9 Relations

9.1 Relations and Their Properties

Def. *1 Let A and B be two set. Then a* **binary relation** *R from A to B is subset of $A \times B$.*

$$R \subseteq A \times B.$$

A: ***domain*** *of the relation R. B:* ***range(codomain)*** *of the relation R.*

Let $a \in A$, $b \in B$, Then $(a, b) \in R$ or $(a, b) \notin R$.

If $(a, b) \in R$, we also write $a\ R\ b$ and we say a is ***related to*** *b by R.*

If $(a, b) \notin R$, we also write $a \not R\ b$ and a is ***not related to*** *b by R.*

Two ***aspects*** *of Relation*

$R \subseteq A \times B$ *relation R is a* ***set*** *of pairs*

$(a, b) \in R$

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$ *relation R is a infix* ***boolean*** *binary* ***operation***

$a\ R\ b$

Ex. $(3, 3) \in =$ and $(3, 4) \notin =$; or $3 = 3$ and $3 \neq 4$.

Function is a Relation but relation is not a function!

$R \subseteq A \times B$ vs $f: A \rightarrow B$ $\forall a \in A: \exists_1 f(a) = b \in B$.

If $f(a) = b$, then we can write $(a, b) \in f$ or $a\ f\ b$.

*. $\therefore$ Function is a (**special** kind of) relation.*

*Relation is **not** a function*

*i) If $\exists a \in A$.∋. $a\ R\ b_1$ and $a\ R\ b_2$, $(b_1 \neq b_2)$, R is **not** a function.*

*function must have an unique **related** image.*

*ii) If $\exists a \in A$ and $\nexists b \in B$.∋. $a\ R\ b$, R is **not** a function.*

all the elements in the domain must have its related image

Let $R \subseteq A \times B$. Then

We write $R(a) = \{b_1, b_2, \ldots, b_n\} \in R$, if $1 \leq {}^{\forall} i \leq n: (a, b_i) \in R$.

*A **function** is a **special kind(subclass)** of a **relation***

***Three** faces of the relation R form A to B.*

*i) R is a sub**set of pairs***

$R \subseteq A \times B$, $(a, b) \in R$ where $a \in A$ and $b \in B$.

*ii) R is a **infix binary boolean(relational) operator***

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$

$a\ R\ b$ where $a \in A$ and $b \in B$.

*iii) R is a **set valued function** from A to 2^B.*

$R: A \to 2^B$.

$R(a) = \{b_1, b_2, \ldots, b_n\}$ where $a \in A$ and $\{b_1, b_2, \ldots, b_n\} \in 2^B$.

if $1 \le {}^\forall i \le n$, $(a, b_i) \in R$ or $a\ R\ b_i$. for $n \ge 0$,

if $n=0$, $R(a) = \{b_1, b_2, \ldots, b_n\} = \varnothing$.

Note that ${}^\forall a \in A$, ${}^{\exists 1}\{b_1, b_2, \ldots, b_n\} \in 2^B$ is unique.

Three notations for the relation

i) ***subset*** *of* $A \times B$, $(a, b) \in R$.

$R \subseteq A \times B$

ii) infix binary boolean ***operation***, $a\ R\ b$,

$R: A \times B \to \{\mathbf{T}, \mathbf{F}\}$, $a\ R\ b$.

Example $=, <, \leq, \ldots \subseteq N \times N$ *or* $N \times N \to \{\mathbf{T}, \mathbf{F}\}$

$3 = 3, 3 \neq 4, 3 \leq 3, \ldots$

iii) ***set valued function***, $R(a) = \{b_1, b_2, \ldots, b_n\}$ *or* $\emptyset$.

$R: A \to 2^B$.

Relation on A Set

Def. *2 Let A be a set and* $R \subseteq A \times A$*. R is called a relation* **on** *A.*

Relation on A is a ***directed graph*** *with* ***vertices*** *A and* ***edges*** *R.*

Properties of Relations

Def. *3 A relation R is* ***reflexive***, *if* $\forall a \in A$, $a\ R\ a$.

