# 9 Relations

## 9.1 Relations and Their Properties

***Def.*** *1 Let A and B be two set. Then a* ***binary relation*** *R from A to B is subset of* $A \times B$.

$$R \subseteq A \times B.$$

*A:* ***domain*** *of the relation R. B:* ***range(codomain)*** *of the relation R.*

*Let* $a \in A$, $b \in B$, *Then* $(a, b) \in R$ *or* $(a, b) \notin R$.

*If* $(a, b) \in R$, *we also write a R b and we say a is* ***related to*** *b by R.*

*If* $(a, b) \notin R$, *we also write* $a \not R b$ *and a is* ***not related to*** *b by R.*

*Two* ***aspects*** *of Relation*

$R \subseteq A \times B$ *relation R is a* ***set*** *of pairs*

$(a, b) \in R$

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$ *relation R is a infix* ***boolean*** *binary* ***operation***

*a R b*

*Ex.* $(3, 3) \in =$ and $(3, 4) \notin =$; or $3 = 3$ and $3 \neq 4$.

## ***Function is a Relation*** ***but relation is not a function!***

$R \subseteq A \times B$ vs $f: A \rightarrow B$ $\forall a \in A: \exists_1 f(a)=b \in B$.

If $f(a) = b$, then we can write $(a, b) \in f$ or $a f b$.

. $\therefore$ Function is a (**special** kind of) relation.

Relation is **not** a function

i) If $\exists a \in A$ .∋. $a R b_1$ and $a R b_2$,$(b_1 \neq b_2)$, R is **not** a function.

function must have an unique **related** image.

ii) If $\exists a \in A$ and $\nexists b \in B$ .∋. $a R b$, R is **not** a function.

all the elements in the domain must have its related image

Let $R \subseteq A \times B$. Then

We write $R(a) = \{b_1, b_2, \ldots, b_n\} \in R$, if $1 \leq {}^{\forall} i \leq n: (a, b_i) \in R$.

# *A* ***function*** *is a* ***special kind****(****subclass****) of a* ***relation***

***Three*** *faces of the relation R form A to B.*

*i) R is a sub****set of pairs***

$R \subseteq A \times B$, $(a, b) \in R$ where $a \in A$ and $b \in B$.

*ii) R is a* ***infix binary boolean(relational) operator***

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$

$a\ R\ b$ where $a \in A$ and $b \in B$.

*iii) R is a* ***set valued function*** *from A to* $2^B$.

$R: A \to 2^B$.

$R(a) = \{b_1, b_2, \ldots, b_n\}$ where $a \in A$ and $\{b_1, b_2, \ldots, b_n\} \in 2^B$.

if $1 \leq {}^\forall i \leq n$, $(a, b_i) \in R$ or $a\ R\ b_i$. for $n \geq 0$,

if $n=0$, $R(a) = \{b_1, b_2, \ldots, b_n\} = \emptyset$.

Note that ${}^\forall a \in A$, ${}^{\exists 1}\{b_1, b_2, \ldots, b_n\} \in 2^B$ is unique.

*Three notations for the relation*

*i)* ***subset*** *of* $A \times B$, $(a, b) \in R$.

$R \subseteq A \times B$

*ii) infix binary boolean* ***operation***, $a \ R \ b$,

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$, $a \ R \ b$.

*Example* $=, <, \leq, \ldots \subseteq N \times N$ *or* $N \times N \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$3 = 3, 3 \neq 4, 3 \leq 3, \ldots$

*iii)* ***set valued function***, $R(a) = \{b_1, b_2, \ldots, b_n\}$ *or* $\emptyset$.

$R: A \rightarrow 2^B$.

