

9 Relations

9.1 Relations and Their Properties

Def. 1 Let A and B be two set. Then a **binary relation** R from A to B is subset of $A \times B$.

$$R \subseteq A \times B.$$

A : **domain** of the relation R . B : **range(codomain)** of the relation R .

Let $a \in A$, $b \in B$, Then $(a, b) \in R$ or $(a, b) \notin R$.

If $(a, b) \in R$, we also write $a R b$ and we say a is **related to** b by R .

If $(a, b) \notin R$, we also write $a \not R b$ and a is **not related to** b by R .

Two **aspects** of Relation

$R \subseteq A \times B$ relation R is a **set of pairs**

$$(a, b) \in R$$

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$ relation R is a infix **boolean binary operation**

$$a R b$$

Ex. $(3, 3) \in =$ and $(3, 4) \notin =$; or $3 = 3$ and $3 \neq 4$.

Function is a Relation but relation is not a function!

$R \subseteq A \times B$ vs $f: A \rightarrow B \quad \forall a \in A: \exists_1 f(a) = b \in B.$

If $f(a) = b$, then we can write $(a, b) \in f$ or $a f b$.

\therefore Function is a (**special** kind of) relation.

Relation is **not** a function

i) If $\exists a \in A . \exists . a R b_1$ and $a R b_2, (b_1 \neq b_2)$, R is **not** a function.

function must have an **unique** related image.

ii) If $\exists a \in A$ and $\nexists b \in B . \exists . a R b$, R is **not** a function.

all the elements in the domain must have its related image

Let $R \subseteq A \times B$. Then

We write $R(a) = \{b_1, b_2, \dots, b_n\} \in R$, if $1 \leq \forall i \leq n: (a, b_i) \in R$.

A function is a special kind(subclass) of a relation

Three faces of the relation R from A to B .

i) R is a subset of pairs

$R \subseteq A \times B, (a, b) \in R$ where $a \in A$ and $b \in B$.

ii) R is a infix binary boolean(relational) operator

$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$a R b$ where $a \in A$ and $b \in B$.

iii) R is a set valued function from A to 2^B .

$R: A \rightarrow 2^B$.

$R(a) = \{b_1, b_2, \dots, b_n\}$ where $a \in A$ and $\{b_1, b_2, \dots, b_n\} \in 2^B$.

if $1 \leq \forall i \leq n, (a, b_i) \in R$ or $a R b_i$ for $n \geq 0$,

if $n=0, R(a) = \{b_1, b_2, \dots, b_n\} = \emptyset$.

Note that $\forall a \in A, \exists^1 \{b_1, b_2, \dots, b_n\} \in 2^B$ is unique.

Three notations for the relation

i) **subset** of $A \times B$, $(a, b) \in R$.

$$R \subseteq A \times B$$

ii) **infix binary boolean operation**, $a R b$,

$$R: A \times B \rightarrow \{\mathbf{T}, \mathbf{F}\}, a R b.$$

Example $=, <, \leq, \dots \subseteq N \times N$ or $N \times N \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$$3 = 3, 3 \neq 4, 3 \leq 3, \dots$$

iii) **set valued function**, $R(a) = \{b_1, b_2, \dots, b_n\}$ or \emptyset .

$$R: A \rightarrow 2^B.$$

Relation on A Set

Def. 2 Let A be a set and $R \subseteq A \times A$. R is called a relation on A .

Relation on A is a directed graph with vertices A and edges R .

Properties of Relations

Def. 3 A relation R is **reflexive**, if $\forall a \in A, a R a$.

A relation R is **irreflexive**, if $\forall a \in A, a \not R a$.

A relation may be neither reflexive nor irreflexive.

Def. 4 A relation R is **symmetric**, if $a R b \Rightarrow b R a$.