A relation R is ***irreflexive***, *if* $\forall a \in A$, $a\ \not R\ a$.

A relation may be ***neither*** *reflexive* ***nor*** *irreflexive.*

Def. *4 A relation R is* ***symmetric***, *if* $a\ R\ b \Rightarrow b\ R\ a$.

A relation R is ***asymmetric***, *if* $a\ R\ b \Rightarrow b\ \not R\ a$.

A relation R is ***antisymmetric***, *if* $(a\ R\ b \wedge b\ R\ a) \Rightarrow (a = b)$.

or if $(a\ R\ b \wedge a \neq b) \Rightarrow b\ \not R\ a$.

If a relation is a ***asymmetric*** *then it is also* ***antisymmetric***.$(\subseteq)$

Def. *5 A relation R is* ***transitive***, *if* $a\ R\ b \wedge b\ R\ c \Rightarrow a\ R\ c$.

Combining Relations

Let $R_1, R_2 \subseteq A \times B$. *Consider* $R_1 \cup R_2, R_1 \cap R_2, R_1 \oplus R_2, R_1 - R_2, R_2 - R_1$.

Def. 6 *Let* $R \subseteq A \times B, S \subseteq B \times C$. *Then* ***composition*** *of R and S, denoted as* $S \circ R = \{(a, c) \in A \times C \mid (a, b) \in R, (b, c) \in S\}$.

Def. 7 *Let* $R \subseteq A \times A$. *Then for* $n \geq 1$,

$R^1 = R$ ***basis***

$R^{n+1} = R^n \circ R$. ***induction***

Def. 6.1 *Let A be a set. We define an* ***identity*** *relation*

$id_A = \{(a, a) \in A \times A \mid a \in A\}$ $= \Delta$ *in this text sec. 9.4 in p. 977*

Col. *0.5 Let* $R \subseteq A \times B$. *Then*

$R \circ id_A = id_A \circ R = R$. $\therefore$ id_A *is a* ***identity*** *element for* ***composition***.

Def. 7.1 *Let* $R \subseteq A \times A$. *Then for* $n \in \mathbf{N}$,

$R^0 = id_A$ ***basis*** $(x^0 = 1)$

$R^{n+1} = R^n \circ R$. ***induction***

Thm. 1 *Let $R \subseteq A \times A$. R is **transitive**, if and only if, $R^n \subseteq R$ for $\forall n \geq 1$.*

***Proof**:*

*1. (if) $R^n \subseteq R$ for $\forall n \geq 1 \Rightarrow R$ is **transitive**.*

Since $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R^2$. $\therefore (a, c) \in R$

*$\therefore$ R is **transitive**.*

*2.(only if) R is **transitive** $\Rightarrow R^n \subseteq R$ for $\forall n \geq 1$.*

mathematical induction on $n \in \mathbf{N}^+$.

***basis** Trivial for n = 1(since $R^1 = R$, by definition).*

***induction** Assume $R^n \subseteq R$ for some $n \in \mathbf{N}^+$ and R is **transitive**.*

Consider $(a, b) \in R^{n+1}$, then $\exists x \in A$.϶. $(a, x) \in R$ and $(x, b) \in R^n$.

$R^n \subseteq R$, $\therefore (x, b) \in R$. ***induction hypothesis***

Since R is transitive; $(a, x) \in R$ and $(x, b) \in R \Rightarrow (a, b) \in R$.

$\therefore R^{n+1} \subseteq R$.