## Relation on A Set

***Def.*** *2 Let A be a set and $R \subseteq A \times A$. R is called a relation* ***on*** *A.*

*Relation on A is a* ***directed graph*** *with* ***vertices*** *A and* ***edges*** *R.*

## Properties of Relations

***Def.*** *3 A relation R is* ***reflexive****, if $\forall a \in A$, a R a.*

*A relation R is* ***irreflexive****, if $\forall a \in A$, a $\not R$ a.*

*A relation may be* ***neither*** *reflexive* ***nor*** *irreflexive.*

***Def.*** *4 A relation R is* ***symmetric****, if $a R b \Rightarrow b R a$.*

*A relation R is* ***asymmetric****, if $a R b \Rightarrow b \not R a$.* *a R a (X)*

*A relation R is* ***antisymmetric****, if $(a R b \wedge b R a) \Rightarrow (a = b)$.*

*or if $(a R b \wedge a \neq b) \Rightarrow b \not R a$* *a R a (O)*

*If a relation is a* ***asymmetric*** *then it is also* ***antisymmetric****.($\subseteq$)*

***Def.*** *5 A relation R is* ***transitive****, if $a R b \wedge b R c \Rightarrow a R c$.*

## Combining Relations

*Let $R_1, R_2 \subseteq A \times B$. Then we define*

$R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, *and* $R_2 - R_1$.

***Def. 6*** *Let $R \subseteq A \times B$, $S \subseteq B \times C$. Then* ***composition*** *of R and S, denoted as* $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.

***Def. 7*** *Let $R \subseteq A \times A$. Then for $n \geq 1$,*

$R^1 =_B R$ $(n=1)$ ***basis***

$R^{n+1} =_R R^n \circ R$ $(n \geq 2)$ ***induction*** $R^{n+1} \overset{?}{=} R \circ R^n$ ***ind.***

$R^3(a) =_R R^2 \circ R(a) =_R R^1 \circ R \circ R(a) =_B R \circ R \circ R(a).$

$R^3(a) =_R R \circ R^2(a) =_R R \circ R \circ R^1(a) =_B R \circ R \circ R(a).$

***Def. 6.1*** *Let A be a set. We define an* ***identity*** *relation*

$id_A = \{(a, a) \in A \times A \mid a \in A\}$ $= \Delta$ *in this text sec. 9.4 in p. 977*

***Col.*** *0.5 Let* $R \subseteq A \times B$. *Then*

$R \circ id_A = id_A \circ R = R$. $\therefore$ $id_A$ *is a* ***identity*** *element for* ***composition***.

***Def. 7.1*** *Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}$,

$R^0 = id_A$ $(n=0)$ ***basis*** $(x^0 = 1)$

$R^{n+1} = R^n \circ R$. $(n \geq 1)$ ***induction***

$R^3(a) =_R R^2 \circ R(a) =_R R^1 \circ R \circ R(a) =_R R^0 \circ R \circ R(a)$

$=_B id_A \circ R \circ R \circ R(a) = R \circ R \circ R(a)$.

***Thm. 1*** *Let* $R \subseteq A \times A$. *R is* ***transitive****, if and only if,* $R^n \subseteq R$ *for* $\forall n \geq 1$.

***Proof****:*

*1. (if)* $R^n \subseteq R$ *for* $\forall n \geq 1 \Rightarrow R$ *is* ***transitive****.*

*Since* $R^2 \subseteq R$. *If* $(a, b) \in R$ *and* $(b, c) \in R$, $(a, c) \in R^2$. $\therefore (a, c) \in R$

$\therefore$ *R is* ***transitive****.*

*2.(only if) R is* ***transitive*** $\Rightarrow R^n \subseteq R$ *for* $\forall n \geq 1$.

*mathematical induction on* $n \in \mathbf{N}^+$.

***basis*** *Trivial for* $n = 1$*(since* $R^1 = R$*, by definition).*

***induction*** *Assume* $R^n \subseteq R$ *for some* $n \in \mathbf{N}^+$ *and R is* ***transitive****.*

*Consider* $(a, b) \in R^{n+1}$*, then* $\exists x \in A$ .϶. $(a, x) \in R$ *and* $(x, b) \in R^n$.

$R^n \subseteq R$, $\therefore (x, b) \in R$. ***induction hypothesis***

*Since R is transitive;* $(a, x) \in R$ *and* $(x, b) \in R \Rightarrow (a, b) \in R$.

$\therefore R^{n+1} \subseteq R$.