A relation R is **asymmetric**, if $a R b \Rightarrow b \not R a$. $a R a (X)$

A relation R is **antisymmetric**, if $(a R b \wedge b R a) \Rightarrow (a = b)$.

or if $(a R b \wedge a \neq b) \Rightarrow b \not R a$ $a R a (O)$

If a relation is asymmetric then it is also antisymmetric. (\subseteq)

Def. 5 A relation R is **transitive**, if $a R b \wedge b R c \Rightarrow a R c$.

Combining Relations

Let $R_1, R_2 \subseteq A \times B$. Then we define

$$R_1 \cup R_2, R_1 \cap R_2, R_1 \oplus R_2, R_1 - R_2, \text{ and } R_2 - R_1.$$

Def. 6 Let $R \subseteq A \times B$, $S \subseteq B \times C$. Then **composition** of R and S , denoted as $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.

Def. 7 Let $R \subseteq A \times A$. Then for $n \geq 1$,

$$R^1 =_B R \quad (n=1) \quad \text{basis}$$

$$R^{n+1} =_R R^n \circ R \quad (n \geq 2) \quad \text{induction} \quad R^{n+1} ? = R \circ R^n \text{ ind.}$$

$$R^3(a) =_R R^2 \circ R(a) =_R R^1 \circ R \circ R(a) =_B R \circ R \circ R(a).$$

$$R^3(a) =_R R \circ R^2(a) =_R R \circ R \circ R^1(a) =_B R \circ R \circ R(a).$$

Def. 6.1 Let A be a set. We define an **identity relation**

$$id_A = \{(a, a) \in A \times A \mid a \in A\} = \Delta \text{ in this text sec. 9.4 in p. 977}$$

Col. 0.5 Let $R \subseteq A \times B$. Then

$$R \circ id_A = id_A \circ R = R. \therefore id_A \text{ is a } \mathbf{identity} \text{ element for } \mathbf{composition}.$$

Def. 7.1 Let $R \subseteq A \times A$. Then for $n \in \mathbf{N}$,

$$R^0 = id_A \quad (n=0) \quad \mathbf{basis} \quad (x^0 = 1)$$

$$R^{n+1} = R^n \circ R. \quad (n \geq 1) \quad \mathbf{induction}$$

$$\begin{aligned} R^3(a) &=_{\mathbf{R}} R^2 \circ R(a) =_{\mathbf{R}} R^1 \circ R \circ R(a) =_{\mathbf{R}} R^0 \circ R \circ R(a) \\ &=_{\mathbf{B}} id_A \circ R \circ R \circ R(a) = R \circ R \circ R(a). \end{aligned}$$

Thm. 1 Let $R \subseteq A \times A$. R is **transitive**, if and only if, $R^n \subseteq R$ for $\forall n \geq 1$.

Proof:

1. (if) $R^n \subseteq R$ for $\forall n \geq 1 \Rightarrow R$ is **transitive**.

Since $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R^2$. $\therefore (a, c) \in R$
 $\therefore R$ is **transitive**.

2. (only if) R is **transitive** $\Rightarrow R^n \subseteq R$ for $\forall n \geq 1$.

mathematical induction on $n \in \mathbf{N}^+$.

basis Trivial for $n = 1$ (since $R^1 = R$, by definition).

induction Assume $R^n \subseteq R$ for some $n \in \mathbf{N}^+$ and R is **transitive**.

Consider $(a, b) \in R^{n+1}$, then $\exists x \in A$. \exists . $(a, x) \in R$ and $(x, b) \in R^n$.

$R^n \subseteq R$, $\therefore (x, b) \in R$. **induction hypothesis**

Since R is **transitive**; $(a, x) \in R$ and $(x, b) \in R \Rightarrow (a, b) \in R$.

$\therefore R^{n+1} \subseteq R$.

9.2 *n*-ary Relations and Their Applications

Def. 1 Let A_1, A_2, \dots, A_n be sets. $R \subseteq A_1 \times A_2 \times \dots \times A_n$ is a ***n*-ary relation** on $A_1 \times A_2 \times \dots \times A_n$. The sets A_1, A_2, \dots, A_n are called the **domains** of the relations, and n is called as the **degree** of R .