9.2 n-ary Relations and Their Applications

Def. *1 Let $A_1, A_2, \ldots, A_n$ be sets. $R \subseteq A_1 \times A_2 \times \ldots \times A_n$ is a* ***n-ary relation*** *on $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the* ***domains*** *of the relations, and n is called as the* ***degree*** *of R.*

Database and Relations

relational *data model: $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$ is a* ***data base***

record $(a_1, a_2, \ldots, a_n) \in R,$

field $1 \le {}^\forall f \le n: a_f \in A_f.$

*(**primary**) **key** field: $1 \le {}^\exists k \le n: {}^\forall K \in A_k$.э. $(a_1, a_2, \ldots, K, \ldots, a_n) \in R,$*

$$|(a_1, a_2, \ldots, K, \ldots, a_n)| \le 1.$$

*The **dadabase** R is said to be **functional** in the (**key**) field A_k.*

Examle Table1 Students (St_name, St_id, Major, GPA) p. 564

Operations on n-ary Relations

Let $R \subseteq A = A_1 \times A_2 \times \ldots \times A_n$. *Then*

Def. 2 ***Selection*** *operator: Let* $C: A \to \{\mathbf{T}, \mathbf{F}\}$. *Then*

$$S_C(R) = \{a \in R \mid C(a), a \in A\}.$$

C**:** *condition* S_C: *selection operator on R*

Def. 3 ***Projection*** *Operator:* $P_{i_1 i_2 \ldots i_m}$ *where* $i_1 < i_2 < \ldots < i_m \leq n$ *maps n-tuple* $(a_1, a_2, \ldots, a_n) \in R$, *to the m-tuple* $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})(i_m \leq n)$.

$P_{\{i_k\}}: A \to A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m}$, *if* $\{i_k\} = (i_1, i_2, \ldots, i_m)$ *for* $1 \leq k \leq m$.

$$P_{\{i_k\}}(a_1, a_2, \ldots, a_n) = (a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \qquad i_m \leq n.$$

Def. 4 ***Join*** *Operator:*

Let $R_1 \subseteq A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p$ *and* $R_2 \subseteq C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

Then $J(R_1, R_2) \in A_1 \times \ldots \times A_{m-p} \times C_1 \times \ldots \times C_p \times B_1 \times \ldots \times B_{n-p}$.

If R_1: *m-tuples,* R_2: *n-tuples, then* $J(R_1, R_2)$: $m+n-p$*-tuples.* $(p \leq m, n)$

9.3 Representing Relations

Representing Relation as a Boolean Matrix

$R: A \times B \rightarrow \{0, 1\}$. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

A **boolean matrix** $M_R = [m_{ij}]$ for the relation R where

$1 \leq {}^{\forall}i \leq m: 1 \leq {}^{\forall}j \leq n:$

$m_{ij} = 1$, if $(a_i, b_j) \in R$,

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

We may write $M_R^{A \times B}[m_{ij}]$ **instead** of M_R, if needed.

Representing Relation as a Directed Graph(Digraph)

Def. 1 A **directed graph** or **digraph** $G = (V, E)$ consistes of a set V of **vertices**, and a set $E \subseteq V \times V$ of **edges(arcs)**.

$R \subseteq A \times A. \quad \Leftrightarrow \quad G = (A, R)$

relation R on A vs. **digraph** with vertices A and edges R

9.4 Closure of Relations

Closures *Let* $R \subseteq A \times A$ *and* $\mathbf{P}$ = {*reflextive, symmetric, transitive*}. *Then We define a* **p** ***closue*** *of R as* $S \subseteq A \times A$.϶. (1) $R \subseteq S$ *and*

$\forall T \subseteq A \times A$ *has the property* $\mathbf{p} \in \mathbf{P}$, $R \subseteq T$, *and* $S \subseteq T$.

(1) $R \subseteq S$ *and* (2) ***smallest one*** *among T's*(**p**)).

Reflexive *closure of R* $R \cup id_A$.

$id_A = \Delta$ *in the text*(Δ_A) ***diagonal*** *relation on A.*

Symetric *closure of R* $R \cup R^{-1}$.