## *9.2 n-ary Relations and Their Applications*

***Def***. *1 Let $A_1, A_2, \ldots, A_n$ be sets.$R \subseteq A_1 \times A_2 \times \ldots \times A_n$ is a **n-ary relation** on $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relations, and n is called as the **degree** of R.*

***<u>Database and Relations</u>***

***relational*** *data model: $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$ is a **data base***

***record*** $(a_1, a_2, \ldots, a_n) \in R$,

***field*** $1 \leq {}^{\forall} f \leq n$: $a_f \in A_f$.

***(primary) key*** *field:* $1 \leq {}^{\exists} k \leq n$: ${}^{\forall} K \in A_k$ .э. $(a_1, a_2, \ldots, K, \ldots, a_n) \in R$,

$$|(a_1, a_2, \ldots, K, \ldots, a_n)| \leq 1.$$

*The **dadabase** R is said to be **functional** in the (**key**) field $A_k$.*

*Examle Table1 Students (St_name, St_id, Major, GPA) p. 564*

## Operations on n-ary Relations

*Let* $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$. *Then*

***Def**. 2* **Selection** *operator: Let* $C: A \rightarrow \{\mathbf{T}, \mathbf{F}\}$. *Then*

$$S_C(R) = \{a \in R \mid C(a), a \in A\}.$$

$C$**:** *condition* $S_C$: *selection operator on R*

***Def**. 3* **Projection** *Operator:* $P_{i_1 i_2 \ldots i_m}$ *where* $i_1 < i_2 < \ldots < i_m \leq n$ *maps*

*n-tuple* $(a_1, a_2, \ldots, a_n) \in R$, *to the m-tuple* $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})(i_m \leq n)$.

$P_{\{i_k\}}: A \rightarrow A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m}$, *if* $\{i_k\} = (i_1, i_2, \ldots, i_m)$ *for* $1 \leq k \leq m$.

$$P_{\{i_k\}}(a_1, a_2, \ldots, a_n) = (a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \qquad i_m \leq n.$$

***Def**. 4* **Join** *Operator:*

*Let* $R_1 \subseteq A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p$ *and* $R_2 \subseteq C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

*Then* $J(R_1, R_2) \in A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

*If* $R_1$: *m-tuples,* $R_2$: *n-tuples, then* $J(R_1, R_2)$: $m+n-p$*-tuples.* $(p \leq m, n)$

# 9.3 Representing Relations

## Representing Relation as a Boolean Matrix

*R*: $A \times B \rightarrow \{0, 1\}$. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

A **boolean matrix** $M_R = [m_{ij}]$ for the relation *R* where

$1 \leq {}^{\forall}i \leq m$: $1 \leq {}^{\forall}j \leq n$:

$m_{ij} = 1$, if $(a_i, b_j) \in R$,

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

We may write $M_R^{A \times B}[m_{ij}]$ ***instead*** of $M_R$, if needed.

## Representing Relation as a Directed Graph(Digraph)

***Def***. 1 A ***directed graph*** or ***digraph*** $G = (V, E)$ consistes of a set *V* of ***vertices***, and a set $E \subseteq V \times V$ of ***edges***(***arcs***).

$R \subseteq A \times A$. $\Leftrightarrow$ $G = (A, R)$

***relation*** *R* on *A* vs. ***digraph*** with vertices *A* and edges *R*

## 9.4 Closure of Relations

***Closures*** *Let* $R \subseteq A \times A$ *and* $\mathbf{P}$ = {*reflextive, symmetric, transitive*}. *Then We define a* **p** ***closue*** *of R as* $S \subseteq A \times A$ .϶. (1) $R \subseteq S$ *and*

$\forall T \subseteq A \times A$ *has the property* $\mathbf{p} \in \mathbf{P}$, $R \subseteq T$, *and* $S \subseteq T$.

(1) $R \subseteq S$ *and* (2) ***smallest one*** *among T's*(**p**)).

***Reflexive*** *closure of R* $R \cup id_A$.

$id_A = \Delta$ *in the text*($\Delta_A$) ***diagonal*** *relation on A.*

***Symetric*** *closure of R* $R \cup R^{-1}$.

***Path in Digraph***

***Def***. *1 Let G = (V, E) be a digrph. For a, b* $\in$ *V, a* ***path*** *from a, b,*

$Path(a,b) = (x_0, \underline{x_1}), (\underline{x_1}, x_2), \ldots, (x_{n-1}, x_n)$

$1 \leq i \leq n: (x_{i-1}, x_i) \in E$, *and* $x_0 = a$, $x_n = b$.

