# 10 Graphs

## 10.1 Graphs and Graph Models

***Def.*** *1 A(n* ***undirected****)* ***graph****(or* ***ugraph****) G = (V, E) consists of*
*V, a nonempty set of* ***vertices*** *(or* ***nodes****) and E, a set of* ***edges****.*
*Each edge has either* ***one***? *or two* ***vertices*** *associated with it, called* ***endpoints****. An* ***edge*** *is said to* ***connect*** *its* ***endpoints****.*

***infinite graph*** *infinite vertices and/**or** edges*
***finite graph*** *finite vertices and edges*
*But finite graph is considered in this text.*

*Let u, v ∈ V. Then we* ***write*** *{u, v} ∈ E,* ***or*** *{u} ∈ E.*
*The edge {u} is called the self-loop edge connectint the vertex u to u.*
*See Fig. 3 of p 618.*

***Def.*** *2 A* ***directed graph(digraph***, ***graph****) on the set of* ***vertices*** *V is,*

$G = (V, E)$ *where* $E \subseteq V \times V$.

*A pair* $(u, v) \in E$ *is called* ***edge(arc)*** *and said to* ***starts*** *at the* ***vertex(node)*** *u and* ***ends*** *at the* ***vertex*** *v.*

$(u, v) \neq (v, u)$ *and* $(u, u) \in E$ ***self-loop***

***simple graph*** ***single edge*** *for each two vertices*

***multigraph*** ***multiple edges*** *for the same (two) vertices*

*m different edges, {u, v} connecting the vertices u and v.*

*We say that the edge {u, v} has muliplicity of n.*

$G = (V, E, f)$ ***Edge weighted (simple) graph simulates multigraph***

$f: E \rightarrow \mathbf{N}$ *or* $V \times V \rightarrow \mathbf{N}$

*Egde {u, v} of* ***multiplicity*** *m, if* $f(\{u, v\}) = m \in \mathbf{N}$.

***Edges*** *of* ***digraph*** *is a binary* ***relation*** *on V* $(E \subseteq V \times V)$.

- ***Reflexivity*** *All vertices have a self-loop*
- ***Irreflexivity*** *No verteces have a self-loop*
- ***Symmetry*** *All edges are bidirectional*
- ***Antisymmetry*** *All edges are unidirectional, self-loop is allowded*
- ***Asymetry*** *All edges are unidirectional, self-loop is* **not** *allowded*
- ***Transitivity*** *All paths should have an (**extra**) edge*

***Digraph simulates (undirected) graph*** *and* ***pseudograph***

- $\{u, v\} \in E$ *in ugraph* $(u, v), (v, u) \in E$ *in digraph* *symmetricity*
- $\{u\} \in E$ *in pseudograph* $(u, u) \in E$ *in digraph* *reflexivity*

***Edge weighted graph simulates multigraph***

## ***Graph Models***

*Read the text book from p620 to p625.*

## *10.2 Graph Terminology and Special Types of Graphs*

***Def.*** *1* ***Adjacency*** *Let G = (V, E) be an* ***ugraph****(undirected graph).*

*If edge {u, v} ∈ E, then*

*Vertices u, v are* ***adjacent****(****neighbors****,* ***connected****)*

*Edge {u, v}* ***connects****(is* ***incedent*** *with) vertices u and v.*

*Vertices u and v are* ***endopints*** *of the edge {u, v}.*

***Def. 2*** ***Neighbohood****: Let G = (V, E) be an* ***ugraph*** *and v ∈ V. Then*

*N(v) = {u ∈ V| {u, v} ∈ E}, called* ***neighbohood*** *of v*

*the set of* ***all*** *neighbors of v.*

*Let A ⊆ V. Then we* ***define*** $N(A) = \cup_{v \in A} N(v)$.