Path in Digraph

Def. *1 Let G = (V, E) be a digrph. For* $a, b \in V$, *a* ***path*** *from a, b,*

$Path(a,b) = (x_0, \underline{x_1}), (\underline{x_1}, x_2), \ldots, (x_{n-1}, x_n)$

$1 \leq i \leq n$: $(x_{i-1}, x_i) \in E$, *and* $x_0 = a$, $x_n = b$.

a ***sequence*** *of edges of* ***length*** *n*

*The path may be also denoted as a **sequence** of vertices*

$Path^V(a,b) = (x_0, x_1, x_2, \ldots, x_n)$ *of **length** n.*

*We view the set of **empty** edges as a path of **length** 0 from* $a \in V$ *to a .*

A path of length $n \geq 1$ *that*

*begins and ends at the same vertex is called **cycle**.*

***Thm**. 1 Let* $R \subseteq A \times A$. *There is a **path** of length* $n \geq 1$ *from a to b,*

if and only if, $(a, b) \in R^n$.

***proof** easy(mathematical induction)*

***Def**. 1 A **connetivity** relation* $R^+ = \{(a, b) \mid (a, b) \in R^n, \forall n \geq 1\}$

$R^+ = \cup_{i \in \mathbf{N}_1} R^i = R^1 \cup R^2 \cup \ldots$

***transitive** closure of R.*

$R^* = \cup_{i \in \mathbf{N}_0} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots$

***reflexive and transitive** closure of R.*

Computing transitive closure R^*.

Let A and B be sets, $R \subseteq A \times A$, $f, g: A \to 2^C$.(set valued functions), and

$$f(a) = \{c \in C \mid c \in g(a)\} \cup \{c \in C \mid a\, R\, b,\ c \in f(b)\}.$$

$$f(a) = \{c \in g(a)\} \cup \{c \in f(b) \mid a\, R\, b\}.$$

$f(a) = g(a) \cup \cup_{a\,R\,b} f(b)$. (**recursive** definition of f) Then

$$f(a) = \{c \in C \mid c \in g(b),\ a\, R^*\, b\}.$$

$f(a) = g^*(a)$. (**iterative** definition of f)

Warshall's algorithm $O(n^3)$

Depth first search $O(n^2)$

Algorithm *Depth first search*

S: ***stack*** *of Vertex; n(Vertex)* ***array*** *of Depth;*

procedure *Traverse(x: Vertex; d: Depth);*

```
    push x onto S; n(x) := d;                       f(x) := g(x);
    for y ∈ Vertex where x R y do
        if n(y) = 0 then Traverse(y, d+1) fi;
        n(x) := min(n(x), n(y));                    f(x) := f(x) ∪ f(y)
        od;
    if n(x) = d then repeat
        y = pop of S; n(y) := infinite;             f(y) := f(x)
        until  y = x
    fi
end procedure Traverse
for x ∈ Vertex do n(x) := 0;                        f(x) := {} od;
for x ∈ Vertex where n(x) = 0 do Traverse(x, 1) od
```

9.5 Equivalence Relations

***Def**. 1 Let $R \subseteq A \times A$. R is called **equivalence** relation,*
if it is reflexive, symmetric, and transitive.

***Def**. 2 a, b $\in$ A are said to be related **equivalent**, written a ~ b, if R is an equivalent relation on A and a R b.*

***Def**. 3 Let $R \subseteq A \times A$ be an **equivalence** relation.*
*$[a]_R = \{b | a R b\}$ is called the **equivalence class** of a w.r.t. R.*
*If $b \in [a]_R$, b is called the **representative** of the equivalent class.*
*Note that $a \in [a]_R$, since R is **reflexive**.*

*Ex. $\equiv_4 \subseteq \mathbf{Z} \times \mathbf{Z}$ is an equivalent relation. **Equivalent classes** are*

$[0]_{\equiv_4} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]_{\equiv_4} = \{\ldots, -7. -3, 1, 5, 9, \ldots\}$

$[2]_{\equiv_4} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3]_{\equiv_4} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Thm. *1 Let $R \subseteq A \times A$ be an* ***quivalence*** *relation. Three statments are logically equivalent.*

i) $a\ R\ b$ ii) $[a]_R = [b]_R$ iii) $[a]_R \cap [b]_R \neq \emptyset$.

proof

1) i) $\rightarrow$ ii)

$\forall c \in [a]_R$, $a\ R\ c$, $a\ R\ b$, $b\ R\ a$. $\therefore$ $b\ R\ c$, $c \in [b]_R$. $\therefore$ $[a]_R \subseteq [b]_R$.