*a* ***sequence*** *of edges of* ***length*** *n*

*The path may be also denoted as a* ***sequence*** *of vertices*

$Path^V(a,b) = (x_0, x_1, x_2, \ldots, x_n)$ *of* ***length*** *n.*

*We view the set of* ***empty*** *edges as a path of* ***length*** *0 from $a \in V$ to a .*

*A path of length $n \geq 1$ that*

*begins and ends at the same vertex is called* ***cycle***.

***Thm***. *1 Let $R \subseteq A \times A$. There is a* ***path*** *of length $n \geq 1$ from a to b,*

*if and only if, $(a, b) \in R^n$.*

***proof*** *easy(mathematical induction)*

***Def***. *1 A* ***connetivity*** *relation $R^+ = \{(a, b) | (a, b) \in R^n, \forall n \geq 1\}$*

$R^+ = \cup_{i \in \mathbf{N}_1} R^i = R^1 \cup R^2 \cup \ldots$

***transitive*** *closure of R.*

$R^* = \cup_{i \in \mathbf{N}_0} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots$

***reflexive and transitive*** *closure of R.*

## Computing transitive closure $R^*$.

*Let A and B be sets,* $R \subseteq A \times A$, $f, g: A \rightarrow 2^C$.*(set valued functions), and*

$f(a) = \{c \in C | \ c \in g(a)\} \cup \{c \in C | \ a \ R \ b, c \in f(b)\}$.

$f(a) = \{c \in g(a)\} \cup \{c \in f(b) | \ a \ R \ b\}$.

$f(a) = g(a) \cup \cup_{a \ R \ b} f(b)$. (**recursive** *definition of f) Then*

$f(a) = \{c \in C | \ c \in g(b), a \ R^* \ b\}$.

$f(a) = g^*(a)$. (**iterative** *definition of f)*

| | |
|---|---|
| *Warshall's algorithm* | $O(n^3)$ |
| *Depth first search* | $O(n^2)$ |

```
Algorithm Depth first search
S: stack of Vertex; n(Vertex) array of Depth;
procedure Traverse(x: Vertex; d: Depth);
    push x onto S; n(x) := d;                    f(x) := g(x);
    for y ∈ Vertex where x R y do
        if n(y) = 0 then Traverse(y, d+1) fi;
        n(x) := min(n(x), n(y));                 f(x) := f(x) ∪ f(y)
        od;
    if n(x) = d then repeat
        y = pop of S; n(y) := infinite;          f(y) := f(x)
        until  y = x
    fi
end procedure Traverse
for x  ∈ Vertex do n(x) := 0;                    f(x) := {} od;
for x  ∈ Vertex where n(x) = 0 do Traverse(x, 1) od
```

## 9.5 Equivalence Relations

***Def***. 1 Let $R \subseteq A \times A$. R is called **equivalence** relation, if it is reflexive, symmetric, and transitive.

***Def***. 2 $a, b \in A$ are said to be related **equivalent**, written $a \sim b$, if R is an equivalent relation on A and a R b.

***Def***. 3 Let $R \subseteq A \times A$ be an **equivalence** relation.

$[a]_R = \{b \mid a \, R \, b\}$ is called the **equivalence class** of a w.r.t. R.

If $b \in [a]_R$, b is called the **representative** of the equivalent class.

Note that $a \in [a]_R$, since R is ***reflexive***.

Ex. $\equiv_4 \subseteq \mathbf{Z} \times \mathbf{Z}$ is an equivalent relation. **Equivalent classes** are

$[0]_{\equiv_4} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_{\equiv_4} = \{\ldots, -7. -3, 1, 5, 9, \ldots\}$

$[2]_{\equiv_4} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_{\equiv_4} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

***Thm.*** *1 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation. Three statments are logically equivalent.*

i) $a\ R\ b$ ii) $[a]_R = [b]_R$ iii) $[a]_R \cap [b]_R \neq \emptyset$.