***Def. 3*** ***<u>Degree of a vertex</u>***: *Let G = (V, E) be an* ***ugraph*** *and v* $\in$ *V. Then*

*deg(v)* $\in$ **N**. *is the* ***number*** *edges* ***incedent*** *with it*

*(except a* ***self-loop*** *counts twice)*

$deg(v) = |\{\{v, u\} \in E\}| + 2 \cdot |\{\{v\} \in E\}|$

*If deg(v) = 0, v is called* ***isolated***.

*If deg(v) = 1, v is called* ***pendant***.*( 목걸이 , 팔찌 )*

***self-loop***

***Def. 3'*** ***<u>Degree of a vertex</u>***: *Let G = (V, E) be a* ***digraph(directed graph)*** *and v* $\in$ *V. Then*

*deg(v)* $\in$ **N** *is the* ***number*** *edges* ***incedent*** *with it*

$deg(v) = |\{(u, v) \in E\}| + |\{(v, u) \in E\}|$

***indegree*** ***outdegree***

***Thm. 1 The Hand Shaking Theorem***

*Let G = (V, E) be an* ***ugraph*** *with |E| = m. Then 2m =* $\sum_{v \in V} deg(v)$.

***proof*** *Every edge contibutes two to the sum of the degrees.*

***Thm*** *2 Let G = (V, E) be an* ***ugraph****. Then G has an* ***even*** *number of* ***vertices*** *of* ***odd*** *degree.*

***proof*** *Let* $V_e$ *be the set of vertices of* ***even*** *degree and* $V_o$ *be the set of vertices of* ***odd*** *degree.*

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in Ve} deg(v) + \sum_{v \in Vo} deg(v)$$

$|\sum_{v \in Vo} deg(v)|$ *is even.* $\therefore$ $|V_o|$ *is even.*

***Def.*** *4* ***<u>Adjacency</u>*** *Let G = (V, E) be a* ***digraph****. If edge (u, v) ∈ E, then u is said to be* ***adjacent to*** *v and v is said to be* ***adjacent from*** *u. The vertex u is called the* ***initial*** *vertex of (u, v) and the vertex v is called the* ***terminal*** *or* ***end*** *vertex of (u, v).*

***Def. 5*** *Let G = (V, E) be a* ***digraph*** *and* $v \in V$.

***Out-degree*** *of v, denoted as,* $deg^{-}(v)$*, is*

*the number of edges with v as their* ***termina*** *vertices.*

$$deg^{-}(v) = |\{(u, v) \in E\}|$$

***In-degree*** *of v, denoted as,* $deg^{+}(v)$*, is*

*the number of edges with v as their* ***initial*** *vertices.*

$$deg^{+}(v) = |\{(v, u) \in E\}|$$

***Thm. 3*** *Let G = (V, E) be a* ***digraph****. Then*

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

## Some Special Simple Graphs

***Ex. 5 Complete graph***, *for* $n \geq 1$, *n vertices* $K_n = (V_n, E_n)$, *has one edge between **each pair** of **distinct** vertices.*

$K_n = (V_n, E_n)$ *where* $V_n = \mathbf{N}_{1,n}$ *and*

$E_n = \{(i, j) \in (\mathbf{N} \times \mathbf{N}) \mid 1 \leq i < j \leq n\}$

$K_n$ *has* $n(n-1)/2$ *edges*

***Ex. 6 Cycle***, *for* $n \geq 3$, $C_n = (V_n, E_n)$ *n vertices and n edges.*

$E_n = \{(i, i+1) \in (\mathbf{N} \times \mathbf{N})\} \cup \{(1, n)\}$

***Ex. 7 Wheel***, *for* $n \geq 3$, $W_n = (V_n, E_n)$

$C_n$ *(a **cycle** of n vertices) and*

*edges from a **hub** vertex* $(v_0)$ *and to* $C_n$.

