
Exactly ***two edges*** *incident to a vertex are in Hamaiton circuit.*

Trvaling Saleman
NP complete

10.6 Shortest-Path Problem

Let G = (V, E) be a graph, and f: $E \to \mathbf{R}^+$ *be a* ***cost*** *of the edges. Then (V, E, f) is an edge* ***weighted*** *graph or multigraph(**N**) in this text.*

Iterative *definition(extension) of* ***cost*** *of* ***path****.*

Let $(v_0, v_1, \ldots, v_n) \in E^*$ *be a path of length* $n \geq 0$*. Then we define*

if $n = 0$, $f^*(v_0, v_0) = 0$.

if $n \geq 1$, $f^*(v_0, v_n) = \sum_{j=0}^{n-1} f(v_j, v_{j+1})$

Recursive *definition(extension) of* ***cost*** *of* ***path****.*

$f^*(u, u) = 0.$

$f^*(u, v) = f(u, x) + f^*(x, v)$ *or* $f^*(u, x) + f(x, v)$.

We may use f instead of f^**, since* $f \subseteq f^*$*.*

$f: E^* \to \mathbf{R}^+$ ***extend*** *the domain of f from E to* E^**.*

Shortest path problem

Let $G = (E, V, f)$ be an edge weighted graph.

Definition $min_f(u, v) = (u, v_1, \ldots, v_n, v) \in E^*$ $n \geq 0$.э.

$f(u, v_1, \ldots, v_n, v) \leq f(u, u_1, \ldots u_m, v)$.э. $^\forall m \geq 0$ $^\forall(u, u_1, \ldots, u_m, v) \in E^*$.

Find u, v ∈ V, find a shortest path such that f(u, v) is minimum.

Definition *Let* $W \subseteq V$, $u, v \in V$. $L_f^W(u, v) = (u, w_1, \ldots w_n, v)$.э.

$f(u, w_1, \ldots w_n, v) \leq f(u, x_1, \ldots, x_m, v)$.э. $\forall m$, $(u, x_1, \ldots, x_m, v) \in E^*$

$n, m \geq 0$, $1 \leq \forall i \leq n$, $w_i \in W$, $1 \leq \forall i \leq m$, $x_i \in W$.

Note that u, v may be in W or not.

Thm. *Let* $W \subseteq V$, $u \in V$, *and* $x \notin W$.э. $L_f^W(u, x)$ *is the* ***minimum***.

$$\forall y \notin W,\ L_f^{W \cup \{x\}}(u, y) = min(L_f^W(u, y),\ L_f^W(u, x) + f(x, y)).$$

procedure *Shortest path* $G = (V, E \subseteq V \times V, f: E \to \mathbf{R}^+)$

for $w \in V$ ***do*** $L(w) :=$ *Infinite* ***od***; $L(u), W := 0, \varnothing$;

Initialization $\forall y \in V, L(y) = L_f^{\{\}}(u, y)$.

do $v \notin W \to$

Loop invariance $\forall y \in V, L(y) = L_f^{W}(u, y)$.

$x := x \notin W \wedge \min(L(x)); W := W \cup \{x\};$

Loop invariance, invalidated $\forall y \in V, L(y) = L_f^{W-\{x\}}(u, y)$.

do $y \notin W \wedge (x, y) \in E \to L(y) := \min(L(y), L(x) + f(x, y))$ ***od***

Loop invariance, validated $\forall y \in V, L(y) = L_f^{W}(u, y)$.

od

After the loop termination $\forall y \in V, L(y) = L_f^{W}(u, y) \wedge v \in W$.

We should also prove that $L_f^{W}(u, v) = \min_f(u, v)$, *if* $v \in W$.

But it is trivial, since we add the shortest path vertex x to W.

10.7 Planar Graph

Def. *1 A graph is* ***planar****, if it can be drawn in the* ***plane without edge crossing****.*

K_4 is planar and Q_3 is planar, but $K_{3,3}$ is not planar.

Region*: planar graph divides regions*

Tree *has* ***one*** *region*

Removing an edge *on the* ***cycle*** *merges* ***two*** *regions into* ***one****.*

Thm *1 Let G be a* ***connected planar graph*** *with v* ***vertices*** *and e* ***edges****.*
Let r be the number of ***regions*** *in planar reprsentation of G. Then*

$$r = e - v + 2.$$

proof *Induction on number of edges, P(e).*

basis *e=0, v=1, r=1.* $\therefore$ *1 = 0 – 1 + 2 = 1. O.K.*

induction *Consider a connected planar graph G with e+1 edges.*

1. If G is ***acyclic****. G is* ***tree****.* $\therefore$ *r = 1, e – v + 2 = –1 + 2 =1.* $\therefore$ *O.K.*

2. If G is not acyclic, G has at least one ***cycle****, C.*

Consider {u, v} ***in*** *C and a* ***spanning tree****, T. .э. {u, v} is* ***not*** *in T.*

There exisits such an edge {u, v} because T is acyclic.

Remove *the* ***edge*** *{u, v} from G, it is called as G'.*

G' ***has one less regions****, since removing an edge {u, v} on the cycle C.*

G' is connected planar graph and has e edges.

$\therefore$ *r = e – v + 2 in G'* *induction hypothesis*

In G, r+1 regions, e+1 edges, and v vertices. $\therefore$ *(r +1) = (e+1) – v + 2.*

Col. *1 If connected planar graph G with e edges and v vertices where $v \geq 3$. Then $e \leq 3v - 6$.*

proof *Consider degree of region number of boundary edges*

Sum of the degree of regions is 2e.

Minimum number of degree of each region r is 3r(triangle)

$\therefore\ 2e \geq 3r = 3(e - v + 2)$ $\therefore\ e \leq 3v - 6$.

Col. *2 If G is a connected planar graph, then G has a vertex not exceeding five.*

proof *If G has one or two vertices, the result is true.*

If G has more that three vertices, $e \leq 3v - 6$. So $2e \leq 6v - 12$.

If every vertex has more than or equal 6, $2e \geq 6v$(Handshaking Theorem)

But it contradict with $2e \leq 6v - 12$.

Ex. *5 K_5 is not planar.*

$v = 5$, $e = 10$. $3v - 6 = 9$.

10.8 Graph Coloring

***Def**. 1 A graph G is k-**colorable**, if each vertex can be assigned one of k colors so that **adjacent vertices** get the **different colors**.*

map coloring problem

*The smallest number of colors are called **chromatic number** of G, written as $\chi(G)$. $\chi(K_n) = n$.*

***Thm**. 1 The **chromatic** number of **planar** graph is no greater than 4.*

***Thm.** 2 A graph with maximum degree at most k is (k+1)-colorable.*

***proof** Induction on number of **vertices**. P(n).*

***basis** 1-vertex graph, maximum degree 0 and 1-colorable. P(1) is true.*

***induction** Let G be (n+1)-**vertex** graph with maximum degree at most k*

***Remove** a vertex v and its incident edges.*

G' has n vertices and max. deg. at most k. ∴ G' is (k+1)-colorable(IH).

v has at most k adjacent vertices. ∴ G is (k+1)-colorable((k+1) – k = 1).