$\forall c \in [b]_R$, $b\ R\ c$, $a\ R\ b$. $\therefore$ $a\ R\ c$, $c \in [a]_R$. $\therefore$ $[b]_R \subseteq [a]_R$.

2) ii) $\rightarrow$ iii) Assume $[a]_R = [b]_R$, $a\ R\ b$. $\therefore$ $a, b \in [a]_R \cap [b]_R \neq \emptyset$.

3) iii) $\rightarrow$ i) Suppose $[a]_R \cap [b]_R \neq \emptyset$, $[a]_R \neq \emptyset$ and $[b]_R \neq \emptyset$.

$\exists c \in [a]_R \wedge c \in [b]_R$. $a\ R\ c$, $b\ R\ c$. $c\ R\ b$, $\therefore$ $a\ R\ b$.

Lemma *1.5 Let $R \subseteq A \times A$ be an* ***equivalence*** *relation.*

$\cup_{a\in A} [a]_R = A$. $a \in [a]_R$.

$\therefore$ $\{[a]_R \subseteq A \mid a \in A\}$ is a ***partition*** of A.

Definition *1.5 Let S be a set. The* ***partition*** *of S, $\{A_i \mid i \in I\}$ I: index set, is*

- i) $A_i \neq \emptyset,\ i \in I.$ ***nonempty***
- ii) $A_i \cap A_j = \emptyset$, when $i \neq j$. ***disjoint***
- iii) $\cup_{i \in I} A_i = A.$ ***exhaustive***

Thm. *2 Let R be a equivalent relation on A.*

Then the ***equivalent classes*** *of R form a* ***partion*** *of A.*

Conversely, given a ***partition*** $\{A_i \mid i \in I\}$ *of the set A,*

there is an ***equivalent relation*** *R that has the set A_i, $i \in I$,*

as its ***equivalent class***.

Realtion $R \subseteq A \times A$ $O(n^2)$ where $|A| = n$.

Equivalent *relation* $R \subseteq A \times A$ $O(n)$

9.6 Partial Ordering

Def. *1 Let R ⊆ A × A. R is called* ***(ir)reflexive*** ***partial order***,
if it is ***(ir)reflexive***, ***antisymmetric***, *and* ***transitive***.
(A, R) is called ***partially odreded*** *set or* ***poset***.

Ex. 1 2 3 (Z, ≤), (Z+, |), (2S, ⊆) are ***posets***.

Def. *2 Let (A, ≤) be poset and a, b ∈ A. Then*
The elements a and b are ***comparable*** *if either a ≤ b or b ≤ a.*
The elements a and b are ***incomparable*** *if* ***neither*** *a ≤ b* ***nor*** *b ≤ a*
.

Def. *3 Let (S, ≤) be poset. If ∀a, b ∈ S, a and b are* ***comparable***,
S is called ***totally ordered set***, ***linearly ordered set***, *or* ***chain***.
≤ is called ***total order*** *or* ***linear order***.

*Ex. 6, 7 (**Z**, ≤) is a* ***total*** *order, but ($\mathbf{Z}^+$, |) is* ***not*** *a* ***total*** *order*

Def. *4 A poset $(S, \leq)$ is a* ***well-ordered*** *set, if $\leq$ is a* ***total order***, *and every nonempty subset of S has a* ***least*** *emelent.*

Ex. 8 $(\mathbf{Z}^+ \times \mathbf{Z}^+, \leq_L)$ *is* ***well-ordered***, *but* $(\mathbf{Z}, \leq)$ *is* ***not*** ***well-****orderd.*

Thm *.1* ***Principle of Well-Ordered Induction***

Let $(S, \leq)$ be a well-ordered set and $\forall x \in S, P(x)$, if

$\forall y \in S$.э. $x < y: P(x) \Rightarrow P(y)$.

proof *Suppose* $\exists y \in S, \neg P(y)$.