$\therefore$ *n+1 vertices, 2n edges.*

$V_n = \mathbf{N}_{0,n}$ *and* $E_n = \{(i, j) \in (\mathbf{N} \times \mathbf{N}) \mid i \neq j\} \cup \{(0, j) \in (\mathbf{N} \times \mathbf{N})\}$

***Ex. 8 n-cube***, For $n \geq 0$, $Q_n = (V_n, E_n)$ with $|V_n| = 2^n$.

$V_n = \{b_n^i \in \{0, 1\}^n | 0 \leq i \leq 2^n-1\}$ $b_n^i$: n-bits binary string of $i \in \mathbf{N}$

$E_n = \{(b_n^i, b_n^j) | b_n^i, b_n^j \in \{0, 1\}^n, b_n^i$ and $b_n^j$ ***differs only one*** bit$\}$.

Construct $Q_{n+1}$ using $Q_n$ and $Q'_n$(copy of $Q_n$).

$Q_0 = (\{v_0^0\}, \{\})$. ***basis***

$Q_{n+1} = (V_{n+1}, E_{n+1})$ where

$V_n^Z = \{\mathbf{0}\cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading ***zero***

$V_n^O = \{\mathbf{1}\cdot b_n^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^n-1\}$ leading ***one***

$V_{n+1} = V_n^Z \cup V_n^O = \{b_{n+1}^i \in \{0, 1\}^{n+1} | 0 \leq i \leq 2^{n+1}-1\}$

$E_{n+1} = E_n^Z \cup E_n^O \cup \{(\mathbf{0}\cdot b_n^i, \mathbf{1}\cdot b_n^i) \in V_n^Z \times V_n^O | 0 \leq i \leq 2^n-1\}$

***recursion***

## Bipartite Graph

*$G = (V, E)$ is bipartite, iff $V = V_1 \cup V_2$ where $V_1 \cap V_2 = \varnothing$ and*

*$\forall\{v_1, v_2\} \in E, v_1 \in V_1, v_2 \in V_2$.*

*Partition $\{V_1, V_2\}$ is called* ***bipartition****.*

***Complete Bipartite Graph****, $K_{n,m} = (V_1 \cup V_2, E)$ where $V_1 \cap V_2 = \varnothing$ and*

*$|V_1| = n, |V_2| = m$, and $E = \{\{v_1, v_2\} | v_1 \in V_1, v_2 \in V_2\}$*

***Thm. 4*** *A simple graph, is bipartite ,if and only if,*

*it is* ***2-colorable****.(see Fig. 7, Fig. 8, and Sec. 10.8)*

*Every* ***path****,* ***tree****, and* ***even length cycles****($C_{2n}$) are bipartite.*

*odd vetecies and even vertecies*

***Theorem*** *A graph is* ***bipartite****, if and only if,*

*it contains* ***no odd lengh cycle****.*

## New Grapgs from old

***Def. 7*** *A* ***subgraph*** *of graph $G = (V, E)$ is $G' = (V', E')$,*

*where $V' \subseteq V$ and $E' \subseteq E$.*

*A* ***proper subgraph*** *of graph $G = (V, E)$ is $G'' = (V'', E'')$,*

*where $V' \subset V$ and $E' \subset E$.*

***Def. 8*** *A* ***subgraph*** *of graph $G = (V, E)$* ***induced*** *subvertices $V'(V' \subseteq V)$ is $G' = (V', E')$, where $E = \{(u, v) | u, v \in V'\}$.*

***Def.*** *9 The* ***union*** *of two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is $G_{G_1 \cup G_2} = (V_1 \cup V_2, E_1 \cup E_2)$.*

*Let G' = (V', E') is a subgraph of G = (V, E). Then*
*G is called* ***spanning subgraph*** *of G, if V' = V.*

*Let G' = (V', E') is a subgraph of G = (V, E). Then*
*G" = (V", E") is called the* ***complement*** *of G' with respect to G*
*where E" = E – E', and* $V'' = \{a \in V \mid (a, b) \in E''\}$