***proof***

1) i) $\rightarrow$ ii)

$\forall c \in [a]_R$, $a\ R\ c$, $a\ R\ b$, $b\ R\ a$. $\therefore$ $b\ R\ c$, $c \in [b]_R$. $\therefore$ $[a]_R \subseteq [b]_R$.

$\forall c \in [b]_R$, $b\ R\ c$, $a\ R\ b$. $\therefore$ $a\ R\ c$, $c \in [a]_R$. $\therefore$ $[b]_R \subseteq [a]_R$.

2) ii) $\rightarrow$ iii) Assume $[a]_R = [b]_R$, $a\ R\ b$. $\therefore$ $a, b \in [a]_R \cap [b]_R \neq \emptyset$.

3) iii) $\rightarrow$ i) Suppose $[a]_R \cap [b]_R \neq \emptyset$, $[a]_R \neq \emptyset$ and $[b]_R \neq \emptyset$.

$\exists c \in [a]_R \wedge c \in [b]_R$. $a\ R\ c$, $b\ R\ c$. $c\ R\ b$, $\therefore$ $a\ R\ b$.

***Lemma*** *1.5 Let $R \subseteq A \times A$ be an* ***equivalence*** *relation.*

$\cup_{a \in A} [a]_R = A$. $a \in [a]_R$.

$\therefore$ $\{[a]_R \subseteq A \mid a \in A\}$ is a ***partition*** of $A$.

***Definition*** *1.5 Let S be a set. The* ***partition*** *of S, $\{A_i | i \in I\}$ I: index set, is*

- i) $A_i \neq \emptyset$, $i \in I$. ***nonempty***
- ii) $A_i \cap A_j = \emptyset$, when $i \neq j$. ***disjoint***
- iii) $\cup_{i \in I} A_i = A$. ***exhaustive***

***Thm***. *2 Let R be a equivalent relation on A.*

*Then the* ***equivalent classes*** *of R form a* ***partion*** *of A.*

*Conversely, given a* ***partition*** *$\{A_i | i \in I\}$ of the set A,*

*there is an* ***equivalent relation*** *R that has the set $A_i$, $i \in I$,*

*as its* ***equivalent class****.*

***Realtion*** $R \subseteq A \times A$ $O(n^2)$ where $|A| = n$.

***Equivalent*** *relation* $R \subseteq A \times A$ $O(n)$

## *9.6 Partial Ordering*

***Def***. *1 Let $R \subseteq A \times A$. R is called* ***(ir)reflexive partial order****,*
*if it is* ***(ir)reflexive****,* ***antisymmetric****, and* ***transitive****.*
*(A, R) is called* ***partially odreded*** *set or* ***poset****.*

*Ex. 1 2 3 (Z, ≤), (Z+, |), ($2^S$, ⊆) are* ***posets****.*

***Def***. *2 Let (A, ≤) be poset and a, b ∈ A. Then*
*The elements a and b are* ***comparable*** *if either a ≤ b or b ≤ a.*
*The elements a and b are* ***incomparable*** *if* ***neither*** *a ≤ b* ***nor*** *b ≤ a*

.

***Def***. *3 Let (S, ≤) be poset. If ∀a, b ∈ S, a and b are* ***comparable****,*
*S is called* ***totally ordered set****,* ***linearly ordered set****, or* ***chain****.*
*≤ is called* ***total order*** *or* ***linear order****.*

*Ex. 6, 7 (**Z**, ≤) is a* ***total*** *order, but ($\mathbf{Z}^+$, |) is* ***not*** *a* ***total*** *order*

***Def**. 4 A poset $(S, \leq)$ is a **well-ordered** set, if $\leq$ is a **total order**,*
*and every nonempty subset of S has a **least** emelent.*

*Ex. 8* $(\mathbf{Z}^+ \times \mathbf{Z}^+, \leq_L)$ *is **well-ordered**, but* $(\mathbf{Z}, \leq)$ *is **not** **well-**orderd.*

***Thm .1 Principle of Well-Ordered Induction***

*Let $(S, \leq)$ be a well-ordered set and $\forall x \in S, P(x)$, if*

$\forall y \in S$ .϶. $x < y: P(x) \Rightarrow P(y)$.