$A = \{x \in S | \neg P(x)\} \neq \emptyset$.

Let $a \in A$ be the least element.

$\exists a \in A, \forall x \in S$.э. $x < a: P(x) \Rightarrow P(a)$.

Contradiction $\exists y \in S, \neg P(y)$.

Lexicographic order(사전순서)

Let $(A_1, \leq_1)$ *and* $(A_2, \leq_2)$ *be* ***well-ordered*** *sets. Then we define* $(A_1 \times A_2, \leq_{1\times 2})$ *a* *lexicographic order*

$$(a_1, a_2) \leq_{1\times 2} (b_1, b_2), \text{ if } (a_1 <_1 b_1) \vee ((a_1 =_1 b_1) \wedge (a_2 \leq_2 b_2)).$$

Ext. *Let* $(A_1, \leq_1), (A_2, \leq_2), \ldots (A_n, \leq_n)$ *be well-ordered sets. Then we define* $(A_1 \times A_2 \times \ldots \times A_n, \leq_{1\times 2 \ldots \times n} = \leq_L)$ *a lexicographic order*

$$(a_1,a_2,\ldots,a_n) \leq_L (b_1,b_2,\ldots,b_n), \text{ if } 1 \leq^{\exists} i < n: [(1 \leq^{\forall} j < i: a_j =_j b_j) \wedge (a_i <_i b_i)].$$

Ext. *Let* $(V, \leq)$ *be a well-ordered set. Then we define* $(V^*, \leq_L)$ *as*

$$(a_1,a_2,\ldots,a_n) \leq_L (b_1,b_2,\ldots,b_n), \text{ if } (1 \leq^{\exists} i < n: [(1 \leq^{\forall} j < i: a_j =_j b_j) \wedge (a_i < b_i)].$$

$$\vee [(n < m) \wedge (1 \leq^{\forall} i \leq n: a_i = b_i)].$$

$(V^*, \leq_L)$ *is a* ***well-ordered*** *set.*

<u>Hasse Diagram</u>

Let (A, ≤) be a poset. We say a **covers** *b, if* $a < b \wedge \nexists c \in A, a \leq c \wedge c \leq b$.

(A, covers) is a Hasse Diagram.

<u>Maximal and Minimal Elements</u>

Let (A, ≤) be a poset.

We say a is a **maximal**, *if* $\nexists b \in A, a < b$.

We say a is a **minimal**, *if* $\nexists b \in A, b < a$.

We say a is the **greatest element**, *if* $\forall b \in A, b \leq a$.

We say a is the **least elements**, *if* $\forall b \in A, a \leq b$.

$ub(S) = \{c \in A \mid a \in S, a \leq c\}$ **upper bound** *of S.*

$lb(S) = \{c \in A \mid a \in S, c \leq a\}$ **lower bound** *of S.*

$lub(S) = \{c \in A \mid c \in ub(S), d \in ub(S), c \leq d\}$

the **least upper bound** *of S.*

$glb(S) = \{c \in A \mid c \leq lb(S), d \in lb(S), d \leq c\}$

the **greatest lower bound** *S.*

Lattice

A poset (A, ≤) is called ***lattice****. If every pair of elements has both a least upper bound, and a greatest lower bound.*

$(\mathbf{Z}, \leq)$ *is a lattice.* *max, min:* $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$

$(\mathbf{Z}^+, |)$ *is a lattice.* *lcm, gcd:* $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$

$(2^S, \subseteq)$ *is a lattice.* $\cup, \cap: 2^S \times 2^S \rightarrow 2^S$

Topological Sorting

Lemma *1 Every* ***finite nonempty*** *poset* $(A, \leq_p)$ *has* ***at least one minimal*** *elements.*

Algorithm *1* ***Topological sorting***

Construct a total ordering $<_t$ *from a partial ordering* $\leq_p$.

$a <_t b$, *if and only if,* $a \leq_p b$ *or a and b are* ***incomparable***.