*The* ***complement*** *of ugraph G = (V, E) where |V| = n,*
*with respect to* $K_n$*, denoted* $\overline{G}$*, is the (**absolute**)* ***complement*** *of G.*

# *10.3 Representing Graphs and Graph Isomorphism*

## ***Adjacecency List***

## ***Adjacecency Matrices***

$|V| \times |V|$ ***boolean*** *matrix*

## ***Isomorphism of Graphs***

***Def****. 1 The graph* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are* ***isomorphic****,*

*iff* $^{\exists}$*a* ***bijection*** $f: V_1 \leftrightarrow V_2$ .э. $^{\forall}u^{\forall}v \in V_1$: $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

*Let* $Q = (V, E)$ *a graph. A* ***bijection*** $f{:}V \rightarrow V'$ *where* $V \cap V' = \varnothing$. *Then*

$Q' = (\{f(v) \in V' | v \in V\}, \{\{f(v), f(u)\} | \{v, u\} \in E\})$

*We may extend f domain and codomain of f from vertices to edges.*

$f((v, u)) = (f(v), f(u))$.

$Q' = (f(V), f(E))$

*If Q and Q' are* ***isomorphic****,*

*Q' is called an* ***isomorphic image*** *of Q.*

***n-cube***, *revisited.*

*i)* $Q = (\{v\}, \{\})$ *be a* ***0-cube***.

*ii) Let Q be an* ***n-cude***. *Then*

$Q' = (V \cup f(V), E \cup f(E) \cup \{(V, f(V))\}$ *is an (n+1)-cube.*

*Graph isomorphism*

*Example 8, 9, 10*

*Difficult to check!*

***Necessary*** *conditions for graph* ***isomophisms***

$|V_1| = |V_2|$ *and* $|E_1| = |E_2|$.

*The number of vertices with degree n is same in both graphs.*

*Subgraph H of one graph is isomorphic to a subgraph of the other*

*isomorphic simple cycle*

## *10.4 Connectivity*

***Def. 1*** *Let G = (V, E) be a graph. Then a **path** of **length** n(≥0) from u to v in G is a **sequence** of n **edges** Path(u, v) = ($e_1$, $e_2$, …, $e_n$) where*

$$1 \leq \forall i < n: e_i = (v_i^s, v_i^e), v_i^e = v_{i+1}^s, e_1^s = u, e_n^e = v.$$

*Consider two sequences of n edges and (n+1) vertices*

*$((v_0, v_1),(v_1, v_2),\ldots, (v_{n-1}, v_n))$ vs. $(v_0, v_1, \ldots, v_n)$ where $v_0 = u$, $v_n = v$.*

*What is the **path** of length 0? ()*

*A path is a **circuit** if u = v, for n ≥ 1.*

*A path **pass through** the vertices $v_1, \ldots, v_{n-1}$ or **traverse** along it.*

*A path is **simple**, if it contains no **edges** more than once.*

***Def. 1*** *Let G = (V, E) be a graph. Then a **path** of **length** n(≥0) from u to v in G is a **sequence** of (n+1) **vertices** Path(u, v) = ($v_0$, $v_1$, …, $v_n$) where*

$$v_0 = u, v_n = v, \text{ and } 0 \leq \forall i < n: (v_i, v_{i+1}) \in E.$$

*Note that Path(u, u) = (u) is a **path** of length 0 **without** edge.*

*A graph, C, is **simple cycle** of **length** n, iff it is isomorphic to $C_n$ for some $n \geq 3$.*

***Def.*** *2.1 Let $G = (V, E)$ be a digraph. Then a **path** is defined as a **sequence** of vertices $(a_0, a_1, \ldots, a_n)$ with $n \geq 0$ such that*

*$(a_i, a_{i+1}) \in E$, $0 \leq \forall i < n$. The path **starts** $a_o$ and **ends** $a_n$.*