Database and Relations

relational data model: $R \subseteq A = A_1 \times A_2 \times \dots \times A_n$ is a **data base**

record $(a_1, a_2, \dots, a_n) \in R,$

field $1 \leq \forall f \leq n: a_f \in A_f$

(primary) key field: $1 \leq \exists k \leq n: \forall K \in A_k .\exists. (a_1, a_2, \dots, K, \dots, a_n) \in R,$
 $|(a_1, a_2, \dots, K, \dots, a_n)| \leq 1.$

The database R is said to be **functional** in the (**key**) field A_k .

Example Table1 Students (St_name, St_id, Major, GPA) p. 564

Operations on n -ary Relations

Let $R \subseteq A = A_1 \times A_2 \times \dots \times A_n$. Then

Def. 2 Selection operator: Let $C: A \rightarrow \{\mathbf{T}, \mathbf{F}\}$. Then

$$S_C(R) = \{a \in R \mid C(a), a \in A\}.$$

C : condition S_C : selection operator on R

Def. 3 Projection Operator: $P_{i_1 i_2 \dots i_m}$ where $i_1 < i_2 < \dots < i_m \leq n$ maps

n -tuple $(a_1, a_2, \dots, a_n) \in R$, to the m -tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ ($i_m \leq n$).

$P_{\{i_k\}}: A \rightarrow A_{i_1} \times A_{i_2} \times \dots \times A_{i_m}$, if $\{i_k\} = (i_1, i_2, \dots, i_m)$ for $1 \leq k \leq m$.

$$P_{\{i_k\}}(a_1, a_2, \dots, a_n) = (a_{i_1}, a_{i_2}, \dots, a_{i_m}) \quad i_m \leq n.$$

Def. 4 Join Operator:

Let $R_1 \subseteq A_1 \times \dots \times A_{m-p} \times C_1 \times \dots \times C_p$ and $R_2 \subseteq C_1 \times \dots \times C_p \times B_1 \times \dots \times B_{n-p}$.

Then $J(R_1, R_2) \in A_1 \times \dots \times A_{m-p} \times C_1 \times \dots \times C_p \times B_1 \times \dots \times B_{n-p}$.

If R_1 : m -tuples, R_2 : n -tuples, then $J(R_1, R_2)$: $m+n-p$ -tuples. ($p \leq m, n$)

9.3 Representing Relations

Representing Relation as a Boolean Matrix

$R: A \times B \rightarrow \{0, 1\}$. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$.

A **boolean matrix** $M_R = [m_{ij}]$ for the relation R where

$$1 \leq \forall i \leq m: 1 \leq \forall j \leq n:$$

$$m_{ij} = 1, \text{ if } (a_i, b_j) \in R,$$

$$m_{ij} = 0, \text{ if } (a_i, b_j) \notin R.$$

We may write $M_R^{A \times B} [m_{ij}]$ instead of M_R , if needed.

Representing Relation as a Directed Graph(Digraph)

Def. 1 A **directed graph** or **digraph** $G = (V, E)$ consists of a set V of **vertices**, and a set $E \subseteq V \times V$ of **edges(arcs)**.

$$R \subseteq A \times A. \quad \Leftrightarrow \quad G = (A, R)$$

relation R on A vs. **digraph** with vertices A and edges R

9.4 Closure of Relations

Closures Let $R \subseteq A \times A$ and $\mathbf{P} = \{\text{reflexive, symmetric, transitive}\}$. Then

We define a **p closure** of R as $S \subseteq A \times A$. \exists . (1) $R \subseteq S$ and

$\forall T \subseteq A \times A$ has the property $\mathbf{p} \in \mathbf{P}$, $R \subseteq T$, and $S \subseteq T$.

(1) $R \subseteq S$ and (2) **smallest one** among T 's(\mathbf{p})).

Reflexive closure of R $R \cup id_A$.