***proof** Suppose $\exists y \in S, \neg P(y)$.*

$A = \{x \in S \mid \neg P(x)\} \neq \emptyset$.

*Let $a \in A$ be the least element.*

$\exists a \in A, \forall x \in S$ .϶. $x < a: P(x) \Rightarrow P(a)$.

*Contradiction $\exists y \in S, \neg P(y)$.*

## <u>Lexicographic order( 사전순서 )</u>

*Let* $(A_1, \leq_1)$ *and* $(A_2, \leq_2)$ *be* ***well-ordered*** *sets. Then we define* $(A_1 \times A_2, \leq_{1\times2})$ *a* <u>*lexicographic order*</u>

$$(a_1, a_2) \leq_{1\times2} (b_1, b_2), \text{ if } (a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2)).$$

***Ext***. *Let* $(A_1, \leq_1), (A_2, \leq_2), \ldots (A_n, \leq_n)$ *be well-ordered sets. Then we define* $(A_1 \times A_2 \times \ldots \times A_n, \leq_{1\times2\ldots\times n} = \leq_L)$ *a lexicographic order*

$$(a_1,a_2,\ldots,a_n) \leq_L (b_1,b_2,\ldots,b_n), \text{ if } 1\leq{}^{\exists}i<n: [(1\leq{}^{\forall}j<i: a_j =_j b_j) \wedge (a_i <_i b_i)].$$

***Ext***. *Let* $(V, \leq)$ *be a well-ordered set. Then we define* $(V^*, \leq_L)$ *as*

$$(a_1,a_2,\ldots,a_n) \leq_L (b_1,b_2,\ldots,b_n), \text{ if } (1\leq{}^{\exists}i<n: [(1\leq{}^{\forall}j<i: a_j =_j b_j) \wedge (a_i < b_i)].$$
$$\vee [(n < m) \wedge (1\leq{}^{\forall}i\leq n: a_i = b_i)].$$

$(V^*, \leq_L)$ *is a* ***well-ordered*** *set.*

## Hasse Diagram

*Let $(A, \leq)$ be a poset. We say a* **covers** *b, if $a < b \wedge \nexists c \in A, a \leq c \wedge c \leq b$.*

*(A, covers) is a Hasse Diagram.*

## Maximal and Minimal Elements

*Let $(A, \leq)$ be a poset.*

*We say $a$ is a* **maximal**, *if $\nexists b \in A, a < b$.*

*We say $a$ is a* **minimal**, *if $\nexists b \in A, b < a$.*

*We say $a$ is the* **greatest element**, *if $\forall b \in A, b \leq a$.*

*We say $a$ is the* **least elements**, *if $\forall b \in A, a \leq b$.*

$ub(S) = \{c \in A \mid a \in S, a \leq c\}$ **upper bound** *of S.*

$lb(S) = \{c \in A \mid a \in S, c \leq a\}$ **lower bound** *of S.dirtn,*

$lub(S) = \{c \in A \mid c \in ub(S), d \in ub(S), c \leq d\}$

*the* **least upper bound** *of S.*

$glb(S) = \{c \in A \mid c \leq lb(S), d \in lb(S), d \leq c\}$

*the* **greatest lower bound** *S.*

## Lattice

*A poset (A, ≤) is called* ***lattice****. If every pair of elements has both a least upper bound, and a greatest lower bound.*

*(**Z**, ≤) is a lattice.* *max, min:* $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$

*($\mathbf{Z}^+$, |) is a lattice.* *lcm, gcd:* $\mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$

*($2^S$, ⊆) is a lattice.* $\cup, \cap: 2^S \times 2^S \to 2^S$

## Topological Sorting

***Lemma*** *1 Every* ***finite nonempty*** *poset (A, $\leq_p$) has*

***at least one minimal*** *elements.*

***Algorithm*** *1* ***Topological sorting*** *(A, $\leq_t$)*

*k := 1;* ***do*** *a ∈ A is* ***minimal*** *of A →* $a_k$ *:= k; A := A –{a}; k++* ***od***

$a_1, a_2, \ldots, a_k, \ldots, a_n$ *(k≤n) is a* ***total*** *order*