*The **length** of the path is defined to be n.*

*$(a) \in V$ path of length 0; $n = 0$, $a_0 = a$.*

*A path is a **cycle** if $a_0 = a_n$.*

*$a\ E^*\ b$ there is a path from a to b*

*$a\ E^+\ b$ there is a positive length path from a to b*

***Def.*** *2.2 A* ***cycle*** *is a positive lengh path that begins and ends* ***same*** *vertex. A* ***directed acyclic graph****(DAG) is a directed graph with* ***no cycle****.*

***Lemma*** *If G is a DAG,* $G^+$ *is a* ***irreflexive partial order****.*

***Def.*** *3 Two vertices u and v are to said to be* ***connected****,*
*if there is a path between them.*
*A graph is said to be* ***connected****,*
*if there* ***every*** *pair of distinct vertices are connected.*
*We write u connects v  or u is connected to v, if u and v are connected.*

*connected* $\subseteq V \times V$ *binary relation on V*

***Lemma*** *0.1 The binary relation* ***connected*** *on V*
*is an* ***equivalence*** *relation*

***Def.*** *3.2 The* ***quivalence class*** *defined by the equivalence relation*
***connected*** *is called* ***connected component.***

$P = \{[v]_{connected} | v \in V\}$ ***partition*** *of V.*

$|P|$ *= number of conncted components*

***Def.*** *3.3 Two vertices in a graph are* ***k-connected*** *if they* ***remain connected*** *in any subgraph by* ***deleting*** *k–1 edges.*

*A graph is* ***k-connected*** *if every pair of vertices are k-connected.*

| | |
|---|---|
| *simple cycle* | *2-connected* |
| $K_n$ | *(k–1)-connected* |

***Thm*** *1 If vertex v is connected to vertex u in a graph,*

*there is a* ***simple path*** *from u to v.*

***proof*** *Consider* ***minimum*** *length path from u to v.*

$v_o, v_1, \ldots, v_k$ *where* $v_o=u$, $v_k=v$ *with* $k \geq 2$ *(if* $k \leq 1$*, simple) is simple.*

*Assume it is not simple,* $0 \leq \exists i < \exists j < n$ *.э.* $v_i = v_j$*. Not minimum path!*

***Corollary*** *1.1 For any path of* ***length*** *k in graph, there is a* ***simple*** *path of length at* ***most*** *k with the same endpoints.*

***Theorem*** *1.1 Every graph G = (V, E) has*
***at least*** *|V| – |E|* ***connected components****.*

***proof*** *P(n): G = (V, E) with |E| = n has at least |V| – n C. C.*
***base****: |E| = 0, |V| connected components.*
***induction****: Consider G = (V, E) with n+1 edges.*
- ***Remove*** *an edge {u, v} and call the resulting graph G'.*
- *G' has at least |V| – n connected components.(I. H.)*
- ***Add back*** *the edge {u, v}.*
- *case 1: u and v are in the* ***same*** *C. C.* ***Same*** *number of components*
  - *G has at least |V| – n > |V| – (n+1) components.*
- *case 2: u and v are in the* ***diffrent*** *C. C. One* ***less*** *component*
  - *G has at least |V| – n – 1 = |V| – (n+1) components.*

***Corollary****1.2 The* ***connected*** *graph with n vertices has* ***at least*** *n-1 edges.*
***proof*** $1 \geq |V| - |E|$ $|E| \geq |V| - 1$

## *10.5 Euler and Hamilton Path*

***Definition*** *1 Let G be a graph.*
*An **Euler path** is a path containing **every edge** of G.*
*An **Euler circuit** is a circuit containing **every edge** of G.*

***Theorem*** *1 A connected graph has **Euler circuit** iff each vertex has **even degree**.*
***proof*** *($\rightarrow$) The circuit contributes 2 to the degree of each node*
*($\leftarrow$) Algorithm 1*