$id_A = \Delta$ in the text(Δ_A) **diagonal relation on A .**

Symetric closure of R $R \cup R^{-1}$.

Path in Digraph

Def. 1 Let $G = (V, E)$ be a digrph. For $a, b \in V$, a **path** from a, b ,

$Path(a, b) = (x_0, \underline{x_1}), (\underline{x_1}, x_2), \dots, (x_{n-1}, x_n)$

$1 \leq i \leq n: (x_{i-1}, x_i) \in E$, and $x_0 = a, x_n = b$.

a **sequence of edges of length n**

The path may be also denoted as a **sequence** of vertices

$$\text{Path}^V(a,b) = (x_0, x_1, x_2, \dots, x_n) \text{ of length } n.$$

We view the set of **empty** edges as a path of **length** 0 from $a \in V$ to a .

A path of length $n \geq 1$ that

begins and ends at the same vertex is called **cycle**.

Thm. 1 Let $R \subseteq A \times A$. There is a **path** of length $n \geq 1$ from a to b ,

if and only if, $(a, b) \in R^n$.

proof easy (mathematical induction)

Def. 1 A **connectivity** relation $R^+ = \{(a, b) \mid (a, b) \in R^n, \forall n \geq 1\}$

$$R^+ = \cup_{i \in \mathbf{N}_1} R^i = R^1 \cup R^2 \cup \dots$$

transitive closure of R .

$$R^* = \cup_{i \in \mathbf{N}_0} R^i = R^0 \cup R^1 \cup R^2 \cup \dots$$

reflexive and transitive closure of R .

Computing transitive closure R^*

Let A and B be sets, $R \subseteq A \times A$, $f, g: A \rightarrow 2^C$. (set valued functions), and

$$f(a) = \{c \in C \mid c \in g(a)\} \cup \{c \in C \mid a R b, c \in f(b)\}.$$

$$f(a) = \{c \in g(a)\} \cup \{c \in f(b) \mid a R b\}.$$

$$f(a) = g(a) \cup \cup_{a R b} f(b). \text{ (recursive definition of } f) \text{ Then}$$

$$f(a) = \{c \in C \mid c \in g(b), a R^* b\}.$$

$$f(a) = g^*(a). \text{ (iterative definition of } f)$$

Warshall's algorithm $O(n^3)$

Depth first search $O(n^2)$

Algorithm *Depth first search**S*: stack of Vertex; *n*(Vertex) array of Depth;**procedure** *Traverse*(*x*: Vertex; *d*: Depth); **push** *x* onto *S*; *n*(*x*) := *d*; *f*(*x*) := *g*(*x*); **for** *y* ∈ Vertex **where** *x* *R* *y* **do** **if** *n*(*y*) = 0 **then** *Traverse*(*y*, *d*+1) **fi**; *n*(*x*) := min(*n*(*x*), *n*(*y*)); *f*(*x*) := *f*(*x*) ∪ *f*(*y*) **od**; **if** *n*(*x*) = *d* **then repeat** *y* = **pop** of *S*; *n*(*y*) := *infinite*; *f*(*y*) := *f*(*x*) **until** *y* = *x* **fi****end procedure** *Traverse***for** *x* ∈ Vertex **do** *n*(*x*) := 0;*f*(*x*) := {} **od**;**for** *x* ∈ Vertex **where** *n*(*x*) = 0 **do** *Traverse*(*x*, 1) **od**

9.5 Equivalence Relations

Def. 1 Let $R \subseteq A \times A$. R is called **equivalence relation**, if it is reflexive, symmetric, and transitive.

Def. 2 $a, b \in A$ are said to be related **equivalent**, written $a \sim b$, if R is an equivalent relation on A and $a R b$.

Def. 3 Let $R \subseteq A \times A$ be an **equivalence relation**.

$[a]_R = \{b \mid a R b\}$ is called the **equivalence class** of a w.r.t. R .

If $b \in [a]_R$, b is called the **representative** of the equivalent class.