***Algorithm*** *1 Constructing Euler Circuit*
*Begins with **arbitrary** node.*
*Construct a **cycle** form the vertex to the vertex.*
*Repeat for each **remaining** subgraph,*
***inserting** the **new cycle** into the **original** one.*

***Theorem*** *2 A connected graph has* ***Euler path*** *iff it has exaclty* ***two*** *vertices of* ***odd degree****.*
***proof*** *One is the start, the other is the end.*

***Definition*** *2 Let G be a graph.*
*An* ***Hamilton path*** *is a path traverse each vertex ib G exaclty* ***once****.*
*An* ***Hamilton circuit*** *s a circuit traverse each vertex ib G exaclty* ***once****.*

$K_n$ *has a Hamilton circuit circuit*$(n \geq 3)$*.*

*If a graph has a vertex of* ***degree one****, there is* ***no Hamilon*** *circuit.*
*Exactly* ***two edges*** *incident to a vertex are in Hamaiton circuit.*

*Trvaling Saleman*
*NP complete*

# *10.6 Shortest-Path Problem*

*Let $G = (V, E)$ be a graph, and $f: E \rightarrow \mathbf{R}^+$ be a **cost** of the edges. Then $(V, E, f)$ is an edge **weighted** graph or multigraph($\mathbf{N}$) in this text.*

***Iterative** definition(extension) of **cost** of **path**.*

*Let $(v_0, v_1, \ldots, v_n) \in E^*$ be a path of length $n \geq 0$. Then we define*

*if $n = 0$, $f^*(v_0, v_0) = 0$.*

*if $n \geq 1$, $f^*(v_0, v_n) = \sum_{j=0}^{n-1} f(v_j, v_{j+1})$*

***Recursive** definition(extension) of **cost** of **path**.*

$f^*(u, u) = 0.$

$f^*(u, v) = f(u, x) + f^*(x, v)$ *or* $f^*(u, x) + f(x, v).$

*We may use f instead of* $f^*$, *since* $f \subseteq f^*$.

$f: E^* \to \mathbf{R}^+$ ***extend*** *the domain of f from E to* $E^*$.

***Shortest path problem***

*Let G = (E, V, f) be an edge weighted graph.*

***Definition*** $min_f(u, v) = (u, v_1, \ldots, v_n, v) \in E^*\ n \geq 0$ .э.

$f(u, v_1, \ldots, v_n, v) \leq f(u, u_1, \ldots u_m, v)$ .э. $^\forall m \geq 0\ ^\forall(u, u_1, \ldots, u_m, v) \in E^*$.

*Find u, v* $\in$ *V, find a shortest path such that f(u, v) is minimum.*

***Definition*** *Let* $W \subseteq V$, $u, v \in V$. $L_f^W(u, v) = (u, w_1, \ldots w_n, v)$ .э.

$f(u, w_1, \ldots w_n, v) \leq f(u, x_1, \ldots, x_m, v)$ .э. $\forall m, (u, x_1, \ldots, x_m, v) \in E^*$

$n, m \geq 0, 1 \leq \forall i \leq n, w_i \in W, 1 \leq \forall i \leq m, x_i \in W.$

*Note that u, v may be in W or not.*

***Theorem*** *Let* $W \subseteq V$, $u \in V$, *and* $x \notin W$ .∍. $L_f^W(u, x)$ *is the* ***minimum***.

$$\forall y \notin W, L_f^{W \cup \{x\}}(u, y) = \min(L_f^W(u, y), L_f^W(u, x)+f(x, y)).$$

***procedure*** *Shortest path* $G = (V, E \subseteq V \times V, f: E \rightarrow \mathbf{R}^+)$

***for*** $w \in V$ ***do*** $L(w) := Infinite$ ***od***; $L(u), W := 0, \varnothing$;

***Initialization*** $\forall y \in V, L(y) = L_f^{\{\}}(u, y)$.

***do*** $v \notin W \rightarrow$

***Loop invariance*** $\forall y \in V, L(y) = L_f^W(u, y)$.

$x := x \notin W \wedge \min(L(x))$; $W := W \cup \{x\}$;

***Loop invariance, invalidated*** $\forall y \in V, L(y) = L_f^{W-\{x\}}(u, y)$.

***do*** $y \notin W \wedge (x, y) \in E \rightarrow L(y) := \min(L(y), L(x) + f(x, y))$ ***od***

***Loop invariance, validated*** $\forall y \in V, L(y) = L_f^W(u, y)$.

***od***

***After the loop termination*** $\forall y \in V, L(y) = L_f^W(u, y) \wedge v \in W.$

*We should also prove that* $L_f^W(u, v) = min_f(u, v)$, *if* $v \in W$.

*But it is trivial, since we add the shortest path vertex x to W.*

## *10.7 Planar Graph*

***Definition*** *1 A graph is* ***planar****, if it can be drawn in the* ***plane without edge crossing****.*

*$K_4$ is planar and $Q_3$ is planar, but $K_{3,3}$ is not planar.*

***Region****: planar graph divides regions*

***Tree*** *has* ***one*** *region*

***Removing an edge*** *on the* ***cycle*** *merges* ***two*** *regions into* ***one****.*

***Theorem*** *1 Let G be a* ***connected planar graph*** *with v* ***vertices*** *and e* ***edges****. Let r be the number of* ***regions*** *in planar reprsentation of G. Then*

$r = e - v + 2.$

***proof*** *Induction on number of edges, P(e).*

***basis*** *e=0, v=1, r=1.* $\therefore$ *1 = 0 – 1 + 2 = 1. O.K.*

***induction*** *Consider a connected planar graph G with e+1 edges.*

*1. If G is* ***acyclic****. G is* ***tree****.* $\therefore$ *r = 1, e – v + 2 = –1 + 2 =1.* $\therefore$ *O.K.*

*2. If G is not acyclic, G has at least one* ***cycle****, C.*

*Consider {u, v}* ***in*** *C and a* ***spanning tree****, T. .э. {u, v} is* ***not*** *in T.*

*There exisits such an edge {u, v} because T is acyclic.*

***Remove*** *the* ***edge*** *{u, v} from G, it is called as G'.*

*G'* ***has one less regions****, since removing an edge {u, v} on the cycle C.*

*G' is connected planar graph and has e edges.*

$\therefore$ *r = e – v + 2 in G'* *induction hypothesis*

*In G, r+1 regions, e+1 edges, and v vertices.* $\therefore$ *(r +1) = (e+1) – v + 2.*

***Corollary*** *1 If connected planar graph G with e edges and v vertices where* $v \geq 3$*. Then* $e \leq 3v - 6$.

***proof*** *Consider degree of region number of boundary edges*

*Sum of the degree of regions is 2e.*

*Minimum number of degree of each region r is 3r(triangle)*

$\therefore\ 2e \geq 3r = 3(e - v + 2)$ $\therefore\ e \leq 3v - 6$.

***Corollary*** *2 If G is a connected planar graph, then G has a vertex not exceeding five.*

***proof*** *If G has one or two vertices, the result is true.*

*If G has more that three vertices,* $e \leq 3v - 6$*. So* $2e \leq 6v - 12$.

*If every vertex has more than or equal 6,* $2e \geq 6v$*(Handshaking Theorem)*

*But it contradict with* $2e \leq 6v - 12$.

***Example*** *5* $K_5$ *is not planar.*

$v = 5$, $e = 10$. $3v - 6 = 9$.