Note that $a \in [a]_R$, since R is **reflexive**.

Ex. $\equiv_4 \subseteq \mathbf{Z} \times \mathbf{Z}$ is an equivalent relation. **Equivalent classes are**

$$[0]_{\equiv_4} = \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1]_{\equiv_4} = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

$$[2]_{\equiv_4} = \{\dots, -6, -2, 2, 6, 10, \dots\}$$

$$[3]_{\equiv_4} = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

Thm. 1 Let $R \subseteq A \times A$ be an **equivalence relation**. Three statements are logically equivalent.

$$i) a R b \qquad ii) [a]_R = [b]_R \qquad iii) [a]_R \cap [b]_R \neq \emptyset.$$

proof

1) $i) \rightarrow ii)$

$$\forall c \in [a]_R, a R c, a R b, b R a. \therefore b R c, c \in [b]_R. \therefore [a]_R \subseteq [b]_R.$$

$$\forall c \in [b]_R, b R c, a R b. \therefore a R c, c \in [a]_R. \therefore [b]_R \subseteq [a]_R.$$

2) $ii) \rightarrow iii)$ Assume $[a]_R = [b]_R, a R b. \therefore a, b \in [a]_R \cap [b]_R \neq \emptyset.$

3) $iii) \rightarrow i)$ Suppose $[a]_R \cap [b]_R \neq \emptyset, [a]_R \neq \emptyset$ and $[b]_R \neq \emptyset.$

$$\exists c \in [a]_R \wedge c \in [b]_R. a R c, b R c. c R b, \therefore a R b.$$

Lemma 1.5 Let $R \subseteq A \times A$ be an **equivalence relation**.

$$\cup_{a \in A} [a]_R = A. \qquad a \in [a]_R.$$

$\therefore \{[a]_R \subseteq A \mid a \in A\}$ is a **partition** of A .

Definition 1.5 Let S be a set. The **partition** of S , $\{A_i \mid i \in I\}$ I : index set, is

- i) $A_i \neq \emptyset, i \in I$. **nonempty**
- ii) $A_i \cap A_j = \emptyset$, when $i \neq j$. **disjoint**
- iii) $\cup_{i \in I} A_i = A$. **exhaustive**

Thm. 2 Let R be a equivalent relation on A .

Then the **equivalent classes** of R form a **partition** of A .

Conversely, given a **partition** $\{A_i \mid i \in I\}$ of the set A ,

there is an **equivalent relation** R that has the set $A_i, i \in I$,
as its **equivalent class**.

Relation $R \subseteq A \times A$ $O(n^2)$ where $|A| = n$.

Equivalent relation $R \subseteq A \times A$ $O(n)$

9.6 Partial Ordering

Def. 1 Let $R \subseteq A \times A$. R is called **(ir)reflexive partial order**, if it is **(ir)reflexive**, **antisymmetric**, and **transitive**.

(A, R) is called **partially ordered set** or **poset**.

Ex. 1 2 3 (\mathbb{Z}, \leq) , $(\mathbb{Z}^+, |)$, $(2S, \subseteq)$ are **posets**.

Def. 2 Let (A, \leq) be poset and $a, b \in A$. Then

The elements a and b are **comparable** if either $a \leq b$ or $b \leq a$.

The elements a and b are **incomparable** if **neither** $a \leq b$ **nor** $b \leq a$

Def. 3 Let (S, \leq) be poset. If $\forall a, b \in S$, a and b are **comparable**, S is called **totally ordered set**, **linearly ordered set**, or **chain**. \leq is called **total order** or **linear order**.

Ex. 6, 7 (\mathbb{Z}, \leq) is a **total order**, **but** $(\mathbb{Z}^+, |)$ is **not** a **total order**

Def. 4 A poset (S, \leq) is a **well-ordered set**, if \leq is a **total order**, and every nonempty subset of S has a **least element**.

Ex. 8 $(\mathbf{Z}^+ \times \mathbf{Z}^+, \leq_L)$ is **well-ordered**, **but** (\mathbf{Z}, \leq) is **not well-ordered**.

Thm .1 Principle of Well-Ordered Induction

Let (S, \leq) be a well-ordered set and $\forall x \in S, P(x)$, if

$$\forall y \in S .\exists. x < y: P(x) \Rightarrow P(y).$$

proof Suppose $\exists y \in S, \neg P(y)$.

$$A = \{x \in S \mid \neg P(x)\} \neq \emptyset.$$

Let $a \in A$ be the least element.

$$\exists a \in A, \forall x \in S .\exists. x < a: P(x) \Rightarrow P(a).$$

Contradiction $\exists y \in S, \neg P(y)$.

Lexicographic order (사전순서)

Let (A_1, \leq_1) and (A_2, \leq_2) be **well-ordered** sets. Then we define $(A_1 \times A_2, \leq_{1 \times 2})$ a lexicographic order

$$(a_1, a_2) \leq_{1 \times 2} (b_1, b_2), \text{ if } (a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2)).$$

Ext. Let $(A_1, \leq_1), (A_2, \leq_2), \dots, (A_n, \leq_n)$ be well-ordered sets. Then we define $(A_1 \times A_2 \times \dots \times A_n, \leq_{1 \times 2 \dots \times n} = \leq_L)$ a lexicographic order

$$(a_1, a_2, \dots, a_n) \leq_L (b_1, b_2, \dots, b_n), \text{ if } 1 \leq \exists i < n: [(1 \leq \forall j < i: a_j =_j b_j) \wedge (a_i <_i b_i)].$$

Ext. Let (V, \leq) be a well-ordered set. Then we define (V^*, \leq_L) as

$$(a_1, a_2, \dots, a_n) \leq_L (b_1, b_2, \dots, b_m), \text{ if } (1 \leq \exists i < n: [(1 \leq \forall j < i: a_j =_j b_j) \wedge (a_i < b_i)]) \vee [(n < m) \wedge (1 \leq \forall i \leq n: a_i = b_i)].$$

(V^*, \leq_L) is a **well-ordered** set.

Hasse Diagram

Let (A, \leq) be a poset. We say a **covers** b , if $a < b \wedge \nexists c \in A, a \leq c \wedge c \leq b$.

(A, covers) is a Hasse Diagram.

Maximal and Minimal Elements

Let (A, \leq) be a poset.

We say a is a **maximal**, if $\nexists b \in A, a < b$.

We say a is a **minimal**, if $\nexists b \in A, b < a$.

We say a is the **greatest element**, if $\forall b \in A, b \leq a$.

We say a is the **least elements**, if $\forall b \in A, a \leq b$.

$ub(S) = \{c \in A \mid a \in S, a \leq c\}$ **upper bound of S.**

$lb(S) = \{c \in A \mid a \in S, c \leq a\}$ **lower bound of S.**

$lub(S) = \{c \in A \mid c \in ub(S), d \in ub(S), c \leq d\}$

the least upper bound of S.

$glb(S) = \{c \in A \mid c \leq lb(S), d \in lb(S), d \leq c\}$

the greatest lower bound S.

Lattice

A poset (A, \leq) is called **lattice**. If every pair of elements has both a least upper bound, and a greatest lower bound.

(\mathbf{Z}, \leq) is a lattice.

$$\max, \min: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$$

$(\mathbf{Z}^+, |)$ is a lattice.

$$\text{lcm, gcd: } \mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$$

$(2^S, \subseteq)$ is a lattice.

$$\cup, \cap: 2^S \times 2^S \rightarrow 2^S$$

Topological Sorting

Lemma 1 Every finite nonempty poset (A, \leq_p) has
at least one minimal elements.

Algorithm 1 Topological sorting

Construct a total ordering $<_t$ from a partial ordering \leq_p .

$a <_t b$, if and only if, $a \leq_p b$ or a and b are **incomparable**.