## *10.8 Graph Coloring*

***Definition*** *1 A graph G is k-**colorable**, if each vertex can be assigned one of k colors so that **adjacent vertices** get the **different colors**.*

*map coloring problem*

*The smallest number of colors are called **chromatic number** of G, written as $\chi(G)$. $\chi(K_n) = n$.*

***Theorem*** *1 The **chromatic** number of **planar** graph is no greater than 4.*

***Theorem*** *2 A graph with maximum degree at most k is (k+1)-colorable.*

***proof*** *Induction on number of **vertices**. P(n).*

***basis*** *1-vertex graph, maximum degree 0 and 1-colorable. P(1) is true.*

***induction*** *Let G be (n+1)-**vertex** graph with maximum degree at most k*

***Remove*** *a vertex v and its incident edges.*

*G' has n vertices and max. deg. at most k. ∴ G' is (k+1)-colorable(IH).*

*v has at most k adjacent vertices. ∴ G is (k+1)-colorable((k+1) – k = 1).*

## *10.1 Introduction to Trees*

***Definition*** *10.1 A **tree** is an **acyclic connected** graph.*

*A vertex of **degree one** is called **leaf**.*

*A **forest** is a set of trees.*

***Theorem*** *Every **tree** has following properties:*

1. *Any **connected subgraph** is a **tree**.*
2. *There is a **unique (simple) path** between **every pair of vetices**.*
3. ***Adding** an **edge** between two vertices create a **(simple) cycle**.*
4. ***Removing** any **edge disconnects** the tree.*
5. *If tree has at **least two vertices**, then it has **at least two leaves**.*
6. *The number of **vertices** is **one larger** than that of **edges**.*

***proof*** *1. Any subgraph of acyclic graph subgraph is also **acyclic**.*

*2. There is **at least one** (simple) path, since connected and acyclic.*

*Assume **two different** simple paths from u to v.*

*Assume x be the first vertex where the path* ***diverge****,*

*y be the next vertex they* ***share****.*

*There is a* ***cycle*** *from x to y and then y to x.*

*3. Additional edge $\{u, v\} \cup$ (simple)* ***path*** *from u to v = (simple)* ***cycle***

*4. Remove $\{u, v\}$.* ***Unique*** *simple path was (u, v). $\therefore$ Not connected*

*5. Let $(v_1, \ldots, v_m)$ be the* ***longest*** *simple path in the tree. Then $m \geq 2$.*

*$2 < \forall i \leq m$, $\{v_1, v_i\} \notin E$, since $(v_1, \ldots, v_i, v_1)$ is a* ***cycle****.*

*$\{u, v_1\} \notin E$ where u is not in the path, since $(v_1, \ldots, v_m)$ is the* ***longest****.*

*$\therefore$ $v_1$ is a* ***leaf****. By* ***symmetric*** *argumant $v_m$ is a* ***second*** *leaf.*

*6. Induction on number of vertices.*

*n=1, no edge. 0 + 1 = 1. O.K.*

*Consider (n+1)-vertex tree T and let v be a* ***leaf*** *of T.*

*Deleting v and its incident edge gives a smaller* ***tree****.*

$(|E| - 1) = (|V| - 1) + 1$

*Adding v and its incident edge gives a larger* ***tree****. $\therefore$ $|E| = |V| + 1$.*

***Theorem*** *Every **connected graph** has **spanning tree**.*

***proof*** *Let T be a **connected spanning** subgraph of G with the **smallest** number of **edges**.*

*Suppose T has a cycle* $(v_0, v_1, \ldots, v_n, v_0)$.

*Suppose we remove the edge* $\{v_n, v_0\}$.

*If arbitrary vertices x and y has a **path not** containing the edge* $\{v_n, v_0\}$,

*x and y has a path containing **that path**.*

*If arbitrary vertices x and y has a path **containing** the edge* $\{v_n, v_0\}$,

*x and y has a path containing the **path*** $(v_0, v_1, \ldots, v_n)$.

*This is a **contradiction** that T has the **smallest** number of edges and **connected**.*

$\therefore$ *T is acyclic.*

*T is a tree.*